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The essential rate of approximation for radial function manifold

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Abstract In this paper, we investigate the radial function manifolds generated by a linear combination of radial functions. Let $W_p^r(\mathbb{B}^d)$ be the usual Sobolev class of functions on the unit ball $\mathbb{B}^d$. We study the deviation from the radial function manifolds to $W_p^r(\mathbb{B}^d)$. Our results show that the upper and lower bounds of approximation by a linear combination of radial functions are asymptotically identical. We also find that the radial function manifolds and ridge function manifolds generated by a linear combination of ridge functions possess the same rate of approximation.

Keywords radial function, rate of approximation, Sobolev class, ridge function

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1 Introduction

Radial function manifolds (RFMs) can be formally described as devices producing input-output functions depending on flexible parameters. Often input-output functions have the form of linear combination of functions computable by units specific for the given type of a manifold. We call a multivariate function f a radial function if there exists a univariate function F such that $f(\cdot) = F(|\cdot|)$, where $|x|$ denotes the Euclidean norm of vector x. Radial functions appear naturally in harmonic analysis, special function theory, and in several applications such as tomography and radial basis function networks (RBFNs).

Approximation by radial functions has been extensively investigated in recent years. A special type of linear combination of radial functions is RBFN which takes the form of

$$\sum_{i=1}^{m} C_i\phi(w_i|x-x_i|^2), \quad C_i, w_i \in \mathbb{R},\ x_i \in \mathbb{R}^d,$$

where ϕ is a fixed univariate function, and is always chosen for some special purposes. The approximation properties of RBFNs have been studied in several papers. For example, by taking the thin-plate spline type function as the activation function, the upper bound of approximation by RBFNs in various metrics has been established in [1, 2] and [19, 20]. Meanwhile, the lower bound of approximation by RBFNs of this type has been given in [8] and [14]. In [14], Maiorov has also proved that the upper bound and

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lower bound of approximation by RBFNs in the usual Sobolev space are asymptotically identical. For RBFNs with multiquadric and other smooth activation function, Yoon [25] has given the upper bound error estimate for interpolating functions in the Sobolev space by RBFNs. In [11], we have proved that if the activation function is analytical and non-polynomial, then the rate of approximation by RBFNs is not slower than that of approximation by algebraic polynomials. For more details of approximation by RBFNs, we refer the readers to [3–7, 15, 16, 18, 21, 24] and references therein.

For RFMs, whose elements possess the form

$$\sum_{i=1}^{m} g_i(|x-x_i|^2), \quad x_i \in \mathbb{R}^d,$$

where g_i are some univariate functions, Maiorov [13] has studied their approximation properties in a Sobolev space. The results revealed that in order to approximate functions in the Sobolev space, radial functions manifolds and ridge functions manifolds have the same approximation capacity (see [12] and [13]).

In [10], the upper and lower bounds of approximation by ridge function manifolds have been established in a larger Sobolev space than that of [12]. Thus it is natural to arise the question: does this upper bound and lower bound also hold for RFMs? In this paper, we devote into giving an affirmative answer to this question. In other words, we will give the upper and lower bounds of approximation by RFMs in a larger Sobolev space than that in [13], and prove that the upper and lower bounds of approximation by RFMs and that by ridge function manifolds are asymptotically identical.

The paper is organized as follows. In Section 2, some auxiliary tools will be given. In Section 3, we will give our main results of this paper, while their proofs will be given in Section 4.

2 Preliminaries

At first, we give some definitions and notations.

Let $d \in \mathbb{N}$ and $\mathbb{B}^d$ be the open unit ball in $\mathbb{R}^d$. For $r \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, we denote by $W_p^r(\mathbb{B}^d)$ the usual Sobolev class consisting of r-times differentiable function $f : \mathbb{B}^d \to \mathbb{R}$ such that

$$\sum_{|\mathbf{k}|=r} \|D^{\mathbf{k}} f\|_{L^p(\mathbb{B}^d)} \leqslant 1,$$

where $\mathbf{k} := (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ and $D^{\mathbf{k}} f$ is the partial derivative, in the Sobolev sense, of order $|\mathbf{k}| := k_1 + \cdots + k_d$.

Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. We denote by $\mathcal{R}_m$ the manifold consisting of all possible linear combinations of m radial functions

$$R_m(x) := \sum_{l=1}^{m} \phi_l(|x-a_l|^2), \quad x \in \mathbb{B}^d,$$

where $\{\phi_l\}_{l=1}^m$ are real-valued functions defined on $[0,4]$, and $\{a_l\}_{l=1}^m \subset \mathbb{S}^{d-1}$ is the set of centers of $R_m(\cdot)$. If $\phi_l \in L^q([0,4])$ $(1 \leqslant q \leqslant \infty)$, then we write $\mathcal{R}_{m,q}(\mathbb{B}^d)$ instead of $\mathcal{R}_m(\mathbb{B}^d)$.

Write $\mathcal{P}_n(\mathbb{B}^d)$ as the space of polynomials

$$P_n(x) := \sum_{|\mathbf{k}| \leqslant n} c_{\mathbf{k}} x^{\mathbf{k}}, \quad x \in \mathbb{B}^d,$$

where $x^{\mathbf{k}} := x_1^{k_1} \cdots x_d^{k_d}$ and $c_{\mathbf{k}} := c_{k_1,k_2,\ldots,k_d} \in \mathbb{R}$. Denote by $\mathcal{P}_n^h(\mathbb{B}^d)$ the space of all homogeneous polynomials with degree n on $\mathbb{B}^d$, i.e.,

$$\mathcal{P}_n^h(\mathbb{B}^d) := \left\{ P_n^h : P_n^h = \sum_{|\mathbf{k}|=n} c_{\mathbf{k}} x^{\mathbf{k}},\ x \in \mathbb{B}^d \right\}.$$

Define $\mathcal{P}_n^h(\mathbb{S}^{d-1})$ as the restriction to $\mathbb{S}^{d-1}$ of the space $\mathcal{P}_n^h(\mathbb{B}^d)$.

Let $\mathbb{T}^d := [0,2\pi)^d := [0,2\pi)\times\cdots\times[0,2\pi)$ be the d-dimensional torus. By $\mathcal{T}_n(\mathbb{T}^d)$ we denote the space of all trigonometric polynomials

$$T_n(\eta) = \sum_{|\mathbf{k}|\leqslant n} a_{\mathbf{k}} e^{i(\mathbf{k}\cdot\eta)}, \quad \eta\in\mathbb{T}^d,$$

where $a_{\mathbf{k}}\in\mathbb{R}$ and $\mathbf{k}\cdot\eta$ denotes the usual inner of the d-dimensional vectors $\mathbf{k}$ and η. Denote by $\mathcal{T}_n^h(\mathbb{T}^d)$ the space of all trigonometric polynomials

$$T_n(\eta) = \sum_{|\mathbf{k}|= n} a_{\mathbf{k}} e^{i(\mathbf{k}\cdot\eta)}, \quad \eta\in\mathbb{T}^d.$$

For any two sets of functions $W, U \in L^q(\mathbb{B}^d)$, we denote by

$$E(W,U)_{L^q(\mathbb{B}^d)} := \sup_{f\in W} E(f,U)_{L^q(\mathbb{B}^d)} := \sup_{f\in W}\inf_{f^*\in U}\|f-f^*\|_{L^q(\mathbb{B}^d)},$$

the deviation in $L^q(\mathbb{B}^d)$, of W form U.

In the following, some auxiliary lemmas will be given. The first one shows how to represent the moment of radial function relative to arbitrary algebraic polynomial, whose proof is the same as that of [13, Theorem 5.1]. For the sake of brevity, we omit it.

Lemma 1. *Let $d,n\in\mathbb{N}$, $n\geqslant\frac{d^2}{2}$, $a\in\mathbb{S}^{d-1}$ and $g\in L^1([0,4])$. Let $g_a(x) := g(|x-a|^2)$ be any radial function with center $a\in\mathbb{S}^{d-1}$ and P_n be arbitrary algebraic polynomial on $\mathbb{B}^d$ with degree n. Then*

$$\langle g_a, P_n\rangle = \sum_{j=0}^{(2d+5)n} \pi_j(a;P_n)\gamma_j(g),$$

where the functions $\pi_m(a;P_n)$ are some polynomials on the vector a of degree n, $\gamma_j(g)$ are some linear functionals acting on $g\in L^1([0,4])$, and $\langle g_a,P_n\rangle := \int_{\mathbb{B}^d} P_n(x)g_a(x)dx$.

The following Lemma 2 reveals that if the number of the linear combination of radial functions m and the degree of the polynomial n are chosen in a special way, then for any $1\leqslant q\leqslant\infty$, the polynomial space belongs to the manifold $\mathcal{R}_{n,q}$ with centers belonging to $\mathbb{S}^{d-1}$.

Lemma 2. *Let m and n be any natural numbers such that $m = \dim\mathcal{P}_n^h$. Then for any $1\leqslant q\leqslant\infty$, there exists a set of centers $\{a_l\}_{l=1}^m\subset\mathbb{S}^{d-1}$ such that $\mathcal{P}_n(\mathbb{B}^d)\subset\mathcal{R}_{n,q}(\mathbb{B}^d)$.*

Proof. The proof of Lemma 2 can be found in [13, Lemma 7.1]. The result here is only a slight extension of [13, Lemma 7.1], and its proof is the same as which of [13, Lemma 7.1]. □

The following Lemma 3 will play an important role in our proof, which can be found in [22] (see also [9]).

Lemma 3. *Let $d,n\in\mathbb{N}$. For each constant $0<c_*<1$ and every subspace $\mathcal{T}_*\subseteq\mathcal{T}_n(\mathbb{T}^d)$ such that $\dim\mathcal{T}_*\geqslant c_*\dim\mathcal{T}_n(\mathbb{T}^d)$, there exists a trigonometric polynomial $T_*\in\mathcal{T}_*$ such that*

$$\|T_*\|_{L^\infty(\mathbb{T}^d)} = 1 \quad \textit{and} \quad \|T_*\|_{L^2(\mathbb{T}^d)}\geqslant c^*,$$

where $0<c^*=c^*(d,c_*)<1$.

Let $W_p^{*,r}(\mathbb{B}^d)$ be the subclass of all radial functions in $W_p^r(\mathbb{B}^d)$. The following Lemma 4 (see [10, Theorem 1]) establishes the orders of best approximation by polynomials of the classes $W_p^{*,r}(\mathbb{B}^d)$ and $W_p^r(\mathbb{B}^d)$ in $L_q(\mathbb{B}^d)$.

Lemma 4. *Let $d,r\in\mathbb{N}$, and $1\leqslant p,q\leqslant\infty$ such that $r-d(1/p-1/q)>0$. Then*

$$E\left(W_p^{*,r}(\mathbb{B}^d),\mathcal{P}_n(\mathbb{B}^d)\right)_{L^q(\mathbb{B}^d)} \sim E\left(W_p^{r}(\mathbb{B}^d),\mathcal{P}_n(\mathbb{B}^d)\right)_{L^q(\mathbb{B}^d)} \sim n^{-r+d(1/p-1/q)_+},$$

where $(a)_+ := \max\{a,0\}$, $a\in\mathbb{R}$, and $A\sim B$ denotes that there exists an absolute constant $C>0$ independent of A and B such that $\frac{1}{C}A\leqslant B\leqslant CA$.

For $d > 1$, we denote by $G_{d,n}$ the Gegenbauer polynomials defined by the generating function (see [17])

$$\sum_{n=0}^{\infty} G_{d,n}(t)s^n := (1-2st+s^2)^{-d/2}, \quad |s|<1, -1 \leqslant t \leqslant 1.$$

Define

$$U_{d,n}(t) := u_{d,n}^{-1/2} G_{d,n}(t), \quad t \in [-1,1],$$

with

$$u_{d,n} := \frac{\pi^{1/2}(d)_n \Gamma(d/2+1/2)}{(n+d/2)n!\Gamma(d/2)},$$

where $(d)_0 := 0$, $(d)_n := d(d+1)\cdots(d+n-1) = \Gamma(d+n)/\Gamma(d)$. Since the Gegenbauer polynomial $G_{d,n}$ is an odd function on $[-1,1]$ when n is odd, it is obvious that $U_{d,n}(t)$ is odd on the interval $[-1,1]$ when n is odd. The following Lemma 5 (see [17, Theorem 3.1]) gives a decomposition of square integrable function defined on $\mathbb{B}^d$.

Lemma 5. *If $d \in \mathbb{N}$, $d > 1$, then each function $f \in L^2(\mathbb{B}^d)$ has a unique representation*

$$f = \sum_{n=0}^{\infty} Q_{d,n}(\cdot; f),$$

where the convergence is in $L^2(\mathbb{B}^d)$, and

$$Q_{d,n}(x;f) := v_{d,n} \int_{\mathbb{S}^{d-1}} A_{d,n}(\xi;f) U_{d,n}(\xi \cdot x) d\xi, \quad x \in \mathbb{B}^d,$$

with

$$A_{d,n}(\xi;f) := \int_{\mathbb{B}^d} f(\tau) U_{d,n}(\xi \cdot \tau) d\tau, \quad \xi \in \mathbb{S}^{d-1} \tag{1}$$

and

$$v_{d,n} := \frac{(n+1)_{d-1}}{2(2\pi)^{d-1}}.$$

Moreover, the operators $Q_{d,n}(\cdot; f)$, $n \in \mathbb{N}$, are orthogonal projectors from $L^2(\mathbb{B}^d)$ onto $\mathcal{P}_n(\mathbb{B}^d) \ominus \mathcal{P}_{n-1}(\mathbb{B}^d)$, and the following Paresval identity holds:

$$\|f\|^2_{L^2(\mathbb{B}^d)} = \sum_{n=0}^{\infty} \|Q_{d,n}(\cdot;f)\|^2_{L^2(\mathbb{B}^d)} = \sum_{n=0}^{\infty} v_{d,n} \|A_{d,n}(\cdot;f)\|^2_{L^2(\mathbb{S}^{d-1})}. \tag{2}$$

3 Main results

In this section, we give the main results of this paper. In [13], Maiorov established the upper and lower bounds of approximation functions in the Sobolev class $W_2^r(\mathbb{B}^d)$. Namely, he proved that

$$E(W_2^r(\mathbb{B}^d), \mathcal{R}_{n,2}(\mathbb{B}^d))_{L^2(\mathbb{B}^d)} \sim n^{-r/(d-1)}.$$

Our first result is to extend the Sobolev class $W_2^r(\mathbb{B}^d)$ to $W_p(\mathbb{B}^d)$ $(1 \leqslant p \leqslant \infty)$. In other words, we are going to establish the upper and lower bounds of approximation functions in $W_p(\mathbb{B}^d)$ by elements of $\mathcal{R}_{n,2}$.

Theorem 1. *Let $d, r \in \mathbb{N}$, $d > 1$, and $1 \leqslant p \leqslant \infty$ be such that $r - d(1/p - 1/2) > 0$. Then*

$$E\left(W_p^{*,r}(\mathbb{B}^d), \mathcal{R}_{n,2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \sim E\left(W_p^r(\mathbb{B}^d), \mathcal{R}_{n,2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \sim n^{-\frac{r-d(1/p-1/2)_+}{d-1}}.$$

Our second result shows that if the target function is radial, then the rate of approximation by algebraic polynomials and that by elements of $\mathcal{R}_{n,2}$ are asymptotically identical.

Theorem 2. *Let $n, d \in \mathbb{N}$ and $d > 1$. There exist constants $C := C(d) > 0$, $C_1 := C_1(d)$ and $C_2 := C_2(d)$ depending only on d such that for any radial function $f \in L^2(\mathbb{B}^d)$,*

$$CE\left(f, \mathcal{P}_{C_1 n}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \leqslant E\left(f, \mathcal{R}_{n^{d-1},2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \leqslant E\left(f, \mathcal{P}_{C_2 n}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)}. \tag{3}$$

4 Proofs of the main results

Proof of Theorem 2. At first we prove the left inequality of (3). For any set of points $\{a_l\}_{l=1}^m \subset \mathbb{S}^{d-1}$, and arbitrary set of univariate functions from $L^2([0,4])$ $\{r_l\}_{l=1}^m$, the function

$$R_m(x) = \sum_{l=1}^m r_l(|x-a_l|^2), \quad x \in \mathbb{B}^d$$

belongs to $\mathcal{R}_{m,2}(\mathbb{B}^d)$. Given a radial function $f \in L^2(\mathbb{B}^d)$, we are going to estimate the norm $\|f - R_m\|_{L^2(\mathbb{B}^d)}$ from below. It follows from Lemma 5 that

$$\begin{aligned}\|f(\cdot)-R_m(\cdot)\|^2_{L^2(\mathbb{B}^d)} &= \sum_{k=0}^\infty \|Q_{d,k}(\cdot;f-R_m)\|^2_{L^2(\mathbb{B}^d)} = \sum_{k=0}^\infty v_{d,k}\|A_{d,k}(\cdot;f-R_m)\|^2_{L^2(\mathbb{S}^{d-1})}\\ &\geqslant \sum_{k=0}^\infty v_{d,2k}\|A_{d,2k}(\cdot;f-R_m)\|^2_{L^2(\mathbb{S}^{d-1})}.\end{aligned}$$

Thus we only need to estimate $\|A_{d,2k}(\cdot;f-R_m)\|^2_{L^2(\mathbb{S}^{d-1})}$ from below.

Since for any $k \geqslant 0$, as a function of $\xi \in \mathbb{S}^{d-1}$, $A_{d,2k}(\xi;f-R_m)$ belongs to the space $\mathbb{H} := \bigoplus_{i=0}^k \mathcal{H}_{2k-2i}(\mathbb{S}^{d-1})$ (see [17] or [23]), where $\mathcal{H}_i(\mathbb{S}^{d-1})$ denotes the space of spherical harmonics of degree i on $\mathbb{S}^{d-1}$. We obtain

$$\|A_{d,2k}(\cdot;f-R_m)\|_{L^2(\mathbb{S}^{d-1})} = \sup_{\|H\|_{L^2(\mathbb{S}^{d-1})}\leqslant 1,\ H\in\mathbb{H}} \int_{\mathbb{S}^{d-1}} A_{d,2k}(\xi;f-R_m)H(\xi)d\xi.$$

The definition of $A_{d,2k}(\xi;f)$ yields that

$$A_{d,2k}(\xi;f-R_m) = A_{d,2k}(\xi;f) - \sum_{l=1}^m A_{d,2k}(\xi;r_l(|\cdot-a_l|^2)).$$

Moreover, since f is radial on $\mathbb{B}^d$, it follows that $A_{d,2k}(\xi;f)$ does not depend on $\xi \in \mathbb{S}^{d-1}$ and we may write

$$A_{d,2k}(f) := A_{d,2k}(\xi;f), \quad \forall\,\xi \in \mathbb{S}^{d-1}.$$

Therefore, we conclude that

$$\begin{aligned}&\|A_{d,2k}(\cdot;f-R_m)\|_{L^2(\mathbb{S}^{d-1})}\\ &\quad= \sup_H \left(A_{d,2k}(f)\int_{\mathbb{S}^{d-1}} H(\xi)d\xi - \sum_{l=1}^m \int_{\mathbb{S}^{d-1}}\int_{\mathbb{B}^d} r_l(|\tau-a_l|^2)U_{d,2k}(\xi\cdot\tau)d\tau H(\xi)d\xi\right)\\ &\quad= \sup_H \left(A_{d,2k}(f)\int_{\mathbb{S}^{d-1}} H(\xi)d\xi - \sum_{l=1}^m \int_{\mathbb{B}^d} r_l(|\tau-a_l|^2)\int_{\mathbb{S}^{d-1}} U_{d,2k}(\xi\cdot\tau)H(\xi)d\xi d\tau\right),\end{aligned}$$

where the supremum is taken over all $H \in \mathbb{H}$ such that $\|H\|_{L^2(\mathbb{S}^{d-1})} \leqslant 1$.

Note that

$$\int_{\mathbb{S}^{d-1}} U_{d,2k}(\xi\cdot\tau)H(\xi)d\xi$$

is a polynomial depending on H with degree not larger than $2k$. Then from Lemma 1 we have

$$\|A_{d,2k}(\cdot;f-R_m)\|_{L^2(\mathbb{S}^{d-1})} = \sup_H \left(A_{d,2k}(f)\int_{\mathbb{S}^{d-1}} H(\xi)d\xi - \sum_{l=1}^m \sum_{j=0}^{(4d+10)k} \pi_j(a_l;H)\gamma_j(r_l)\right), \tag{4}$$

where $\pi_j(a_l;H)$ are some polynomials depending on H with variable a_l $(1 \leqslant l \leqslant m)$ of degree $2k$, and $\gamma_j(r_l)$ are some linear functionals on $C([0,4])$.

In order to estimate the right hand of (4) from below, we will construct a subset of $\mathbb{H}$ in which the right hand of (4) can be computed easily.

Let $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1})$ denote the subspace of $\mathcal{P}_{2j}^h(\mathbb{S}^{d-1})$, consisting of all spherical polynomial of the form

$$S_{2j}^h(\xi) = \sum_{j_1+\cdots+j_d=j} \alpha_{j_1,\ldots,j_d}\xi_1^{2j_1}\cdots\xi_d^{2j_d}, \quad \xi=(\xi_1,\ldots,\xi_d)\in\mathbb{S}^{d-1}, \tag{5}$$

where $(j_1,\ldots,j_d)\in\mathbb{Z}^d$ and $\alpha_{j_1,\ldots,j_d}\in\mathbb{R}$. We denote by $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\})$ the subspace of all spherical polynomials $S_{2j}\in\mathcal{S}_{2j}^h(\mathbb{S}^{d-1})$ which satisfy

$$\sum_{i=0}^{(4d+10)j}\pi_i(a_l;S_{2j})=0, \quad l=1,\ldots,m.$$

The space $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\})$ is the desired subspace of H. Hence,

$$\begin{aligned}\|A_{d,2k}(\cdot;f-R_m)\|_{L^2(\mathbb{S}^{d-1})} &\geqslant \max_{0\leqslant j\leqslant k}\sup_{S_{2j}}\left(A_{d,2k}(f)\int_{\mathbb{S}^{d-1}}S_{2j}(\xi)d\xi-\sum_{l=1}^{m}\sum_{i=0}^{(4d+10)k}\pi_i(a_l;S_{2j})\gamma_i(r_l)\right)\\ &= A_{d,2k}(f)\max_{0\leqslant j\leqslant k}\sup_{S_{2j}}\int_{\mathbb{S}^{d-1}}S_{2j}(\xi)d\xi,\end{aligned}$$

where the supremum is taken over the spherical polynomials $S_{2j}\in\mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\})$, $0\leqslant j\leqslant k$, such that $\|S_{2j}\|_{L^2(\mathbb{S}^{d-1})}\leqslant 1$.

In order to estimate

$$\max_{0\leqslant j\leqslant k}\sup_{S_{2j}}\int_{\mathbb{S}^{d-1}}S_{2j}(\xi)d\xi$$

from below, Lemma 3 will be used as a main tool. Thus we need to transfer spherical polynomials to trigonometric polynomials on $\mathbb{T}^{d-1}$ at first. Let $\phi:=(\phi_1,\ldots,\phi_{d-1})$ be the spherical coordinates on $\mathbb{S}^{d-1}$ defined by

$$\begin{aligned}\xi_1&=\cos\phi_1,\\ \xi_2&=\sin\phi_1\cos\phi_2,\\ &\vdots\\ \xi_{d-2}&=\sin\phi_1\cdots\sin\phi_{d-3}\cos\phi_{d-2},\\ \xi_{d-1}&=\sin\phi_1\cdots\sin\phi_{d-2}\cos\phi_{d-1},\\ \xi_d&=\sin\phi_1\cdots\sin\phi_{d-2}\sin\phi_{d-1},\end{aligned}$$

where $0\leqslant\phi\leqslant\pi$ $(1\leqslant i\leqslant d-2)$, and $0\leqslant\phi_{d-1}<2\pi$. With $\xi=\xi(\phi)$, the surface element $d\xi$ of $\mathbb{S}^{d-1}$ becomes

$$d\xi=J(\phi)d\phi,$$

where the Jacobian is given by

$$J(\phi):=(\sin\phi_1)^{d-2}(\sin\phi_2)^{d-3}\cdots\sin\phi_{d-2}.$$

It is easy to verify that for each $S_{2j}\in\mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\})$ the function $T_{2j}(\phi):=S_{2j}(\xi(\phi))$, $\phi\in\mathbb{T}^{d-1}$, belongs to the space $\mathcal{T}_{2k}^h(\mathbb{T}^{d-1})$ of trigonometric polynomials on the torus $\mathbb{T}^{d-1}$. We denote the collection of these functions by $\mathcal{T}_{2j}^h(\mathbb{T}^{d-1};\{a_l\})$.

Since $\sum_{i=0}^{(4d+10)k}\pi_i(a_l;S_{2j})$ is a spherical polynomial with variable a_l and degree not larger than $2k$, we impose at most m linear restrictions on the coefficients of the spherical polynomials in $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\})$. Noting that

$$\dim\mathcal{S}_{2j}^h(\mathbb{S}^{d-1})=\begin{pmatrix} j+d-1\\ j\end{pmatrix},$$

we get

$$\dim \mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\}) \geqslant \begin{pmatrix} j+d-1 \\ j \end{pmatrix} - m, \quad j=0,\ldots,k.$$

Therefore,

$$\dim \mathcal{T}_{2j}^h(\mathbb{T}^{d-1};\{a_l\}) \geqslant \dim \mathcal{S}_{2j}^h(\mathbb{S}^{d-1};\{a_l\}) \geqslant \begin{pmatrix} j+d-1 \\ j \end{pmatrix} - m, \quad j=0,\ldots,k.$$

It follows from (5) that $T_{2j}(\phi)$ is even with respect to each variable ϕ_i $(i=1,\ldots,d-2)$. Hence

$$\int_{\mathbb{S}^{d-1}} S_{2j}(\xi)d\xi = \frac{1}{2^{d-2}}\int_{\mathbb{T}^{d-1}} T_{2j}(\phi)|J(\phi)|d\phi,$$
$$\int_{\mathbb{S}^{d-1}} (S_{2j}^h(\xi))^2 d\xi = \frac{1}{2^{d-2}}\int_{\mathbb{T}^{d-1}} (T_{2j}^h(\phi))^2|J(\phi)|d\phi.$$

Hence

$$\|A_{d,2k}(\cdot;f-R_m)\|_{L^2(\mathbb{S}^{d-1})} \geqslant \frac{A_{d,2k}(f)}{2^{d-2}} \max_{0\leqslant j\leqslant k} \sup_{T_{2j}^h} \int_{\mathbb{T}^{d-1}} T_{2j}^h(\phi)|J(\phi)|d\phi,$$

where the supremum is taken over all $T_{2j} \in \mathcal{T}_{2j}^h(\mathbb{T}^{d-1};\{a_l\})$ such that

$$\frac{1}{2^{d-2}}\int_{\mathbb{T}^{d-1}} (T_{2j}(\phi))^2|J(\phi)|d\phi \leqslant 1, \quad j=0,\ldots,k. \tag{6}$$

Let $k(d,m) \in \mathbb{N}$ be the smallest k satisfying $k^{d-1} \geqslant 2^{2d-1}(d-1)!m$. Thus for every $k \geqslant k(d,m)$ we have

$$m \leqslant \frac{k^{d-1}}{2^{2d-1}(d-1)!}.$$

Let $k' := 2[k/2]$, and consider the subspace

$$\mathcal{T}_* := \mathcal{T}_{k'}^h(\mathbb{T}^{d-1};\{a_l\}).$$

Since

$$\begin{pmatrix} k'/2+d-1 \\ k'/2 \end{pmatrix} \geqslant \frac{k^{d-1}}{4^{d-1}(d-1)!},$$

we obtain

$$\dim \mathcal{T}_{k'}^h(\mathbb{T}^{d-1};\{a_l\}) \geqslant \begin{pmatrix} k'/2+d-1 \\ k'/2 \end{pmatrix} - m \geqslant \frac{k^{d-1}}{4^{d-1}(d-1)!} - \frac{k^{d-1}}{2^{2d-1}(d-1)!} = \frac{k^{d-1}}{2^{2d-1}(d-1)!}.$$

On the other hand, it is obvious that

$$\dim \mathcal{T}_{2k}(\mathbb{T}^{d-1}) \leqslant 5^{d-1}k^{d-1}.$$

Thus,

$$\dim \mathcal{T}_* = \dim \mathcal{T}_{k'}^h(\mathbb{T}^{d-1};\{a_l\}) \geqslant c_* \dim \mathcal{T}_{2k}(\mathbb{T}^{d-1}),$$

where $c_* := \frac{1}{5^{d-1}2^{2d-1}(d-1)!}$. Therefore, the assumptions of Lemma 3 are satisfied. Then there exists a trigonometric polynomial $T_* \in \mathcal{T}_*$ such that

$$\|T_*\|_{L^\infty(\mathbb{T}^{d-1})} = 1 \text{ and } \|T_*\|_{L^2(\mathbb{T}^{d-1})} \geqslant c_1, \tag{7}$$

where $0 < c_1 = c_1(d, c_*) < 1$.

Let Φ be a subset of $\mathbb{T}^{d-1}$, and $T^*(\phi) := (T_*(\phi))^2,\ \phi \in \mathbb{T}^{d-1}$. Denote by $|\Phi|$ the Lebesgue measure of Φ. Then $T^* \in \mathcal{T}^h_{2k}(\mathbb{T}^{d-1}; \{a_l\})$. Moreover, it follows from (7) that

$$\|T^*\|_{L^\infty(\mathbb{T}^{d-1})} = 1 \text{ and } \int_{\mathbb{T}^{d-1}} T^*(\phi)d\phi \geqslant c_1|\mathbb{T}^{d-1}|, \tag{8}$$

where $c_1 := (c^*)^2/|\mathbb{T}^{d-1}|$.

If Φ^* is the subset in $\mathbb{T}^{d-1}$ of all points ϕ such that $T^*(\phi) \geqslant c_1/4$, then it follows from (8) that

$$\begin{aligned}|\Phi^*| = \int_{\Phi^*} d\phi \geqslant \int_{\Phi^*} T^*(\phi)d\phi &= \int_{\mathbb{T}^{d-1}} T^*(\phi)d\phi - \int_{\mathbb{T}^{d-1}/\Phi^*} T^*(\phi)d\phi \\ &\geqslant c_1|\mathbb{T}^{d-1}| - \frac{c_1}{4}(|\mathbb{T}^{d-1}| - |\Phi^*|) \geqslant c_1|\mathbb{T}^{d-1}| - \frac{c_1}{4}|\mathbb{T}^{d-1}| = \frac{3c_1}{4}|\mathbb{T}^{d-1}|.\end{aligned}$$

For $\alpha \in [0, 1]$, let $\Phi(\alpha) \subset \mathbb{T}^{d-1}$ be the subset of all points ϕ such that the Jacobian $J(\phi)$ satisfies $|J(\phi)| \geqslant \alpha$. If $d = 2$, then $J(\phi) = 1$, and if $d > 2$, then $|\Phi(\alpha)|$ is a continuous nonincreasing function in α assuming all values from $|\mathbb{T}^{d-1}|$ to 0. Therefore, there exists $\alpha_* \in (0, 1)$, such that

$$|\Phi(\alpha_*)| \geqslant (1 - c_1/2)|\mathbb{T}^{d-1}|.$$

Since $c_1 < 1$, we have $1 - \frac{c_1}{2} > \frac{c_1}{2}$. Therefore

$$|\Phi(\alpha_*)| \geqslant \frac{c_1}{2}|\mathbb{T}^{d-1}|.$$

This together with $|\Phi^*| \geqslant \frac{3c_1}{4}|\mathbb{T}^{d-1}|$ yields that

$$|\Phi^* \cap \Phi(\alpha_*)| \geqslant \frac{c_1}{4}|\mathbb{T}^{d-1}|.$$

Hence,

$$\int_{\mathbb{T}^{d-1}} T^*(\phi)|J(\phi)|d\phi \geqslant \int_{\Phi^*\cap\Phi(\alpha_*)} T^*(\phi)|J(\phi)|d\phi \geqslant \frac{c^*\alpha_*}{4}|\Phi^* \cap \Phi(\alpha_*)| \geqslant \frac{(c^*)^2\alpha_*}{16}|\mathbb{T}^{d-1}|.$$

If we set $T_1(\phi) := c_2T^*(\phi), \phi \in \mathbb{T}^{d-1}$, where

$$c_2 := \left(2^{2-d}\int_{\mathbb{T}^{d-1}} |J(\phi)|d\phi\right)^{-1/2},$$

then $T_1 \in \mathcal{T}^h_{2k}(\mathbb{T}^{d-1}; \{a_l\})$, and by (8),

$$\frac{1}{2^{d-2}}\int_{\mathbb{T}^{d-1}} (T_1(\phi))^2|J(\phi)|d\phi \leqslant 1.$$

Therefore, T_1 satisfies (6). Moreover,

$$\int_{\mathbb{T}^{d-1}} T_1(\phi)|J(\phi)|d\phi = c_2\int_{\mathbb{T}^{d-1}} T^*(\phi)|J(\phi)|d\phi \geqslant \frac{c_2c_1^2\alpha_*}{16}|\mathbb{T}^{d-1}| > 0.$$

Thus

$$\begin{aligned}\|A_{d,2k}(\cdot; f - R_m)\|_{L^2(\mathbb{S}^{d-1})} &\geqslant \frac{A_{d,2k}(f)}{2^{d-2}}\max_{0\leqslant j\leqslant k}\sup_{T_{2j}}\int_{\mathbb{T}^{d-1}} T_{2j}(\phi)|J(\phi)|d\phi \\ &\geqslant \frac{A_{d,2k}(f)}{2^{d-2}}\int_{\mathbb{T}^{d-1}} T_1(\phi)|J(\phi)|d\phi \geqslant c_3A_{d,2k}(f),\end{aligned}$$

where $c_3 := \frac{c_2c_1^2\alpha_*}{2^{d+2}}|\mathbb{T}^{d-1}|$. Thus,

$$\|f(\cdot) - R_m(\cdot)\|^2_{L^2(\mathbb{B}^d)} \geqslant \sum_{k=0}^{\infty} v_{d,2k}\|A_{d,2k}(\cdot; f - R_m)\|^2_{L^2(\mathbb{S}^{d-1})}$$

$$\begin{aligned}&\geqslant \sum_{k>k(d,m)} v_{d,2k}\|A_{d,2k}(\cdot;f-R_m)\|^2_{L^2(\mathbb{S}^{d-1})}\\&\geqslant c_3\sum_{k>k(d,m)} v_{d,2k}A_{d,2k}(f).\end{aligned}$$

Since the polynomial $U_{d,2j-1}(t)$ is odd on the interval $[-1,1]$ and f is radial, it follows from (1) that for arbitrary $j\in\mathbb{N}$, $A_{d,2j-1}(f)$=0. The above assertion together with the fact $A_{d,k}(\xi;f)=A_{d,k}(f)$ for all $\xi\in\mathbb{S}^{d-1}$ yields that

$$\begin{aligned}\|f(\cdot)-R_m(\cdot)\|^2_{L^2(\mathbb{B}^d)}&\geqslant c_3\sum_{k>2k(d,m)} v_{d,2k}A_{d,2k}(f)\\&=c_3\sum_{k>2k(d,m)} v_{d,k}A_{d,k}(f)\\&=\frac{c_3}{|\mathbb{S}^{d-1}|}\sum_{k>2k(d,m)} v_{d,k}\|A_{d,k}(\cdot;f)\|_{L^2(\mathbb{S}^{d-1})}\\&=\frac{c_3}{|\mathbb{S}^{d-1}|}\sum_{k>2k(d,m)} \|Q_{d,k}(\cdot;f)\|_{L^2(\mathbb{S}^{d-1})}\\&=\frac{c_3}{|\mathbb{S}^{d-1}|}\left\|f(\cdot)-\sum_{k=0}^{2k(d,m)}Q_{d,k}(\cdot;f)\right\|_{L^2(\mathbb{S}^{d-1})}.\end{aligned}$$

It is easy to deduce that $Q_{d,k}(\cdot;f)\in\mathcal{P}_k(\mathbb{B}^d)$, $k\in\mathbb{Z}_+$ (see [17, p. 163]). Therefore,

$$P(\cdot;f):=\sum_{k=0}^{2k(d,m)}Q_{d,k}(\cdot;f), \tag{9}$$

is a polynomial of degree not larger than $2k(d,m)$. The only thing remainder is to prove that there exists a constant c_4 depending only on d such that the polynomial defined in (9) is a polynomial with degree not larger than c_4n. Since

$$(4(d-1)m^{\frac{1}{d-1}})^{d-1}=4^{d-1}(d-1)^{d-1}m\geqslant 2^{2d-1}(d-1)!m,$$

we have $2k(d,m)\leqslant 2\lceil 4(d-1)m^{\frac{1}{d-1}}\rceil$, where $\lceil t\rceil$ denotes the smallest integer not smaller than t. Take $m=n^{d-1}$ so that (9) is a polynomial of the degree not larger than $8(d-1)n$.

All the above yields that there exist constants C,C_1 depending only on d such that

$$CE(f,\mathcal{P}_{C_1n}(\mathbb{B}^d))_{L^2(\mathbb{B}^d)}\leqslant\|f(\cdot)-R_{n^{d-1}}(\cdot)\|_{L^2(\mathbb{B}^d)}.$$

Since $R_{n^{d-1}}$ is arbitrarily chosen from $\mathcal{R}_{n^{d-1},2}$, we obtain that

$$CE\left(f,\mathcal{P}_{C_1n}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)}\leqslant E\left(f,\mathcal{R}_{n^{d-1},2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)},$$

which finishes the first inequality of (3).

Now, we turn to prove the second inequality of (3). Since there exists an absolute constant C such that $\dim\mathcal{P}^h_n\leqslant Cn^{d-1}$, it follows from Lemma 2 that $\mathcal{P}_n\in\mathcal{R}_{Cn^{d-1}}$. Thus for arbitrary $f\in L^2(\mathbb{B}^d)$, there holds

$$E(f,\mathcal{R}_{Cn^{d-1},2}(\mathbb{B}^d))_{L^2(\mathbb{B}^d)}\leqslant E(f,\mathcal{P}_n(\mathbb{B}^d))_{L^2(\mathbb{B}^d)}.$$

This completes the proof of Theorem 2. □

Proof of Theorem 1. It follows from Lemmas 2 and 4 with $q=2$ that there exist constants $C:=C(d,p)$ and $C_1:=C_1(p,d)$ such that

$$E\left(W^r_p(\mathbb{B}^d),\mathcal{R}_{C_1n^{d-1},2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)}\leqslant E\left(W^r_p(\mathbb{B}^d),\mathcal{P}_n(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)}\leqslant Cn^{-r+d(1/p-1/2)_+}.$$

From (3) and Lemma 4, it is easy to deduce that there exist constants $C_2 := C_2(p,d)$ and $C_3 := C_3(p,d)$ such that

$$E\left(W_p^{*,r}(\mathbb{B}^d), \mathcal{R}_{C_2 n^{d-1},2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \geqslant E\left(W_p^{*,r}(\mathbb{B}^d), \mathcal{P}_n(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \geqslant C_3 n^{-r+d(1/p-1/2)_+}.$$

Thus the fact

$$E\left(W_p^{*,r}(\mathbb{B}^d), \mathcal{R}_{C_2 n^{d-1},2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)} \leqslant E\left(W_p^{r}(\mathbb{B}^d), \mathcal{R}_{C_1 n^{d-1},2}(\mathbb{B}^d)\right)_{L^2(\mathbb{B}^d)}$$

yields the desired result. This completes the proof of Theorem 1. □

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