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Symmetry of flexoelectric coefficients in crystalline medium

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Flexoelectric effect, which is defined as strain gradient–induced polarization or electric gradient–induced strain in crystalline solids, can be presented as a fourth-rank tensor. The symmetry of the flexoelectric coefficients in matrix form is studied. The results indicate that the direct flexoelectric coefficients should be presented in 3×18 form and the converse flexoelectric coefficients in 6×9 form, rather than 6×6 form, like elastic constants. In addition, non-zero and independent elements in the matrices have been calculated for 32 point groups and 7 Ci groups. These results will provide valuable reference to the theoretical and application studies of flexoelectric effect. © *2011 American Institute of Physics*. [doi:10.1063/1.3662196]

INTRODUCTION

Flexoelectric effect is an electromechanical coupling phenomenon in crystalline solids. Unlike the piezoelectric effect, this effect has no crystallographic constraint. Direct flexoelectricity describes the coupling between electric polarization and strain gradient and can be characterized by the following formula:

$$P_i = \mu_{ijkl} \frac{\partial S_{jk}}{\partial x_l}, \tag{1}$$

where P is the induced polarization, μ is the flexoelectric coefficient, a fourth-rank tensor, s is elastic strain, and x is the axis.[1]

Flexoelectricity was firstly named by Indenbom in analogy with charge separation in non-piezoelectric liquid crystals.[2] Later, it was regarded as an intrinsic property of the materials. Tagantsev firstly made an exhaustive theoretical analysis about the mechanism of this effect, which contained four parts: (1) dynamic bulk flexoelectricity, (2) static bulk flexoelectricity, (3) surface flexoelectricity, and (4) surface piezoelectricity.[3–5]

For solid crystals and dielectrics, the flexoelectric coefficient was previously estimated to have the order of magnitude of e/a, which is approximately equal to 10^{-10} C/m.[6,7] Recently, the coefficients value in certain ferroelectrics, incipient ferroelectric, and relaxor ferroelectric perovskites were measured experimentally by Cross and his co-workers.[8,9] The enhanced flexoelectric coefficients by at least 4–5 orders have been obtained in some specific ferroelectric solid-state materials with nano-polarized clusters, such as $BaTiO_3$-based material systems. The applied device, such as the truncated pyramid structure, has a reasonable and attractive superior equivalent piezoelectric coefficient, which is more than 100-200 pC/N.[10] Another attractive feature of the flexoelectricity phenomenon in solids is its equivalent piezoelectric coefficient, which is inversely proportional to the thickness of the active layer d. If the size of a designed sample is scaled down in the same ratio, the equivalent piezoelectric flexoelectric coefficient will be increased drastically.[11]

With proper and careful design, this new flexoelectric phenomenon in solids inherently exhibits great advantages over piezoelectric materials used currently in either having sensing without actuating, or even having actuating without sensing, or having both sensing and actuating properties.

But the mechanics of this effect are not very clear. It seems a fundamental problem must be solved promptly: the symmetry of the flexoelectric coefficients. In 2006, Cross proposed that flexoelectric coefficients have the same symmetry with the electrostrictive constant.[1] For cubic crystal, it has only three independent non-zero components: μ_{11}, μ_{12}, and μ_{44}. Recently, a different viewpoint has been raised by H. Le Quang and Q.-C. He after analyzing the number and types of all possible rotational symmetries for flexoelectric tensors.[12]

To solve this problem, this paper bases from the fundamental tensor relationship of the flexoelectricity and expresses the symmetry of the flexoelectric coefficients in matrices form.

SYMMETRY OF DIRECT FLEXOELECTRICITY

The flexoelectric coefficient and the electrostrictive constant usually have no comparable properties either in form or nature.

$$x_{ij} = Q_{ijkl} P_k P_l. \tag{2}$$

The strain tensor and the product terms $P_k P_l$ are symmetrical, while the strain gradient term is a partial derivative.

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Actually, it is a third rank tensor in Cartesian coordinates. So the most significant difference between the coefficients is that electrostrictive constant Q combines two second-rank tensors, while μ combines a first-rank tensor and a third-rank tensor.

Because of the terms of the strain gradient, the equation differs greatly from the common dielectric effects. It is necessary to discuss whether the coefficient is a tensor. As we know, all tensors should subject to the principle of coordinate transformation; therefore, when an old coordinate changes, the tensors P, S, and x should be transformed too.

$$P_{i'} = a_{i'}^{i} P_i, \tag{3}$$

$$S_{j'k'} = a_{j'}^{j} a_{k'}^{k} S_{jk}, \tag{4}$$

$$x_{l'} = a_{l'}^{l} x_l \tag{5}$$

combines with Eq. (1),

$$P_{i'} = a_{i'}^{i} P_i = a_{i'}^{i} \mu_{ijkl} \frac{\partial S_{jk}}{\partial x_l} = a_{i'}^{i} \mu_{ijkl} \frac{\partial (a_j^{j'} a_k^{k'} S_{j'k'})}{\partial (a_l^{l'} x_{l'})}. \tag{6}$$

Obviously, the transformation matrix will not always be regular matrix only when all coordinates are the same kinds of the coordinate systems, such as Cartesian coordinates and cylindrical coordinates. But different coordinate system transformation is not involved in the area of crystals and dielectrics.

Thus, we can obtain directly,

$$\partial a_j^{j'} a_k^{k'} S_{j'k'} = a_j^{j'} a_k^{k'} \partial S_{j'k'}, \tag{7}$$

$$\partial a_l^{l'} x_{l'} = a_l^{l'} \partial x_{l'}, \tag{8}$$

TABLE I. Symmetry of coefficient combination.

s: elastic constant	Q: electrostrictive	μ: flexoelectric
21	36	54

$$\mu_{i'jk'l'} = a_{i'}^{i} a_j^{j'} a_k^{k'} a_{l'}^{l} \mu_{ijkl}. \tag{9}$$

The tensor nature of the μ_{ijkl} can be demonstrated by the above equation.

The matrix μ should reflect the relationship between the induced polarization and applied strain gradient. Focusing on Eq. (1), the polarization term is a vector (first rank tensor) which indicates that the matrix μ is more suitable to be written as the form of $3 \times n$ rather than $6 \times m$. In terms of physics, the polarization (the subscript i) is an electrical quantity, but the strain gradient (the subscript j, k, l) is a dynamical variable. The subscript i is not commutative with the subscript l, so, by taking the subscript l and i as a whole, to describe the effect may conflict with its physical implication.

For strain gradient tensor, only i and j is commutative. A unique method is used to decrease its rank.

To simplify the formulation, let

$$\frac{\partial S_{jk}}{\partial x_l} = e_{jkl}. \tag{10}$$

e_n	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
	e_{111}	e_{112}	e_{113}	e_{221}	e_{222}	e_{223}	e_{331}	e_{332}	e_{333}
e_n	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
	$2e_{121}$	$2e_{122}$	$2e_{123}$	$2e_{131}$	$2e_{132}$	$2e_{133}$	$2e_{231}$	$2e_{232}$	$2e_{233}$

Then, Eq. (1) can be written as

$$P_i = \mu_{in} e_n. \tag{11}$$

For cubic crystal, after the subscript transformation, the matrix presentation should qualify the following form:

$$\mu_{3\times 18} = \begin{pmatrix} \mu_{11} & 0 & 0 & \mu_{14} & 0 & 0 & \mu_{14} & 0 & 0 & 0 & \mu_{111} & 0 & 0 & 0 & \mu_{111} & 0 & 0 & 0 \\ 0 & \mu_{14} & 0 & 0 & \mu_{11} & 0 & 0 & \mu_{14} & 0 & \mu_{111} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_{111} \\ 0 & 0 & \mu_{14} & 0 & 0 & \mu_{14} & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{111} & 0 & 0 & 0 & \mu_{111} & 0 \end{pmatrix},$$

$$\mu_{1111} = \mu_{2222} = \mu_{3333} = \mu_{11},$$

$$\mu_{1133} = \mu_{2233} = \mu_{1122} = \mu_{2121} = \mu_{3232} = \mu_{3131} = \mu_{111},$$

$$\mu_{1221} = \mu_{1331} = \mu_{2112} = \mu_{2332} = \mu_{3223} = \mu_{3113} = \mu_{14}.$$

For isotropic material, the number of the non-zero independent components reduces to two further though the matrix transformation.

A is defined as a symmetry operation matrix, which is an arbitrary rotation around the x axis,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & n \\ 0 & -n & m \end{pmatrix} \quad m = cos\theta, \quad n = sin\theta.$$

Owning to this operation, the strain gradient tensor e_n changes to e'_n,

$$e_n = N^T . e'_n,$$

where

$$N^T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m & -n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & n & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m^2 & 0 & 0 & n^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -mn & 0 & 0 \\
0 & 0 & 0 & 0 & m^3 & -m^2n & 0 & mn^2 & -n^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m^2n & mn^2 \\
0 & 0 & 0 & 0 & m^2n & m^3 & 0 & n^3 & mn^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -mn^2 & -m^2n \\
0 & 0 & 0 & n^2 & 0 & 0 & m^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & mn & 0 \\
0 & 0 & 0 & 0 & mn^2 & -n^3 & 0 & m^3 & -m^2n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m^2n & -mn^2 \\
0 & 0 & 0 & 0 & n^3 & mn^2 & 0 & m^2n & m^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & mn^2 & m^2n \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & -n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m^2 & -mn & 0 & -mn & n^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & mn & m^2 & 0 & -n^2 & -mn & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & mn & -n^2 & 0 & m^2 & -mn & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n^2 & mn & 0 & mn & m^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2mn & 0 & 0 & -2mn & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m^2-n^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2m^2n & -2mn^2 & 0 & -2m^2n & 2mn^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m^3-mn^2 & n^3-m^2n \\
0 & 0 & 0 & 0 & 2mn^2 & 2m^2n & 0 & -2mn^2 & -2m^2n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m^2n-n^3 & m^3-mn^2
\end{pmatrix}$$

combined with the identical equation

$$\mu' = A\mu.N^T.$$

Then, the relationship between the coefficients can be written as

$$2m^2n^2(\mu_{11} - \mu_{111}) + (m^4 + n^4 - 2m^2n^2)\mu_{14} = 0 \qquad (12)$$

combined with

$$m^2 + n^2 = 1, \qquad (13)$$

so

$$2m^2n^2(\mu_{11} - \mu_{111} - 2\mu_{14}) = 0. \qquad (14)$$

Thus, for arbitrary m, n ($mn \neq 0$),

$$\mu_{14} = \frac{1}{2}(\mu_{11} - \mu_{111}). \qquad (15)$$

SYMMETRY OF CONVERSE FLEXOELECTRICITY

Converse flexoelectricity is the coupling between the applied gradient of the electric field intensity and the induced elastic strain. This can be characterized by the following formula:

$$x_{ij} = \mu_{ijkl}\frac{\partial E_k}{\partial x_l}. \qquad (16)$$

Converse flexoelectric coefficients are also a low symmetrical fourth-rank tensor. Obviously, the subscript k and l is not commutative, so it can be characterized by a 6×9 matrix. Actually, the converse flexoelectric coefficients have the same symmetry with the direct flexoelectric coefficients. The numbers of both the direct and converse matrix components are 54. Take the analogous method and definition

$$\frac{\partial E_k}{\partial x_l} = e_m(m = 1,2,,,9)(kl = 11,12,13,21,,,33)$$
$$x_{ij} = x_n(i = j)2x_{ij} = x_n(i \neq j). \qquad (17)$$

For cubic crystal, it satisfies the following character:

$$\begin{bmatrix}
\mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{15} \\
\mu_{15} & 0 & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{15} \\
\mu_{15} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{11} \\
0 & 0 & 0 & 0 & 0 & \mu_{46} & 0 & \mu_{46} & 0 \\
0 & 0 & \mu_{46} & 0 & 0 & 0 & \mu_{46} & 0 & 0 \\
0 & \mu_{46} & 0 & \mu_{46} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\mu_{1111} = \mu_{2222} = \mu_{3333} = \mu_{11},$$
$$\mu_{1133} = \mu_{2233} = \mu_{1122} = \mu_{3311} = \mu_{3322} = \mu_{2211} = \mu_{15},$$
$$\mu_{2323} = \mu_{2332} = \mu_{1212} = \mu_{1221} = \mu_{1331} = \mu_{1313} = \mu_{46}.$$

For isotropic medium, the relationship between the non-zero components is

$$\mu_{46} = \mu_{11} - \mu_{15}. \qquad (18)$$

The difference between Eqs. (15) and (18) will be emphasized in the discussion.

DISCUSSION

Maybe these notations, calculated in cubic crystals and isotropic medium, are according with Ref. 1, but the

TABLE II. Different coefficient relationship for isotropic medium.

Compliance s	Stiffness c	Electrostriction Q	Direct μ	Converse μ
s_{11}, s_{12}, s_{44} $s_{44} = 2(s_{11} - s_{12})$	c_{11}, c_{12}, c_{44} $c_{44} = \frac{1}{2}(c_{11} - c_{12})$	Q_{11}, Q_{12}, Q_{44} $Q_{44} = 2(Q_{11} - Q_{12})$	$\mu_{11}, \mu_{111}, \mu_{14}$ $\mu_{14} = \frac{1}{2}(\mu_{11} - \mu_{111})$	$\mu_{11}, \mu_{15}, \mu_{46}$ $\mu_{46} = (\mu_{11} - \mu_{15})$

difference can be revealed and illustrated in Table I. The key point is that both the direct and converse flexoelectric coefficients are very low symmetrical, because only the subscripts of the strain term are commutative, which means that the four subscripts i, j, k, l will have 54 different combinations. However, the electrostrictive constant only has 36 different subscript combinations, because both i, j and k, l are commutative; the elastic compliance constant has 21 different subscript combinations, since i, j as a whole can exchange with the whole of k, l further.

Most of the crystalline materials studied recently are in cubic phase. But the fourth-rank coefficients' symmetry is almost hidden in cubic crystals due to Neumann's law.[13] For cubic crystals, the three different coefficients Q, s, and μ have absolutely the same non-zero independent components (subscript form 1111, 1122, and 1221).

For the isotropic medium, the relationship of the non-zero independent components is different.

Actually, the origin of this difference derives from matrix form. If we take them into the fourth-rank tensor form, then a universal relationship will be found.

$$A_{1221} = \frac{1}{2}(A_{1111} - A_{1122}). \tag{19}$$

A stands for any of the coefficient in Table II.

The unity of the fourth-rank tensor in isotropic medium can be concluded through formula (19). In addition, Table III expresses the demonstrations of the different coefficients, respectively.

Then, we discuss the coefficients in various crystalline point groups and Curie groups. The details are shown in the Appendix. In contrast with the previous work, we also analyze the numbers of the non-zero independent components for different symmetry class.

Table IV is in accordance with the results calculated by H. Le Quang and Q.-C. He,[12] who used the harmonic decomposition and Cartan decomposition of the group theory to solve the problem of the number and types of all rotational symmetries for flexoelectric tensor. Based on the characters of symmetry groups and basis vectors, the group theory is unique and systematic in calculating the non-zero independent components. Comparing with their rotational group theory, our method describes the symmetry more directly and visually and provides the specific component position and the relationship between the non-zero components for each point group and Curie group in the matrices.

These results will provide valuable reference to the theoretical and application studies of flexoelectric effect. It reveals clearly that the flexoelectric coefficient is lower symmetrical than the elastic constant and electrostrictive constant.

TABLE III. Relationship between tensor components and matrix components.

Relationship	Demonstrations
$s_{44} = 4s_{1212}$	$x_4 = 2x_{23} = 2(s_{2323}X_{23} + s_{2332}X_{32}) = 4s_{1212}X_4$
$c_{44} = c_{1212}$	$X_4 = X_{23} = (c_{2323}x_{23} + c_{2332}x_{32}) = 2c_{1212} \times 1/2\, x_4$ $= c_{1212}x_4$
$Q_{44} = 4Q_{1212}$	$x_4 = 2x_{12} = 2(Q_{1212}P_1P_2 + Q_{1221}P_2P_1) = 4Q_{1212}P_1P_2$
$\mu_{14} = \mu_{1221}$	$P_1 = \mu_{1212}\frac{\partial S_{22}}{\partial x_1}$
$\mu_{46} = 2\mu_{1212}$	$x_4 = 2x_{12} = 2\left(\mu_{1212}\frac{\partial E_1}{\partial x_2} + \mu_{1221}\frac{\partial E_2}{\partial x_1}\right)$ $= 2\mu_{1212}\left(\frac{\partial E_1}{\partial x_2} + \frac{\partial E_2}{\partial x_1}\right)$

SUMMARY

This paper discusses the proper notation of the flexoelectric coefficient in matrix form. The flexoelectric coefficient is an asymmetrical tensor, due to the gradient term of its definition. Compared with the electrostrictive constant and the elastic constant, it has more non-zero independent components, except for the cubic crystals and the isotropic medium. In fact, most fourth-rank tensors of cubic crystals have only three independent components due to its extremely high symmetry. For isotropic medium, widely applied nowadays, the accurate number of the non-zero independent components further reduces to only two.

The traditional 6×6 matrix form cannot express the symmetrical quality of flexoelectric coefficients. The correct form for direct flexoelectric and converse flexoelectric coefficients is 3×18 and 6×9. The matrices calculated in the Appendix express more new information about the flexoelectric effect, which may provide guidelines for future research

TABLE IV. Numbers of the non-zero independent components for different point groups and Curie groups.

Point and Curie groups	s/c	Q	μ direct/converse
1, $\bar{1}$	21	36	54
2, m, 2/m	13	20	28
222, mm2, mmm	9	12	15
3, $\bar{3}$	7	11	18
32, 3m, $\bar{3}$ m	6	8	10
4, $\bar{4}$, 4/m	7	11	14
4mm, $\bar{4}2m$422, 4/mmm	6	7	8
6, $\bar{6}$, 6/m, $\infty, \infty/m$	5	7	13
622, 6mm, $\bar{6}$m2, 6/mmm, $\infty 2, \infty m, \infty/mm$	5	6	7
23, m3	3	4	5
432, $\bar{4}3m, m3m$	3	3	3
$\infty\infty, \infty\infty m$	2	2	2

work. In addition, for the origin of the flexoelectricity, we believe flexoelectricity exists in such materials, where existing random polarization or polarization can be aligned by stress gradient. If, in a material, no dipole or polarization formed under stress gradient, such as in some covalent bonded solids, the flexoelectric effect would not exist.

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APPENDIX: DIFFERENCES BETWEEN S, Q, AND μ FOR SPECIFIC POINT GROUPS AND CURIE GROUPS

Point groups (1, $\bar{1}$)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} \\ Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66} \end{bmatrix} \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} & \mu_{17} & \mu_{18} & \mu_{19} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} & \mu_{27} & \mu_{28} & \mu_{29} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} & \mu_{35} & \mu_{36} & \mu_{37} & \mu_{38} & \mu_{39} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{44} & \mu_{45} & \mu_{46} & \mu_{47} & \mu_{48} & \mu_{49} \\ \mu_{51} & \mu_{52} & \mu_{53} & \mu_{54} & \mu_{55} & \mu_{56} & \mu_{57} & \mu_{58} & \mu_{59} \\ \mu_{61} & \mu_{62} & \mu_{63} & \mu_{64} & \mu_{65} & \mu_{66} & \mu_{67} & \mu_{68} & \mu_{69} \end{bmatrix}$$

Point groups (2,m,2/m)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & s_{15} & 0 \\ s_{12} & s_{22} & s_{23} & 0 & s_{25} & 0 \\ s_{13} & s_{23} & s_{33} & 0 & s_{35} & 0 \\ 0 & 0 & 0 & s_{44} & 0 & s_{46} \\ s_{15} & s_{25} & s_{35} & 0 & s_{55} & 0 \\ 0 & 0 & 0 & s_{46} & 0 & s_{66} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & Q_{15} & 0 \\ Q_{21} & Q_{22} & Q_{23} & 0 & Q_{25} & 0 \\ Q_{31} & Q_{32} & Q_{33} & 0 & Q_{35} & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & Q_{46} \\ Q_{51} & Q_{52} & Q_{53} & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & Q_{64} & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \mu_{11} & 0 & \mu_{13} & 0 & \mu_{15} & 0 & \mu_{17} & 0 & \mu_{19} \\ \mu_{21} & 0 & \mu_{23} & 0 & \mu_{25} & 0 & \mu_{27} & 0 & \mu_{29} \\ \mu_{31} & 0 & \mu_{33} & 0 & \mu_{35} & 0 & \mu_{37} & 0 & \mu_{39} \\ 0 & \mu_{42} & 0 & \mu_{44} & 0 & \mu_{46} & 0 & \mu_{48} & 0 \\ \mu_{51} & 0 & \mu_{53} & 0 & \mu_{55} & 0 & \mu_{57} & 0 & \mu_{59} \\ 0 & \mu_{62} & 0 & \mu_{64} & 0 & \mu_{66} & 0 & \mu_{68} & 0 \end{bmatrix}$$

Point groups (222, mm2, mmm)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{19} \\ \mu_{21} & 0 & 0 & 0 & \mu_{25} & 0 & 0 & 0 & \mu_{29} \\ \mu_{31} & 0 & 0 & 0 & \mu_{35} & 0 & 0 & 0 & \mu_{39} \\ 0 & 0 & 0 & 0 & 0 & \mu_{46} & 0 & \mu_{48} & 0 \\ 0 & 0 & \mu_{53} & 0 & 0 & 0 & \mu_{57} & 0 & 0 \\ 0 & \mu_{62} & 0 & \mu_{64} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Point groups (3, $\bar{3}$)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & -s_{25} & 0 \\ s_{12} & s_{11} & s_{13} & -s_{14} & s_{25} & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ s_{14} & -s_{14} & 0 & s_{44} & 0 & 0 \\ -s_{25} & s_{25} & 0 & 0 & s_{44} & 2s_{14} \\ 0 & 0 & 0 & s_{14} & 2s_{14} & 2(s_{11}-s_{12}) \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & -Q_{25} & Q_{16} \\ Q_{12} & Q_{11} & Q_{13} & -Q_{14} & Q_{25} & -Q_{16} \\ Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\ Q_{41} & -Q_{41} & 0 & Q_{44} & Q_{45} & Q_{52} \\ -Q_{52} & Q_{52} & 0 & Q_{45} & Q_{44} & 2Q_{41} \\ -Q_{16} & Q_{16} & 0 & Q_{25} & 2Q_{14} & 2(Q_{11}-Q_{12}) \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & -\mu_{26} & \mu_{17} & -\mu_{28} & \mu_{19} \\ \mu_{15} & -\mu_{14} & -\mu_{17} & -\mu_{12} & \mu_{11} & \mu_{26} & -\mu_{13} & \mu_{28} & \mu_{19} \\ \mu_{31} & 0 & 0 & 0 & \mu_{31} & 0 & 0 & 0 & \mu_{39} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{52} & -\mu_{41} & \mu_{46} & \mu_{47} & \mu_{48} & 0 \\ -\mu_{41} & -\mu_{52} & \mu_{46} & -\mu_{42} & -\mu_{43} & -\mu_{43} & \mu_{48} & -\mu_{47} & 0 \\ \mu_{61} & \mu_{11}-\mu_{15} & -2\mu_{46} & \mu_{11}-\mu_{15} & -\mu_{61} & 2\mu_{46} & -2\mu_{48} & 2\mu_{48} & 0 \end{bmatrix}$$

Point groups (32,3 m,$\bar{3}$m)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & 0 \\ s_{12} & s_{11} & s_{13} & -s_{14} & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ s_{14} & -s_{14} & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 2s_{14} \\ 0 & 0 & 0 & 0 & 2s_{14} & 2(s_{11}-s_{12}) \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ Q_{12} & Q_{11} & Q_{13} & -Q_{14} & 0 & 0 \\ Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\ Q_{41} & -Q_{41} & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{44} & 2Q_{41} \\ 0 & 0 & 0 & 0 & 2Q_{14} & Q_{44} \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & -\mu_{26} & 0 & -\mu_{28} & \mu_{19} \\ \mu_{15} & 0 & 0 & 0 & \mu_{11} & \mu_{26} & 0 & \mu_{28} & \mu_{19} \\ \mu_{31} & 0 & 0 & 0 & \mu_{31} & 0 & 0 & 0 & \mu_{39} \\ \mu_{41} & 0 & 0 & & -\mu_{41} & \mu_{46} & 0 & \mu_{48} & 0 \\ 0 & \mu_{52} & \mu_{46} & -\mu_{52} & 0 & 0 & \mu_{48} & 0 & 0 \\ 0 & \mu_{11}-\mu_{15} & -2\mu_{46} & \mu_{11}-\mu_{15} & 0 & 0 & -2\mu_{48} & 0 & 0 \end{bmatrix}$$

Point groups (4, $\bar{4}$,4/m)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & s_{16} \\ s_{12} & s_{11} & s_{13} & 0 & 0 & -s_{16} \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ s_{16} & -s_{16} & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{11} & Q_{13} & 0 & 0 & -Q_{16} \\ Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & 0 & -Q_{45} & Q_{44} & 0 \\ Q_{61} & -Q_{61} & 0 & 0 & 0 & Q_{66} \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_{11} & \mu_{12} & 0 & \mu_{14} & \mu_{15} & 0 & 0 & 0 & \mu_{19} \\ \mu_{15} & -\mu_{14} & 0 & -\mu_{12} & \mu_{11} & 0 & 0 & 0 & \mu_{19} \\ \mu_{31} & \mu_{32} & 0 & 0 & \mu_{31} & 0 & 0 & 0 & \mu_{39} \\ 0 & 0 & \mu_{43} & 0 & 0 & \mu_{46} & \mu_{47} & \mu_{48} & 0 \\ 0 & 0 & \mu_{46} & 0 & 0 & -\mu_{43} & \mu_{48} & -\mu_{47} & 0 \\ \mu_{61} & \mu_{62} & 0 & \mu_{62} & -\mu_{61} & 0 & 0 & 0 & -\mu_{32} \end{bmatrix}$$

Point groups (4 mm, $\bar{4}2m$ 422,4/mmm)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{13} & 0 & 0 & 0 \\ Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{19} \\ \mu_{15} & 0 & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{19} \\ \mu_{31} & 0 & 0 & 0 & \mu_{31} & 0 & 0 & 0 & \mu_{39} \\ 0 & 0 & 0 & 0 & 0 & \mu_{46} & 0 & \mu_{48} & 0 \\ 0 & 0 & \mu_{46} & 0 & 0 & 0 & \mu_{48} & 0 & 0 \\ 0 & \mu_{62} & 0 & \mu_{62} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Point groups and Curie groups (6, $\bar{6}$,6/m, $\infty, \infty/m$)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{11} & Q_{13} & 0 & 0 & -Q_{16} \\ Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & 0 & -Q_{45} & Q_{44} & 0 \\ Q_{16} & Q_{16} & 0 & 0 & 0 & 2(Q_{11}-Q_{12}) \end{bmatrix}$$
$$\times \begin{bmatrix} \mu_{11} & \mu_{12} & 0 & \mu_{14} & \mu_{15} & 0 & 0 & 0 & \mu_{19} \\ \mu_{15} & -\mu_{14} & 0 & -\mu_{12} & \mu_{11} & 0 & 0 & 0 & \mu_{19} \\ \mu_{31} & \mu_{32} & 0 & 0 & \mu_{31} & 0 & 0 & 0 & \mu_{39} \\ 0 & 0 & \mu_{43} & 0 & 0 & \mu_{46} & \mu_{47} & \mu_{48} & 0 \\ 0 & 0 & \mu_{46} & 0 & 0 & -\mu_{43} & \mu_{48} & -\mu_{47} & 0 \\ \mu_{61} & \mu_{11}-\mu_{15} & 0 & \mu_{11}-\mu_{15} & -\mu_{61} & 0 & 0 & 0 & -\mu_{32} \end{bmatrix}$$

Point groups and Curie groups (622,6 mm, $\bar{6}$ m2,6/mmm, $\infty m, \infty 2, \infty/mm$)

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{13} & 0 & 0 & 0 \\ Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(Q_{11}-Q_{12}) \end{bmatrix}$$
$$\times \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{19} \\ \mu_{11} & 0 & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{19} \\ \mu_{31} & 0 & 0 & 0 & \mu_{31} & 0 & 0 & 0 & \mu_{39} \\ 0 & 0 & 0 & 0 & 0 & \mu_{46} & 0 & \mu_{48} & 0 \\ 0 & 0 & \mu_{46} & 0 & 0 & 0 & \mu_{48} & 0 & 0 \\ 0 & \mu_{11}-\mu_{15} & 0 & \mu_{11}-\mu_{15} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Point groups (23, m3)

$$\begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{44} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{21} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{21} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{15} \\ \mu_{15} & 0 & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{15} \\ \mu_{15} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{11} \\ 0 & 0 & 0 & 0 & 0 & \mu_{46} & 0 & \mu_{46} & 0 \\ 0 & 0 & \mu_{53} & 0 & 0 & 0 & \mu_{53} & 0 & 0 \\ 0 & \mu_{62} & 0 & \mu_{62} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Point groups (432, $\bar{4}3m, m3m$)

$$\begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{44} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{12} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{15} \\ \mu_{15} & 0 & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{15} \\ \mu_{15} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{11} \\ 0 & 0 & 0 & 0 & 0 & \mu_{46} & 0 & \mu_{46} & 0 \\ 0 & 0 & \mu_{46} & 0 & 0 & 0 & \mu_{46} & 0 & 0 \\ 0 & \mu_{46} & 0 & \mu_{46} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Curie groups ($\infty\infty$, $\infty\infty m$)

$$\begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(s_{11}-s_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{12} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(Q_{11}-Q_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(Q_{11}-Q_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(Q_{11}-Q_{12}) \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_{11} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{15} \\ \mu_{15} & 0 & 0 & 0 & \mu_{11} & 0 & 0 & 0 & \mu_{15} \\ \mu_{15} & 0 & 0 & 0 & \mu_{15} & 0 & 0 & 0 & \mu_{11} \\ 0 & 0 & 0 & 0 & 0 & \mu_{11}-\mu_{15} & 0 & \mu_{11}-\mu_{15} & 0 \\ 0 & 0 & \mu_{11}-\mu_{15} & 0 & 0 & 0 & \mu_{11}-\mu_{15} & 0 & 0 \\ 0 & \mu_{11}-\mu_{15} & 0 & \mu_{11}-\mu_{15} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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