[5] E. FENNER und O. SCHOTT, *Möglichkeiten und Grenzen der Bildverstärkung*, Z. angew. Phys. *6*, H. 2 (1954).
[6] F. ECKART, *Elektronenoptische Bildwandler und Röntgenbildverstärker* (Leipzig 1956).

Résumé

Le présent article a pour objet la détermination des limites et des possibilités de l'observation infra-rouge au moyen d'un dispositif optique déterminé. On décrit des possibilités susceptibles de réduire au maximum la limitation de l'information basée sur la statistique du phénomène visuel.

(Eingegangen: 13. Dezember 1957.)

On Prager's Hardening Rule

By RICH. THORPE SHIELD, Providence, R. I., USA[1]), and HANS ZIEGLER, Zürich[2])

1. Introduction

In order to describe the behaviour of a rigid-work-hardening material, one needs

(a) an initial yield condition, specifying the states of stress for which plastic flow first sets in;

(b) a flow rule, connecting the plastic strain increment with the stress and the stress increment;

(c) a hardening rule, specifying the modification of the yield condition in the course of plastic flow.

It is customary to represent the yield condition as a surface in stress space, convex [1][3]) and initially containing the origin. The current yield conditions for a metal are those of v. MISES [2] and of TRESCA [3]. The flow rule generally accepted is also due to v. MISES [4]. It is justified to a certain extent by physical reasons [5, 1], and it states that the strain increment vector lies in the exterior normal of the yield surface at the stress point. As to the hardening rule, there are various versions in use. The rule of isotropic work-hardening [6, 7] assumes that the yield surface expands during plastic flow, retaining its shape and situation with respect to the origin. Another rule, developed by PRAGER [8], assumes that the yield surface is rigid but undergoes a translation in the direction of the strain increment. This rule accounts for the Bauschinger effect

---

[1]) Brown University, Division of Applied Mathematics.

[2]) ETH; 1956/57 Brown University, Providence, R. I., USA, Division of Applied Mathematics. The results presented in this paper were obtained in the course of research conducted under Contract Nonr 562(10) sponsored by the Office of Naval Research.

[3]) Numbers in square brackets refer to References, page 275.

observed in the materials in question. The main advantage of the rule is that for piecewise linear yield conditions, such as that of TRESCA, the law exhibits a limited path independence of the final plastic strain with a resulting simplification in the mathematical analysis.

The following sections contain a discussion of PRAGER's hardening rule and its implications for special states of stress prevalent in practical applications.

Mention should be made of the work of HODGE (see [10], for example), which uses a strain-hardening rule which is a combination of the PRAGER rule and isotropic hardening.

2. Treatment in 9-Space

Let us consider an element of a rigid-work-hardening solid, referred to an orthogonal coordinate system x_i. The state of stress of this element can be represented by a stress point P in a 9-space $\sigma_{i k}$. In this space, the initial yield surface is represented by an equation

$$F(\sigma_{i k}) = k^2 = \text{const.} \tag{2.1}$$

In the following, for simplicity attention will be confined to initially isotropic materials for which the form of the function F is invariant with respect to a rotation of the stress state. An initially anisotropic material can be treated in an analogous manner.

The hardening rule suggested by PRAGER assumes that during plastic deformation the yield surface moves in translation. After a certain amount of plastic flow, it is given by

$$F(\sigma_{i k} - \alpha_{i k}) = k^2\,, \tag{2.2}$$

where the tensor $\alpha_{i k}$ represents the total translation. Because $\alpha_{i k}$ is not necessarily the isotopic tensor $\delta_{i k}$, where $\delta_{i k}$ is the Kronecker delta, *the material becomes anisotropic as a result of the hardening process*. Accordingly, direction is important and we shall fix the coordinate system x_i with respect to the element, small deformations being assumed.

Due to the flow rule of v. MISES, the plastic strain increment $d\varepsilon_{i k}$, considered as a vector in the space $\sigma_{i k}$, lies in the exterior normal of the surface (2.2) at P. Thus, it is represented by

$$d\varepsilon_{i k} = \frac{\partial F}{\partial \sigma_{i k}}\, d\lambda\,, \quad d\lambda > 0\,. \tag{2.3}$$

The definition of a Prager-hardening material is completed by assuming that the surface (2.2) moves in the direction of $d\varepsilon_{i k}$; more explicitly

$$d\alpha_{i k} = c\, d\varepsilon_{i k}\,, \tag{2.4}$$

where c is a constant characterizing the material. This work-hardening law is a generalization to complex states of stress of a linear work-hardening law in simple tension (Figure 1), which exhibits a Bauschinger effect. [The work-hardening modulus c_1 in simple tension (Figure 1) is related to the work-hardening modulus c by $c_1 = (3/2)\, c$.]

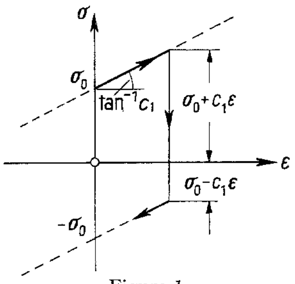

Figure 1

Response of the material considered in simple tension or compression.

The hardening rule described is physically acceptable because the components

$$\alpha_{i k} = c\, \varepsilon_{i k} \tag{2.5}$$

form a tensor of the second order, and the law is therefore independent of the particular coordinate system x_i chosen.

The scalar $d\lambda$ in (2.3) is determined by the condition that P remains on the yield surface in plastic flow. From this condition,

$$(d\sigma_{i k} - d\alpha_{i k}) \frac{\partial F}{\partial \sigma_{i k}} = 0 , \tag{2.6}$$

and from (2.4) and (2.3) follows at once

$$d\lambda = \frac{1}{c} \cdot \frac{(\partial F/\partial \sigma_{i j})\, d\sigma_{i j}}{(dF/\partial \sigma_{k l})\, (\partial F/\partial \sigma_{k l})} , \tag{2.7}$$

if the summation convention is adopted in 9-space.

In an initially isotropic solid the yield function takes the form

$$F(\sigma_{i k}) = G\, [I_1(\sigma_{i k}),\, I_2(\sigma_{i k}),\, I_3(\sigma_{i k})] , \tag{2.8}$$

where

$$I_1 = \sigma_{i i} , \quad I_2 = \frac{1}{2}\, \sigma_{i j}\, \sigma_{j i} , \quad I_3 = \frac{1}{3}\, \sigma_{i j}\, \sigma_{j k}\, \sigma_{k i} \tag{2.9}$$

are the invariants of the stress tensor. Moreover, if the initial yield is independent of the mean normal stress,

$$F(\sigma_{i k} + \beta\, \delta_{i k}) = F(\sigma_{i k}) , \tag{2.10}$$

where β is an arbitrary scalar. When plastic flow has set in, the yield function becomes, on account of (2.2) and (2.8),

$$F(\sigma_{ik} - \alpha_{ik}) = G\,[I_1(\sigma_{ik} - \alpha_{ik}),\, I_2(\sigma_{ik} - \alpha_{ik}),\, I_3(\sigma_{ik} - \alpha_{ik})]\,. \qquad (2.11)$$

From (2.10) it follows that the values of (2.11) remain unchanged when σ_{ik} is replaced by $\sigma_{ik} + \beta\,\delta_{ik}$: PRAGER's hardening rule implies that *during the whole hardening process yield is independent of the mean normal stress.*

With (2.11), the flow rule (2.3) reads

$$d\varepsilon_{ik} = \frac{\partial G}{\partial \sigma_{ik}}\, d\lambda = \left(\frac{\partial G}{\partial I_1} \cdot \frac{\partial I_1}{\partial \sigma_{ik}} + \cdots\right) d\lambda\,. \qquad (2.12)$$

Since

$$\frac{\partial I_1}{\partial \sigma_{ik}} = \delta_{ik}\,, \quad \frac{\partial I_2}{\partial \sigma_{ik}} = \sigma_{ik} - \alpha_{ik}\,, \quad \frac{\partial I_3}{\partial \sigma_{ik}} = (\sigma_{ij} - \alpha_{ij})\,(\sigma_{jk} - \alpha_{jk})\,, \quad (2.13)$$

we obtain from (2.12)

$$d\varepsilon_{ik} = \left[\frac{\partial G}{\partial I_1}\,\delta_{ik} + \frac{\partial G}{\partial I_2}\,(\sigma_{ik} - \alpha_{ik}) + \frac{\partial G}{\partial I_3}\,(\sigma_{ij} - \alpha_{ij})\,(\sigma_{jk} - \alpha_{jk})\right] d\lambda\,. \quad (2.14)$$

Let us assume now that the physical coordinate axes originally coincide with the principal axes of stress. Then we have first

$$\sigma_{ik} = 0 \quad (i \neq k) \quad \text{and} \quad \alpha_{ik} = 0\,. \qquad (2.15)$$

From (2.14) follows

$$d\varepsilon_{ik} = 0 \quad (i \neq k)\,. \qquad (2.16)$$

I.e., since the material is isotropic at the beginning, the strain increment tensor is coaxial with the stress tensor. By (2.4) and (2.16), also

$$d\alpha_{ik} = 0 \quad (i \neq k)\,. \qquad (2.17)$$

The last result remains valid if the second assumption (2.15) is replaced by the weaker assumption

$$\alpha_{ik} = 0 \quad (i \neq k)\,. \qquad (2.18)$$

It follows that, *if the principal axes of stress remain fixed in the element from the start, the strain increment tensor and thus the strain tensor remain coaxial with the stress tensor.*

If the principal axes of stress rotate, (2.16) holds only in a first step, provided the principal system of stress is used as the physical coordinate system. If (2.16) shall hold in a second step, the coordinate system must be rotated between the first step and the second one. This rotation, however, violates (2.18):

Due to the anisotropy caused by strain hardening, the strain increment tensor is in general not coaxial with the stress tensor.

Many problems of practical importance can be treated in a space of less than 9 dimensions. In certain cases, e.g., a 3-space defined by the principal stresses is useful. From our last result follows, however, that this 3-space is inadequate where the principal axes of stress are not fixed in the element. In addition, we shall see in the next sections that the reduction in dimensions is not without influence on the *form* of the hardening rule.

3. Treatment in 6-Space

On account of the symmetry of the stress and strain tensors, the problem may as well be treated in 6-space. It is convenient here and particularly for the subsequent specializations to denote the physical coordinates by x, y, z, the stresses by $\sigma_x, \ldots, \tau_{yz}, \ldots$, and the strains by $\varepsilon_x, \ldots, \varepsilon_{yz}, \ldots$, where the dots indicate cyclic permutations.

In the new notations the yield condition (2.2) reads

$$F(\sigma_x - \alpha_x, \ldots, \tau_{yz} - \alpha_{yz}, \ldots, \tau_{zy} - \alpha_{zy}, \ldots) = k^2, \tag{3.1}$$

where $\tau_{yz}, \tau_{zy}, \ldots$ have to be considered as independent variables. The flow rule (2.3) becomes

$$d\varepsilon_x = \frac{\partial F}{\partial \sigma_x} d\lambda, \ldots, \quad d\varepsilon_{yz} = \frac{\partial F}{\partial \tau_{yz}} d\lambda, \ldots, \quad d\varepsilon_{zy} = \frac{\partial F}{\partial \tau_{zy}} d\lambda, \ldots, \tag{3.2}$$

and the hardening rule (2.4) takes the form

$$d\alpha_x = c\, d\varepsilon_x, \ldots, \quad d\alpha_{yz} = c\, d\varepsilon_{yz}, \ldots, \quad d\alpha_{zy} = c\, d\varepsilon_{zy}, \ldots. \tag{3.3}$$

Treatment in 6-space, however, requires the elimination of the stress components $\tau_{zy}, \ldots$, of the strain components $\varepsilon_{zy}, \ldots$, and of the displacements $\alpha_{zy}, \ldots$.

Because of the symmetry of the stress tensor

$$F(\sigma_x, \ldots, \tau_{yz}, \ldots, \tau_{zy}, \ldots) = f(\sigma_x, \ldots, \tau_{yz}, \ldots). \tag{3.4}$$

Thus, the yield surface in 6-space is given by

$$f(\sigma_x - \alpha_x, \ldots, \tau_{yz} - \alpha_{yz}, \ldots) = F[\sigma_x - \alpha_x, \ldots, \tau_{yz} - \alpha_{yz}, \ldots, \tau_{zy} - \alpha_{zy}, \ldots] = k^2. \tag{3.5}$$

From (3.2) and (3.5) we obtain

$$d\varepsilon_x = \frac{\partial f}{\partial \sigma_x} d\lambda, \ldots, \quad d\gamma_{yz} = 2\, d\varepsilon_{yz} = \frac{\partial f}{\partial \tau_{yz}} d\lambda, \ldots. \tag{3.6}$$

This is the well-known result that *the flow rule of* v. MISES *remains valid in 6-space, if the state of strain is represented by the engineering components* $\varepsilon_x, \dots, \gamma_{yz}, \dots$.

If PRAGER's hardening rule holds in 9-space, the yield surface (3.5) in 6-space also moves in a translation. On account of (3.3), this translation is given by

$$d\alpha_x = c\, d\varepsilon_x, \dots, \quad d\alpha_{yz} = \frac{1}{2}\, c\, d\gamma_{yz}, \dots ; \tag{3.7}$$

in general it is not in the direction of the exterior normal at the point P. (3.7) is the form that PRAGER's hardening rule takes in the new strain components in 6-space.

It might seem that, dropping the factors 1/2 in (3.7), one might postulate the validity of PRAGER's rule in its original form in 6-space, thereby renouncing its validity in this form in 9-space. Since both sides of (3.7) represent tensors, such a procedure would involve the sacrifice of the invariance of the rule with respect to rotations of the physical coordinate system. It is clear that this is inacceptable, and that we have to accept, conversely, the fact that the form of PRAGER's rule is apt to deteriorate in a subspace. The next sections will show different stages of this process.

4. Special Cases

In many practically important cases some of the stress components are absent. Starting once more in 9-space, we may denote the stress components present by σ'_{ik}, the zero ones by σ''_{ik}. The initial yield condition is then

$$F(\sigma'_{ik}, \sigma''_{ik} = 0) = H(\sigma'_{ik}) = k^2 . \tag{4.1}$$

If we are not interested in the strains ε''_{ik} corresponding to the zero stresses σ''_{ik}, we may treat the problem in a subspace σ'_{ik}. Here, $H(\sigma'_{ik})$ defines a new yield surface.

After plastic flow has set in, the yield surface is given by

$$F(\sigma'_{ik} - \alpha'_{ik}, -\alpha''_{ik}) = k^2 . \tag{4.2}$$

We will not be able, in general, to express (4.2) by means of the function H: *in general, hardening implies a deformation of the yield surface in a subspace.*

From (4.1) follows

$$d\varepsilon'_{ik} = \frac{\partial F}{\partial \sigma'_{ik}}\, d\lambda \left(= \frac{\partial H}{\partial \sigma'_{ik}}\, d\lambda \right). \tag{4.3}$$

Thus, *the flow rule remains valid in any subspace.* However, it supplies only the strain components ε'_{ik} defined in this subspace although the ε''_{ik}, too, may be different from zero. It is clear, therefore, that *even in cases where the yield surface undergoes a translation, it may move in a direction different from the outward normal at the stress point P.*

The cases where the new yield surface does not deform are those in which it is possible to convert the left hand side of (4.2) such that the terms $-\alpha''_{ik}$ vanish.

When the initial yield is independent of the mean normal stress, the initial yield function can be written

$$f(\sigma_x, \dots, \tau_{yz}, \dots) = g(J_2, J_3) , \tag{4.4}$$

where

$$\left.\begin{aligned} J_2 &= \frac{1}{3}\,\sigma_x^2 + \cdots - \frac{1}{3}\,\sigma_y\,\sigma_z - \cdots + \tau_{yz}^2 + \cdots , \\ J_3 &= \frac{2}{27}\,\sigma_x^3 + \cdots - \frac{1}{9}\,\sigma_y\,\sigma_z\,(\sigma_y + \sigma_z) - \cdots + \frac{4}{9}\,\sigma_x\,\sigma_y\,\sigma_z \\ &\qquad - \frac{1}{3}\,(2\,\sigma_x - \sigma_y - \sigma_z)\,\tau_{yz}^2 - \cdots + 2\,\tau_{yz}\,\tau_{zx}\,\tau_{xy} \end{aligned}\right\} \tag{4.5}$$

are the invariants of the stress deviator. When plastic flow has set in, $\sigma_x, \dots,$ $\tau_{yz}, \dots$ have to be replaced by $\sigma_x - \alpha_x, \dots, \tau_{yz} - \alpha_{yz}, \dots$. It follows that *yield remains independent of the mean normal stress.*

By (3.6)

$$\left.\begin{aligned} d\varepsilon_x &= \left(\frac{\partial g}{\partial J_2}\cdot\frac{\partial J_2}{\partial \sigma_x} + \frac{\partial g}{\partial J_3}\cdot\frac{\partial J_3}{\partial \sigma_x}\right) d\lambda, \dots , \\ d\gamma_{yz} &= \left(\frac{\partial g}{\partial J_2}\cdot\frac{\partial J_2}{\partial \tau_{yz}} + \frac{\partial g}{\partial J_3}\cdot\frac{\partial J_3}{\partial \tau_{yz}}\right) d\lambda, \dots , \end{aligned}\right\} \tag{4.6}$$

where, on account of (4.5) and under the assumption that plastic flow has taken place,

$$\left.\begin{aligned} \frac{\partial J_2}{\partial \sigma_x} &= \frac{1}{3}\,[2\,(\sigma_x - \alpha_x) - (\sigma_y - \alpha_y) - (\sigma_z - \alpha_z)], \dots , \\ \frac{\partial J_2}{\partial \tau_{yz}} &= 2\,(\tau_{yz} - \alpha_{yz}), \dots , \\ \frac{\partial J_3}{\partial \sigma_x} &= \frac{1}{9}\,[2\,(\sigma_x - \alpha_x)^2 - (\sigma_y - \alpha_y)^2 - (\sigma_z - \alpha_z)^2 + 4\,(\sigma_y - \alpha_y)\,(\sigma_z - \alpha_z) \\ &\qquad - 2\,(\sigma_z - \alpha_z)\,(\sigma_x - \alpha_x) - 2\,(\sigma_x - \alpha_x)\,(\sigma_y - \alpha_y)] \\ &\qquad - \frac{1}{3}\,[2\,(\tau_{yz} - \alpha_{yz})^2 - (\tau_{zx} - \alpha_{zx})^2 - (\tau_{xy} - \alpha_{xy})^2], \dots , \\ \frac{\partial J_3}{\partial \tau_{yz}} &= -\frac{2}{3}\,[2\,(\sigma_x - \alpha_x) - (\sigma_y - \alpha_y) - (\sigma_z - \alpha_z)]\,(\tau_{yz} - \alpha_{yz}) \\ &\qquad + 2\,(\tau_{zx} - \alpha_{zx})\,(\tau_{xy} - \alpha_{xy}) , \dots . \end{aligned}\right\} \tag{4.7}$$

From (2.10) follows

$$f(\sigma_x - \alpha_x + \beta\,, \ldots\,, \tau_{yz} - \alpha_{yz}, \ldots) = f(\sigma_x - \alpha_x, \ldots\,, \tau_{yz} - \alpha_{yz}\,, \ldots)\,. \quad (4.8)$$

Hence, in cases where one of the normal stresses, e.g. σ_z, is absent, the term $-\alpha_z$ can be eliminated by addition of α_z to the normal stresses.

If $\tau_{yz} = \tau_{zx} = 0$, (4.6) and (4.7) yield, in connection with (3.7),

$$d\gamma_{yz} = d\gamma_{zx} = 0 \quad (\gamma_{yz} = \gamma_{zx} = 0)\,. \quad (4.9)$$

This is the proof, by complete induction, that

$$\alpha_{yz} = \frac{1}{2}\,c\,\gamma_{yz} = 0\,, \quad \alpha_{zx} = \frac{1}{2}\,c\,\gamma_{zx} = 0 \quad (\tau_{yz} \equiv \tau_{zx} \equiv 0)\,, \quad (4.10)$$

i.e. that, in the absence of at least two shear stresses, the corresponding shear strains and thus the corresponding displacements of the yield surface, are absent. This result might have been inferred from the symmetry (Figure 2) of

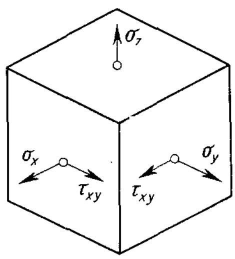

Figure 2
State of stress with $\tau_{yz} = \tau_{zx} = 0$.

the state of stress with respect to the middle plane x, y of an element. It is clear that a similar result does not hold in general if only one shear stress is absent; on the other hand, (4.10) is the reason why PRAGER's hardening rule applies without change of form in the space of principal stresses.

While the foregoing results hold for any form of the yield function (4.4), we can obtain some more results by restricting ourselves to the more common types of g. In v. MISES' case, the function g reduces to J_2, and the initial yield condition is

$$J_2 = \frac{1}{3}\,\sigma_0^2\,, \quad (4.11)$$

where σ_0 is the initial yield limit in simple tension or compression. In TRESCA's case, g also depends on J_3. Here, the yield condition is better discussed in terms of maximum shear stress.

In (4.10), it is essential that two shear stresses vanish: in cases where one shear stress only, i.e. τ_{xy}, is absent, it may be impossible to eliminate the corresponding displacement, α_{xy}, from the right hand side of (4.8), and this means that the yield surface deforms in the process of hardening. In TRESCA's case, this actually happens. However, if v. MISES' yield condition applies, it follows from (4.6), (4.7) and (3.7) that

$$\alpha_{xy} = \frac{1}{2} c \gamma_{xy} = 0 \quad (\tau_{xy} \equiv 0) . \tag{4.12}$$

If $\sigma_y = \sigma_z = 0$, we obtain from the same relations

$$\varepsilon_y = \varepsilon_z , \quad \alpha_y = \alpha_z , \quad (\sigma_y \equiv \sigma_z \equiv 0) , \tag{4.13}$$

provided the material obeys the yield condition of v. MISES. It is easy to see, however, that under TRESCA's yield condition (4.13) does not apply. In the last case it is not possible to eliminate both α_y and α_z from the right hand side of (4.8): the yield surface deforms in the course of hardening.

In the next sections we shall make use of the fact that the plastic volume change of the material is zero. That is,

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0 , \tag{4.14}$$

a consequence of the flow rule and the independence of the yield function on the mean normal stress.

5. Plane Strain

Here, per definitionem,

$$\tau_{yz} = \tau_{zx} = 0 , \quad \varepsilon_z = 0 . \tag{5.1}$$

From (4.10) we obtain

$$\gamma_{yz} = \gamma_{zx} = 0 . \tag{5.2}$$

It follows from (3.7) that the yield function has the form

$$g\left(\sigma_x - c\,\varepsilon_x,\ \sigma_y - c\,\varepsilon_y,\ \sigma_z,\ \tau_{xy} - \frac{1}{2} c\,\gamma_{xy}\right) . \tag{5.3}$$

On account of (3.6),

$$d\varepsilon_z = \frac{\partial g}{\partial \sigma_z} d\lambda = 0 , \tag{5.4}$$

and by (4.14)

$$\varepsilon_y = -\varepsilon_x . \tag{5.5}$$

Thus, we finally get the yield condition

$$h\left(\sigma_x - c\,\varepsilon_x,\ \sigma_y + c\,\varepsilon_x,\ \tau_{xy} - \frac{1}{2}\,c\,\gamma_{xy}\right) = k^2\,. \tag{5.6}$$

It follows that the yield surface moves in a translation, but, on account of the factor 1/2 in the last argument, not in the direction of the exterior normal.

In many cases where the original form of PRAGER's hardening rule does not hold in the subspace appropriate to the problem, this defect can be remedied by a transformation. Here, for instance, the new quantities

$$t_{xy} = \sqrt{2}\,\tau_{xy}\,,\quad g_{xy} = \frac{1}{\sqrt{2}}\,\gamma_{xy} \tag{5.7}$$

can be introduced. The yield condition (5.6) becomes

$$p(\sigma_x - c\,\varepsilon_x,\ \sigma_y + c\,\varepsilon_x,\ t_{xy} - c\,g_{xy}) = k^2\,, \tag{5.8}$$

and the form of PRAGER's rule in 9-space is regained.

In order to specialize (5.8) for v. MISES' yield condition, we start from (4.5) and obtain first

$$\left.\begin{aligned} f = J_2 = \frac{1}{3}\,(\sigma_x - c\,\varepsilon_x)^2 + \cdots - \frac{1}{3}\,(\sigma_y - c\,\varepsilon_y)\,(\sigma_z - c\,\varepsilon_z) - \cdots \\ + \left(\tau_{xy} - \frac{1}{2}\,c\,\gamma_{xy}\right)^2 + \cdots\,. \end{aligned}\right\} \tag{5.9}$$

Carrying out the steps (5.3) through (5.8) with (5.9), we get

$$p = [(\sigma_x - c\,\varepsilon_x) - (\sigma_y + c\,\varepsilon_x)]^2 + 2\,(t_{xy} - c\,g_{xy})^2 = \frac{4}{3}\,\sigma_0^2\,. \tag{5.10}$$

Figure 3 shows the yield surface, a circular cylinder of radius $(2/3)^{1/2}\,\sigma_0$ the axis of which is parallel to the plane σ_x, σ_y and bisects the angle between the axes σ_x and σ_y.

In order to specialize (5.8) for TRESCA's case, we start from the principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2},\quad \sigma_3 = \sigma_z\,, \tag{5.11}$$

where σ_3 lies between σ_1 and σ_2. The material yields initially when the maximum shear stress reaches a critical value,

$$\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = \frac{1}{4}\,\sigma_0^2\,. \tag{5.12}$$

After plastic flow has set in, (5.12) takes the form (5.10) with σ_0^2 replacing

$4\,\sigma_0^2/3$ on the right hand side. The yield surface is a circular cylinder (Figure 3) of radius $\sigma_0/\sqrt{2}$, moving in the direction of the outward normal.

Subcase a: If $\tau_{xy} = 0$, (4.10) yields $\gamma_{xy} = 0$. The problem can be treated in a (σ_x, σ_y)-plane and is a degenerate case of principal stress space with the principal strain ε_z zero. The yield locus is the strip obtained by bisecting the cylinder of Figure 3 parallel to the plane σ_x, σ_y.

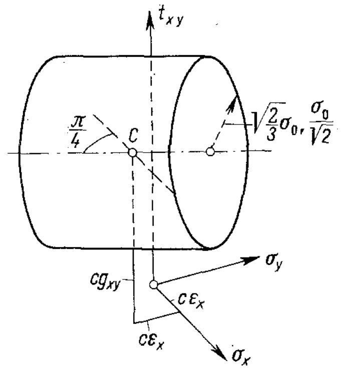

Figure 3
Yield surface in plane strain.

Subcase b: If $\sigma_y = 0$, the problem can be treated in a (σ_x, τ_{xy})-plane. The yield locus, obtained by intersecting the cylinder of Figure 3 with a plane parallel to the plane σ_x, t_{xy}, does not move in the direction of the exterior normal. However, if besides (5.7) the transformation

$$s_x = \frac{1}{\sqrt{2}}\,\sigma_x\,, \quad e_x = \sqrt{2}\,\varepsilon_x \tag{5.13}$$

is used, the yield condition (5.10) with $\sigma_y = 0$ becomes

$$(s_x - c\,e_x)^2 + (t_{xy} - c\,g_{xy})^2 = \frac{2}{3}\,\sigma_0^2\,. \tag{5.14}$$

The yield locus is a circle of radius $(2/3)^{1/2}\,\sigma_0$ in v. MISES' case and $(1/2)^{1/2}\,\sigma_0$ in TRESCA's case, moving in the direction of the exterior normal.

6. Plane Stress

Here, per definitionem,

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0\,. \tag{6.1}$$

On account of (4.10),

$$\gamma_{yz} = \gamma_{zx} = 0\,, \tag{6.2}$$

and from (4.14) follows

$$\varepsilon_z = -(\varepsilon_x + \varepsilon_y) . \tag{6.3}$$

Hence, the yield function is

$$g\left[\sigma_x - c\,\varepsilon_x, \sigma_y - c\,\varepsilon_y, c\,(\varepsilon_x + \varepsilon_y), \tau_{xy} - \frac{1}{2}\,c\,\gamma_{xy}\right] . \tag{6.4}$$

After subtraction of $c(\varepsilon_x + \varepsilon_y)$ from the normal stresses in accordance with (4.8) we obtain the yield condition

$$h\left[\sigma_x - c\,(2\,\varepsilon_x + \varepsilon_y), \sigma_y - c\,(\varepsilon_x + 2\,\varepsilon_y), \tau_{xy} - \frac{1}{2}\,c\,\gamma_{xy}\right] = k^2 . \tag{6.5}$$

Again, the yield surface moves in a translation, but not in the direction of the exterior normal.

If we make use of the transformations

$$\left.\begin{aligned} s_\xi &= \frac{1}{2}\,(\sigma_x + \sigma_y)\,, \quad s_\eta = \frac{\sqrt{3}}{2}\,(\sigma_y - \sigma_x)\,, \quad t_{\xi\eta} = \sqrt{3}\,\tau_{xy}\,, \\ e_\xi &= \varepsilon_x + \varepsilon_y\,, \qquad e_\eta = \frac{1}{\sqrt{3}}\,(\varepsilon_y - \varepsilon_x)\,, \quad g_{\xi\eta} = \frac{1}{\sqrt{3}}\,\gamma_{xy}\,, \end{aligned}\right\} \tag{6.6}$$

the yield surface becomes

$$p\left(s_\xi - \frac{3}{2}\,c\,e_\xi\,, \quad s_\eta - \frac{3}{2}\,c\,e_\eta\,, \quad t_{\xi\eta} - \frac{3}{2}\,c\,g_{\xi\eta}\right) = k^2 \tag{6.7}$$

and moves in the direction of the outward normal.

In the case of v. MISES' yield condition, (6.7) takes the form

$$p = \left(s_\xi - \frac{3}{2}\,c\,e_\xi\right)^2 + \left(s_\eta - \frac{3}{2}\,c\,e_\eta\right)^2 + \left(t_{\xi\eta} - \frac{3}{2}\,c\,g_{\xi\eta}\right)^2 = \sigma_0^2 . \tag{6.8}$$

The yield surface (an ellipsoid in the original stresses) is a sphere of radius σ_0.

In TRESCA's case, we obtain three sheets

$$2\left(\frac{s_\eta^2 + t_{\xi\eta}^2}{3}\right)^{1/2} = \sigma_0\,, \qquad \left| s_\xi \pm \left(\frac{s_\eta^2 + t_{\xi\eta}^2}{3}\right)^{1/2} \right| = \sigma_0 . \tag{6.9}$$

Thus, the yield surface (Figure 4) is a circular cylinder, closed by two circular cones such that the intersections of the whole surface with the middle planes parallel to s_ξ, s_η and s_ξ, $t_{\xi\eta}$ are regular hexagons with sides σ_0.

It often happens that the yield surface contains singularities. In 3-space such singularities are edges where two smooth surfaces intersect (in Figure 4 the circles of intersection between cylinder and cones) or corners where more than two smooth surfaces meet. An isolated singularity or vertex (in Figure 4) the vertex of either cone) may be considered as a limiting case of a corner.

If P is to remain in a corner or vertex, the yield surface must move with P; hence, the displacement $d\vec{\alpha}$ coincides with the stress increment $d\vec{\sigma}$. The strain increment $d\vec{\varepsilon}$ follows from $d\vec{\alpha}$, since the components of $\vec{\alpha}$ are the factors of c in the general equation [here (6.5) or (6.7)] of the yield surface. If $d\vec{\alpha} = c\, d\vec{\varepsilon}$, then P remains in the corner or vertex as long as $d\vec{\sigma}$ lies in the pyramid or cone enclosed by the exterior normals of the yield surface in the vicinity of the singularity. If $d\vec{\alpha} \neq c\, d\vec{\varepsilon}$, the pyramid or cone is obtained from the vectors $d\vec{\alpha}$ instead of the normals.

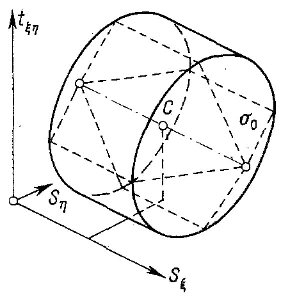

Figure 4
Tresca yield surface in plane stress.

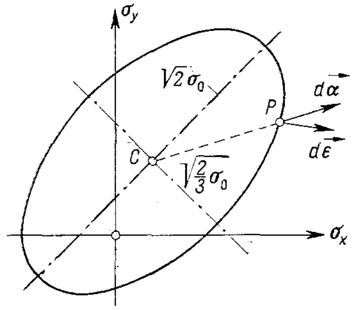

Figure 5
v. Mises yield locus in plane stress with $\tau_{xy} = 0$.

If P is to remain in an edge, the components of $d\vec{\sigma}$ and $d\vec{\alpha}$ normal to the edge must coincide. Further, according to the flow rule, $d\vec{\varepsilon}$ has no component along the tangent of the edge. Thus, two components of $d\vec{\alpha}$ and the third one of $d\vec{\varepsilon}$ are known as soon as $d\vec{\sigma}$ is given. The remaining components follow again from the equation of the yield surface. If $d\vec{\alpha} = c\, d\vec{\varepsilon}$, P remains in the edge as long as $d\vec{\sigma}$ lies in the wedge enclosed by the normals of the yield surface in the vicinity of the edge. If $d\vec{\alpha} \neq c\, d\vec{\varepsilon}$, the vectors $d\vec{\alpha}$ take the function of the normals.

Subcase a: If $\tau_{xy} = 0$, (4.10) yields $\gamma_{xy} = 0$. The problem can be treated in a (σ_x, σ_y)-plane, the section of principal stress space by the plane $\sigma_z = 0$.

In v. MISES' case, the yield locus is the well-known ellipse illustrated in Figure 5 with the equation

$$\left.\begin{aligned} &[\sigma_x - c\,(2\,\varepsilon_x + \varepsilon_y)]^2 + [\sigma_y - c\,(\varepsilon_x + 2\,\varepsilon_y)]^2 \\ &\qquad - [\sigma_x - c\,(2\,\varepsilon_x + \varepsilon_y)]\,[\sigma_y - c\,(\varepsilon_x + 2\,\varepsilon_y)] = \sigma_0^2\,. \end{aligned}\right\} \quad (6.10)$$

Comparing the vectors $d\alpha_x$, $d\alpha_y$ and $d\varepsilon_x$, $d\varepsilon_y$ following from (6.10), we easily obtain

$$d\alpha_x = 3\,c\,(\sigma_x - \alpha_x)\,d\lambda\,, \quad d\alpha_y = 3\,c\,(\sigma_y - \alpha_y)\,d\lambda\,. \qquad (6.11)$$

Hence, the ellipse moves in the direction of the radius $C\,P$.

Using the transformations (6.6), we obtain a representation in a plane s_ξ, s_η. Here, the yield locus is a circle of radius σ_0, moving in the direction of the outward normal.

In TRESCA's case, the yield locus in the (σ_x, σ_y)-plane is the hexagon of Figure 6. The arrows indicate the vectors $d\vec{\alpha}$, of constant direction on each side, given by the line connecting the center of the side in question with the center

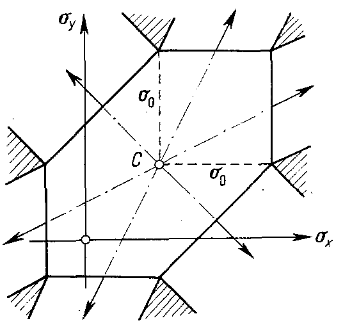

Figure 6
Tresca yield locus in plane stress with $\tau_{xy} = 0$.

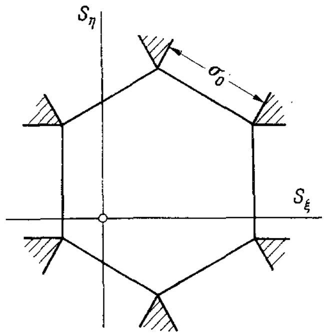

Figure 7
Yield locus of Figure 6 in s_ξ, s_η.

of the opposite side. If $d\vec{\sigma}$ lies in one of the shaded regions, the hexagon moves with P. Taking $d\alpha_x$, $d\alpha_y$ from (6.5), solving for $d\varepsilon_x$, $d\varepsilon_y$ and setting $d\vec{\alpha} = d\vec{\sigma}$, we obtain

$$d\varepsilon_x = \frac{1}{3\,c}\,(2\,d\sigma_x - d\sigma_y)\,, \quad d\varepsilon_y = \frac{1}{3\,c}\,(2\,d\sigma_y - d\sigma_x) \tag{6.12}$$

for the strain increment in a corner of the hexagon.

In the (s_ξ, s_η)-plane the yield locus becomes a regular hexagon (Figure 7) with side σ_0 moving in the direction of the exterior normal. It is clear that this hexagon is the section of the yield surface of Figure 4 with the plane of symmetry parallel to s_ξ, s_η.

If P remains in a corner,

$$de_\xi = \frac{2}{3\,c}\,ds_\xi, \quad de_\eta = \frac{2}{3\,c}\,ds_\eta; \tag{6.13}$$

thus, $d\vec{e}$ has the same direction as $d\vec{s} = d\vec{\alpha}$.

Subcase a: If $\sigma_y = 0$, the problem can be treated in a (σ_x, τ_{xy})-plane. However, since (4.13) only applies in v. MISES' case, the yield locus deforms in any other case in the process of hardening.

In TRESCA's case, for instance, the initial yield locus is the intersection of the yield surface of Figure 4 with one of the vertical planes passing through C and touching the two circular edges. The displacement of C, however, generally does not lie in this plane, and it becomes clear, therefore, that the yield locus deforms.

In v. MISES' case, (4.13) applies. Hence, the yield locus does not deform. This follows also from the fact that the yield surface corresponding to Figure 4 is a sphere. The yield locus is the circle

$$\left(\sigma_x - \frac{3}{2}\, c\, \varepsilon_x\right)^2 + \left(t_{\xi\eta} - \frac{3}{2}\, c\, g_{\xi\eta}\right)^2 = \sigma_0^2 \tag{6.14}$$

in the plane σ_x, $t_{\xi\eta}$, where $t_{\xi\eta}$ and $g_{\xi\eta}$ follow from τ_{xy} and γ_{xy} by means of (6.6). The circle (6.14) is of radius σ_0 and has been discussed by PRAGER [9]. It moves in the direction of the outward normal.

7. Another Special Case

In certain cases, e.g., if a cylinder is subjected to torsion and simple tension, we have $\sigma_x = \sigma_y = \tau_{xy} = 0$. From (4.12) through (4.14) follows

$$\varepsilon_x = \varepsilon_y = -\frac{1}{2}\, \varepsilon_z\,, \quad \gamma_{xy} = 0\,, \tag{7.1}$$

provided the material obeys v. MISES' yield condition. For any other yield condition (4.12) and (4.13) do not generally hold; hence, the yield surface in 3-space deforms in the process of hardening.

Restricting ourselves to materials obeying v. MISES' condition, we start from (4.5), i.e., from the yield condition

$$\left.\begin{aligned} f = J_2 = \frac{1}{3}\,(\sigma_x - c\,\varepsilon_x)^2 + \cdots - \frac{1}{3}\,(\sigma_y - c\,\varepsilon_y)\,(\sigma_z - c\,\varepsilon_z) - \cdots \\ + \left(\tau_{yz} - \frac{1}{2}\, c\,\gamma_{yz}\right)^2 + \cdots = \frac{1}{3}\,\sigma_0^2\,. \end{aligned}\right\} \tag{7.2}$$

Inserting (7.1) and subtracting $c\,\varepsilon_z/2$ from the normal stresses, we obtain in this particular case

$$g = \left(\sigma_z - \frac{3}{2}\, c\,\varepsilon_z\right)^2 + 3\left(\tau_{yz} - \frac{1}{2}\, c\,\gamma_{yz}\right)^2 + 3\left(\tau_{zx} - \frac{1}{2}\, c\,\gamma_{zx}\right)^2 = \sigma_0^2\,. \tag{7.3}$$

This yield surface (an ellipsoid of rotation) moves in a translation, but not in the direction of the outward normal.

If we use the transformations

$$t_{yz} = \sqrt{3}\,\tau_{yz}\,, \quad t_{zx} = \sqrt{3}\,\tau_{zx}\,, \quad g_{yz} = \frac{1}{\sqrt{3}}\,\gamma_{yz}\,, \quad g_{zx} = \frac{1}{\sqrt{3}}\,\gamma_{zx}\,, \tag{7.4}$$

the yield surface becomes a sphere

$$p = \left(\sigma_z - \frac{3}{2}\,c\,\varepsilon_z\right)^2 + \left(t_{yz} - \frac{3}{2}\,c\,g_{yz}\right)^2 + \left(t_{zx} - \frac{3}{2}\,c\,g_{zx}\right)^2 = \sigma_0^2 \tag{7.5}$$

of radius σ_0, moving in the direction of the exterior normal.

Subcase a: If $\sigma_z = 0$, (4.13) and (4.14) yield $\varepsilon_z = 0$. For v. Mises' yield condition, the problem can be treated in a (τ_{yz}, τ_{zx})-plane. The yield locus is the circle

$$\left(\tau_{yz} - \frac{1}{2}\,c\,\gamma_{yz}\right)^2 + \left(\tau_{zx} - \frac{1}{2}\,c\,\gamma_{zx}\right)^2 = \frac{1}{3}\,\sigma_0^2 \tag{7.6}$$

of radius $3^{-1/2}\,\sigma_0$, moving in the direction of its outward normal.

It is interesting to note that under Tresca's yield condition the yield surface deforms even in this comparatively simple case and that γ_{xy} does not remain zero.

Subcase b: If $\tau_{yz} = 0$, (4.10) yields $\gamma_{yz} = 0$. If v. Mises' yield condition holds, the problem can be treated in a (σ_z, t_{zx})-plane. The yield locus is the circle

$$\left(\sigma_z - \frac{3}{2}\,c\,\varepsilon_z\right)^2 + \left(t_{zx} - \frac{3}{2}\,c\,g_{zx}\right)^2 = \sigma_0^2 \tag{7.7}$$

of radius σ_0, moving in the direction of the exterior normal.

It is clear that the last result, apart from the difference in notation, is the one already obtained in the last subcase of section 6.

8. Conclusion

In sections 5 through 7 we have encountered the various possibilities discussed already in section 4. In most of the cases considered the yield surface moves in a translation, and a simple transformation at most suffices to make the original form of Prager's rule apply in the subspace appropriate to the problem. In certain cases, however, the yield surface deforms in the course of hardening. Incidentally, these exceptions occur, as far as our examples are concerned, in those cases where at the same time (a) more than one normal stress is different from zero and (b) the material obeys Tresca's yield condition.

In section 1 the advantages of Tresca's yield condition have been emphasized. The deformation of the yield surface which has been found in many cases represents a serious drawback in this respect.

REFERENCES

[1] H. Ziegler, *An Attempt to Generalize Onsager's Principle, and Its Significance for Rheological Problems*, Z. angew. Math. Phys. *9b*, 748 (1958).

[2] R. v. Mises, *Mechanik der festen Körper im plastisch deformablen Zustand*, Göttinger Nachrichten, Math. phys. Kl. *1913*, 582 (1913).

[3] H. TRESCA, *Mémoire sur l'écoulement des corps solides*, Mém. prés. Acad. Sci., Paris *18*, 733 (1868).
[4] R. v. MISES, *Mechanik der plastischen Formänderung von Kristallen*, Z. angew. Math. Mech. *8*, 161 (1928).
[5] D. C. DRUCKER, *Some Implications of Work Hardening and Ideal Plasticity*, Quart. appl. Math. *7*, 411 (1950).
[6] R. HILL, *The Mathematical Theory of Plasticity* (Oxford 1950).
[7] P. G. HODGE, Jr., *The Theory of Piecewise Linear Isotropic Plasticity*, IUTAM Colloquium Madrid 1955, *Deformation and Flow of Solids* (Berlin 1956).
[8] W. PRAGER, *The Theory of Plasticity: A Survey of Recent Achievements* (James Clayton Lecture), Proc. Inst. Mech. Eng. *169*, 41 (1955).
[9] W. PRAGER, *Probleme der Plastizitätstheorie* (Basel 1955), p. 16.
[10] P. G. HODGE, Jr., *Piecewise Linear Plasticity*, Proc. 9th Intern. Congr. Appl. Mech., Brussels, 1956.

Zusammenfassung

Um das Verhalten eines Metalls mit dem Spannungs-Dehnungs-Diagramm der Figur 1 unter einem beliebigen räumlichen Spannungszustand zu beschreiben, wird neben der Annahme isotroper Verfestigung, welche aber den Bauschinger-Effekt nicht erklärt, die Pragersche Verfestigungsregel [8] verwendet. Es wird hier untersucht, welche Formen diese Regel in den wichtigsten Spannungsräumen von weniger als neun Dimensionen annimmt.

(Received: November 26, 1957.)

Kurze Mitteilungen – Brief Reports – Communications brèves

On the Free Convection from a Horizontal Plate

By KEITH STEWARTSON, Durham, England[1])

The free convection of heat from a heated vertical plate in a fluid has been extensively studied in recent years. A review of the work done has been given by SQUIRE [4][2]) and subsequently numerical solutions of the governing equations has been given by OSTRACH [2] for a wide range of values of the Prandtl number σ. The convection takes place in boundary layers originating at the lower edge of the plate. Fluid is drawn into them, is heated and gaining buoyancy moves upwards. On the other hand if the plate is cooled relative to the surrounding fluid the situation is reversed for the boundary layers originate at the top of the plate, and the fluid drawn into them is forced downwards. When the plate is inclined to the vertical there is no change in the flow pattern, since the vertical buoyancy force has a component along the plate which drives the fluid thus generating the boundary layer. However, if the plate is horizontal the buoyancy has no component along its length and the boundary layer, if it exists, must be of a different character.

[1]) Department of Mathematics, The University.
[2]) Numbers in brackets refer to References, page 281.