It can easily be shown that the same limit locus results from an application of the upper-bound theorem. Consider a velocity field in which only one of the flanges elongates while the other remains rigid. In order for the mean elongation rate of the beam to be $\dot{\Delta}$, that of the deforming flange must be $2\dot{\Delta}$, and the rotation rate, to within a sign, is $\dot{\theta} = 2\dot{\Delta}/h$. For the sake of definiteness, let us take both $\dot{\Delta}$ and $\dot{\theta}$ as positive. The strain rate in the deforming flange is $2\dot{\Delta}/L$, so that the total plastic dissipation — the numerator on the right-hand side of (3.5.1) — is $\sigma_Y(2\dot{\Delta}/L)(AL) = P_U\dot{\Delta}$. We take P as the reference load, and pick a loading direction by letting $M = \alpha Ph/2$. The denominator in (3.5.1) is thus $\dot{\Delta} + (\alpha h/2)(2\dot{\Delta}/h) = (1+\alpha)\dot{\Delta}$, and the upper bound for P is $P_U/(1+\alpha)$. Since $M_U = P_U h/2$, the upper bound for M is $\alpha M_U/(1+\alpha)$. The upper-bound values satisfy $M/M_U + P/P_U = 1$, an equation describing the first quadrant of the previously found limit locus. The remaining quadrants are found by varying the signs of $\dot{\Delta}$ and $\dot{\theta}$.

A velocity field with both flanges deforming leads to an upper-bound load point lying outside the limit locus just found, with two exceptions: one where the elongation rates of the flanges are the same, and one where they are equal and opposite. Details are left to an exercise.

3.5.2. Nonstandard Limit-Analysis Theorems

The theorems of limit analysis can be stated in a form that does not directly refer to any concepts from plasticity theory:

A body will not collapse under a given loading if a possible stress field can be found that is in equilibrium with a loading greater than the given loading.

A body will collapse under a given loading if a velocity field obeying the constraints (or a mechanism) can be found that so that the internal dissipation is less than the rate of work of the given loading.

In this form, the theorems appear intuitively obvious. In fact, the concepts underlying the theorems were used long before the development of plasticity theory. Use of what is essentially the upper-bound theorem goes back to the eighteenth century: it was used in 1741 by a group of Italian mathematicians to design a reinforcement method for the crumbling dome of Saint Peter's Church, and in 1773 by Coulomb to investigate the collapse strength of soil. The latter problem was also studied by Rankine in the mid-nineteenth century by means of a technique equivalent to the lower-bound theorem.

The simple form of the theorems given above hides the fact that the postulate of maximum plastic dissipation (and therefore the normality of the flow rule) is an essential ingredient of the proof. It was therefore necessary to find a counterexample showing that the theorems are not universally applicable to nonstandard materials. One such counterexample, in which plasticity is combined with Coulomb friction at an interface, was presented by Drucker [1954a]. Another was shown by Salençon [1973].

Radenkovic's Theorems

A theory of limit analysis for nonstandard materials, with a view toward its application to soils, was formulated by Radenkovic [1961, 1962], with modifications by Josselin de Jong [1965, 1974], Palmer [1966], Sacchi and Save [1968], Collins [1969], and Salençon [1972, 1977]. **Radenkovic's first theorem** may be stated simply as follows: *The limit loading for a body made of a nonstandard material is bounded from above by the limit loading for the standard material obeying the same yield criterion.*

The proof is straightforward. Let $\mathbf{v}^*$ denote any kinematically admissible velocity field, and $\mathbf{P}^*$ the upper-bound load point obtained for the standard material on the basis of this velocity field. If $\boldsymbol{\sigma}$ is the actual stress field at collapse in the real material, then, since this stress field is also statically and plastically admissible in the standard material,

$$D_p(\dot{\boldsymbol{\varepsilon}}^*) \geq \sigma_{ij}\dot{\varepsilon}_{ij}{}^*,$$

and therefore, by virtual work,

$$\mathbf{P}^* \cdot \dot{\mathbf{p}}^* \geq \mathbf{P} \cdot \dot{\mathbf{p}}^*.$$

Since $\mathbf{v}^*$ may, as a special case, coincide with the correct collapse velocity field in the fictitious material, $\mathbf{P}^*$ may be the correct collapse loading in this material, and the theorem follows.

Radenkovic's second theorem, as modified by Josselin de Jong [1965], is based on the existence of a function $g(\boldsymbol{\sigma})$ with the following properties:

1. $g(\boldsymbol{\sigma})$ is a convex function (so that any surface $g(\boldsymbol{\sigma}) = \mathit{constant}$ is convex);

2. $g(\boldsymbol{\sigma}) = 0$ implies $f(\boldsymbol{\sigma}) \leq 0$ (so that the surface $g(\boldsymbol{\sigma}) = 0$ lies entirely within the yield surface $f(\boldsymbol{\sigma}) = 0$);

3. to any $\boldsymbol{\sigma}$ with $f(\boldsymbol{\sigma}) = 0$ there corresponds a $\boldsymbol{\sigma}'$ such that (a) $\dot{\boldsymbol{\varepsilon}}^p$ is normal to the surface $g(\boldsymbol{\sigma}) = 0$ at $\boldsymbol{\sigma}'$, and (b)

$$(\sigma_{ij} - \sigma'_{ij})\dot{\varepsilon}_{ij} \geq 0. \tag{3.5.4}$$

The theorem may then be stated thus: *The limit loading for a body made of a nonstandard material is bounded from below by the limit loading for the standard material obeying the yield criterion* $g(\boldsymbol{\sigma}) = 0$.

The proof is as follows. Let $\boldsymbol{\sigma}$ denote the actual stress field at collapse, $\mathbf{P}$ the limit loading, $\mathbf{v}$ the actual velocity field at collapse, $\dot{\boldsymbol{\varepsilon}}$ the strain-rate field, and $\dot{\mathbf{p}}$ the generalized velocity vector conjugate to $\mathbf{P}$. Thus, by virtual work,

$$\mathbf{P} \cdot \dot{\mathbf{p}} = \int_R \sigma_{ij} \dot{\varepsilon}_{ij} \, dV.$$

Now, the velocity field $\mathbf{v}$ is kinematically admissible in the fictitious standard material. If $\boldsymbol{\sigma}'$ is the stress field corresponding to $\boldsymbol{\sigma}$ in accordance with the definition of $g(\boldsymbol{\sigma})$, then it is the stress field in the fictitious material that is plastically associated with $\dot{\varepsilon}$, and, if $\mathbf{P}'$ is the loading that is in equilibrium with $\boldsymbol{\sigma}'$, then

$$\mathbf{P}' \cdot \dot{\mathbf{p}} = \int_R \sigma'_{ij} \dot{\varepsilon}_{ij} \, dV.$$

It follows from inequality (3.5.4) that

$$\mathbf{P}' \cdot \dot{\mathbf{p}} \leq \mathbf{P} \cdot \dot{\mathbf{p}}.$$

Again, $\boldsymbol{\sigma}'$ may, as a special case, coincide with the correct stress field at collapse in the standard material, and therefore $\mathbf{P}'$ may be the correct limit loading in this material. The theorem is thus proved.

In the case of a Mohr–Coulomb material, the function $g(\boldsymbol{\sigma})$ may be identified with the plastic potential if this is of the same form as the yield function, but with an angle of dilatation that is less than the angle of internal friction (in fact, the original statement of the theorem by Radenkovic [1962] referred to the plastic potential only). The same is true of the Drucker–Prager material.

It should be noted that neither the function g, nor the assignment of $\boldsymbol{\sigma}'$ to $\boldsymbol{\sigma}$, is unique. In order to achieve the best possible lower bound, g should be chosen so that the surface $g(\boldsymbol{\sigma}) = 0$ is as close as possible to the yield surface $f(\boldsymbol{\sigma}) = 0$, at least in the range of stresses that are expected to be encountered in the problem studied. Since the two surfaces do not coincide, however, it follows that the lower and upper bounds on the limit loading, being based on two different standard materials, cannot be made to coincide. The correct limit loading in the nonstandard material cannot, therefore, be determined in general. This result is consistent with the absence of a uniqueness proof for the stress field in a body made of a nonstandard perfectly plastic material (see 3.4.1).

3.5.3. Shakedown Theorems

The collapse discussed thus far in the present section is known as *static collapse*, since it represents unlimited plastic deformation while the loads remain constant in time. If the loads are applied in a cyclic manner, without ever reaching the static collapse condition, other forms of collapse may occur. If the strain increments change sign in every cycle, with yielding on both sides of the cycle, then *alternating plasticity* is said to occur; the *net* plastic