Quantitative Studies on Asymptotic Growth Behaviors of Trajectories of Nonlinear Discrete Dynamical Systems

Lisheng Wang and Zongben Xu

Abstract—**This technical note studies quantitatively asymptotic growth behaviors of trajectories (AGBT) of nonlinear autonomous discrete dynamical system that has unbounded domain, non-Lipschitz continuous nonlinear operator, and stable or unstable equilibrium point. We explain how trajectory motion speed is quantitatively determined in the system, and study exact computation and sharp estimation of the smallest exponential bound of trajectories. We characterize exponential stability and asymptotic stability of the system from a new point of view, and provide a simple condition to distinguish them from each other. These results extend existing results that were obtained in some special cases of the system, and are helpful for quantitative analysis and understanding of AGBT of the system.**

Index Terms—**Asymptotic growth behaviors, nonlinear discrete dynamical systems, trajectory motion speed.**

I. Introduction

Suppose that X is a Banach space, $E \subset X$ is an arbitrary subset containing the origin O as an interior point, $T: E \rightarrow E$ is a nonlinear operator. This technical note considers the following nonlinear discrete dynamical system (NDDS):

$$x(k+1) = T(x(k)), \quad x(0) \in E, \quad k = 0, 1, 2 \dots \tag{1}$$

where, O is an equilibrium point (i.e., $T(O) = O$), trajectories of the system converge to or diverge from the equilibrium point exponentially as $k \rightarrow \infty$. This means that there exist constants $\beta \in (0, \infty)$ and $M(\beta) > 0$ such that the trajectory motion of the system satisfies

$$\|x(k)\| \leq M(\beta) \cdot \beta^k \cdot \|x(0)\|, \quad \forall x(0) \in E, \quad k = 1, 2 \dots \tag{2}$$

here, β is called an exponential bound of trajectories, and $M(\beta)$ the growth coefficient corresponding to β. Any positive number larger than β is an exponential bound as well. β can be less than one if system (1) is exponentially stable in E [1], but cannot be less than one in all other cases. System (1) models many different NDDS in practice, such as ones in neural networks, economic or biological systems, numerical analysis [2], [3], [18], etc. Thus, it is useful to study dynamical behaviors and trajectory motion of system (1).

In many cases, users attempt to know or estimate how fast trajectories of system (1) converge to or diverge from the equilibrium point (namely, the speed of trajectory motion) [1], [2], [14], [15]. One possible way to the problem is to compute appropriate β and $M(\beta)$ in (2). By such computation, asymptotic growth behaviors of trajectories (AGBT) of system (1) can be well described. In system (1), it is the infimum of all possible exponential bounds of trajectories rather than any given β that can describe the fastest speed of trajectory motion or real AGBT. Thus, this technical note focuses on the discussion of the infimum. The infimum is denoted by η and called the essential exponential bound (e-EB) of trajectories starting from E.

η is an intrinsic quantity of system (1) [15]. While there is no exponential bound less than η, each exponential bound can be represented as $\eta + \varepsilon_0$, here $\varepsilon_0 > 0$ is varied for different exponential bound. Furthermore, for any sufficiently small $\varepsilon > 0$, there must exist a growth coefficient $M(\varepsilon) > 0$ ($M(\varepsilon)$ is the function of ε) such that the trajectory motion of system (1) satisfies

$$\|x(k)\| \leq M(\varepsilon) \cdot (\eta + \varepsilon)^k \cdot \|x(0)\|, \quad \forall x(0) \in E, \quad k = 1, 2 \dots. \tag{3}$$

Naturally, there are problems: how η is determined in system (1) and how $M(\varepsilon)$ is changed with $\eta + \varepsilon$ and/or with different vector norms $\|\cdot\|$? Answering these two problems facilitates us to understand clearly the mechanism of how trajectory motion speed (TMS) or AGBT is quantitatively determined in system (1), and possibly is helpful for the computation of η and $M(\varepsilon)$. Thus, this technical note will discuss the e-EB η, $M(\varepsilon)$ and several problems related to η. These problems include:

(P1) What are the factors affecting η and $M(\varepsilon)$ and how η and $M(\varepsilon)$ are determined by them?

(P2) Is there a simple condition to distinguish asymptotic stability from exponential stability of system (1)?

(P3) Whether there are equivalent relationships between stability properties of system (1) *and contraction properties of T?*

(P4) Computation or sharp estimation of η.

The system with exponential stability is more robust to various disturbances than the system with only asymptotic stability. Whenever system (1) is asymptotically stable in E, users usually attempt to know whether or not it is exponentially stable in E. Thus, it is important to study the problem (P2). In linear systems, stability properties are usually equivalent to contraction properties of linear operators. Hence, we try to study whether similar results exist in the system (1) or not [i.e., the problem (P3)]. This will provide a different view to understand different stability properties of system (1).

TMS or AGBT has been deeply studied in linear systems. For example, by the spectral radius of a matrix A and equivalent vector norms in R^n, the e-EB of trajectories of $x(k+1) = A(x(k))$ can be well characterized [2]. By the joint spectral radius of a matrix set and equivalent norms of the matrix set, the e-EB or maximal Lyapunov exponent (namely $\ln \eta$) of trajectories of linear time-varying discrete dynamical systems can be well described [4], [5]. By the pseudo-spectra of the matrix A, the power growth of $\|A^n\|$, which is closely related to the estimation of the growth coefficient, can be quantitatively described [6]. These useful concepts or tools developed in linear cases, however, usually cannot be applied directly in nonlinear system (1).

In past two decades, stability properties of NDDS have been studied widely by Lyapunov functions, see [7]–[13] and references therein. Some researchers studied sufficient conditions for different stability properties [7]–[10], and others studied the converse Lyapunov problem for different nonlinear systems with different stability properties [11]–[13]. These results are important and useful for us to understand stability properties of various NDDS. Differing from the existing results, this technical note tries to provide general understanding and quantitative analyses of TMS or AGBT of the system (1) that might be stable or unstable in E. Some researchers have discussed the estimation of β and/or $M(\beta)$ of different NDDS [1], [2], [14], [21]. However, exact computation or sharp estimation of η

Manuscript received May 28, 2012; revised November 21, 2013; accepted November 22, 2013. Date of publication January 02, 2014; date of current version June 19, 2014. This work was supported in part by the National 973 fund (2013CB329401) and NSFC funds (61375020, 61221003) of China. Recommended by Associate Editor C. Moog. (*Corresponding Author: L. Wang*).

L. Wang is with the Department of Automation, Shanghai Jiao Tong University, and Key Laboratory of System Control and Information Processing, Ministry of Education, Shanghai 200240, China (e-mail: lswang@sjtu.edu.cn).

Z. Xu is with the Institute for Information and System Science, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China (e-mail: zbxu@mail.xjtu.edu.cn).

Digital Object Identifier 10.1109/TAC.2013.2297201

and $M(\varepsilon)$, the mechanism of how TMS of system (1) is quantitatively determined, are seldom studied.

Recently, some attempts have been made to describe or estimate η in some special cases of system (1) [14], [18], [21]. Nonlinear spectral radius and lub Lipschitz constant, introduced in [14] and [21], were used to estimate exponential bounds of the NDDS with Lipschitz continuity. But these values are only upper bounds rather than equal to η. In [15] and [17], we discussed η and $M(\varepsilon)$ in a special case of the system (1) where E is bounded and T Lipschitz continuous. However, methods in [15] and [17] cannot be applied to the NDDS that has no Lipschitz continuity or/and its domain E is unbounded. In [16], we studied the quantitative relation between η and Lyapunov functions of the globally exponentially stable system (1). However, the ideas in [16] cannot be applied to the NDDS that is not globally exponentially stable.

The stability problem is very important. While the linear case is well developed, the nonlinear case is still growing up. So, this technical note tries to develop a general framework to describe quantitatively TMS or AGBT of general nonlinear discrete dynamical system (1). Based on our preliminary results in [17], [18], this technical note introduces a set of equivalent measure functions of $\|\cdot\|$ in E and defines the nonlinear operator modulus for T. By the tools, we provide a theoretical explanation on how η and $M(\varepsilon)$ are quantitatively determined in system (1), study the exact computation and sharp estimation of η, and reveal that equivalence relations really exist between exponential and asymptotic stability of system (1) and different contraction properties of T. We also show that in many cases, the exponential stability and asymptotic stability of system (1) can be distinguished from each other based merely on the information at a single point—the equilibrium point. These results are helpful in understanding TMS or AGBT of system (1). They extend main results in [15]–[18], and present a theoretical basis for an important aspect of discrete-time model development. Such quantitative studies allow for more accurate system analysis and modeling. Researchers in the control engineering field will benefit from the results in this technical note, and potentially be able to use these results for discrete-time system analysis and design.

II. Quantitative Descriptions of TMS

In X, let Ψ_1 denote the set of all equivalent vector norms of $\|\cdot\|$, and Ψ_2 the set of all topologically equivalent metrics of $\|\cdot\|$. Then $\Psi_1 \subset \Psi_2$ and they are two classes of different equivalent measure functions of $\|\cdot\|$. In this section, we will introduce new equivalent measure functions of $\|\cdot\|$ in E.

Suppose that $F(x) : E \subset X \to [0, \infty)$ is a positive definite function. $F(x)$ is called a strongly equivalent function (SEF) of $\|\cdot\|$ in E if there exist two constants $C_2 \ge C_1 > 0$ such that

$$C_1 \cdot \|x\| \le F(x) \le C_2 \cdot \|x\|, \quad \forall x \in E. \tag{4}$$

Here, C_2 and C_1 are called strongly equivalent coefficients between $F(\cdot)$ and $\|\cdot\|$. They describe quantitatively the metric-based equivalent relation between $\|\cdot\|$ and $F(\cdot)$. The infimum of all possible $\frac{C_2}{C_1}$ is called the equivalent ratio between $\|\cdot\|$ and $F(\cdot)$, denoted by $R(\|\cdot\|, F(\cdot))$. $R(\|\cdot\|, F(\cdot))$ may be replaced approximately by a certain choice of $\frac{C_2}{C_1}$.

Let Ψ_3 denote the set of all SEFs of $\|\cdot\|$ in E. It is easy to see that different equivalent norms in Ψ_1 have the same set of SEFs in E. Further, SEFs have the following properties:

(i). $\Psi_1 \subset \Psi_3$, but there are some SEFs that are not vector norms. A metric function is a mapping from $X \times X$ to $[0, \infty)$. Hence, SEFs are a class of new equivalent measure functions of $\|\cdot\|$ in E, differing from the strongly equivalent metrics in [15], topologically equivalent metrics of $\|\cdot\|$ and equivalent vector norms of $\|\cdot\|$. Ψ_3 may be regarded as a special case of the strongly equivalent functional in [18]. But strongly equivalent functional are varied with the equilibrium point x^* of nonlinear systems, and even some equivalent vector norms do not belong to them if $x^* \neq O$.

(ii). Ψ_1, Ψ_2 and Ψ_3 are measure functions determined by $\|\cdot\|$ and E, but independent of nonlinear systems defined on E. Differing from Ψ_1, Ψ_2 and Ψ_3, Lyapunov functions will not exist if system (1) is not stable, and Lyapunov functions of stable system (1) are varied when T represents different systems. Hence, Lyapunov function based methods (such as [16]) actually describe AGBT of system (1) by different sets of functions when T represents different systems. By SEFs, we may describe the TMS of system (1) in a uniform manner, no matter whether the system is stable in E and T represents different systems.

In system (1), $\|T(x)\| \le M(\beta) \cdot \beta \cdot \|x\|$ holds for any $x \in E$. This implies that for any SEF $F(\cdot) \in \Psi_3$, $F(T(x)) \le R(\|\cdot\|, F(\cdot)) \cdot M(\beta) \cdot \beta \cdot F(x)$ holds for all $x \in E$. This means that for each given SEF $F(\cdot) \in \Psi_3$, we can define a functional $L_F(T, O, E) = \sup_{x \neq O, x \in E} \frac{F(T(x))}{F(x)}$. Here, $L_F(T, O, E)$ can be regarded as the nonlinear operator modulus of T. We have $L_{\|\cdot\|}(T, O, E) = \sup_{x \neq O, x \in E} \frac{\|T(x)\|}{\|x\|}$ if $F(\cdot)$ is a norm $\|\cdot\|$. For any positive integers k, m, we have

$$L_F(T^{k+m}, O, E) \le L_F(T^k, O, E) \cdot L_F(T^m, O, E). \tag{5}$$

This implies that the limit $\lim_{k \to \infty} L_F(T^k, O, E)^{\frac{1}{k}}$ exists [16]. Denote

$$Lip(T, O, E) = \lim_{k \to \infty} L_F(T^k, O, E)^{1/k}. \tag{6}$$

According to (4), $Lip(T, O, E)$ is an invariant quantity derived from different SEFs and different equivalent norms of $\|\cdot\|$. This means that for any $\|\cdot\|_* \in \Psi_1$

$$Lip(T, O, E) = \lim_{k \to \infty} L_{\|\cdot\|_*}(T^k, O, E)^{1/k}. \tag{7}$$

To some extents, $L_F(T, O, E)$ and $Lip(T, O, E)$ can be regarded as nonlinear generalizations of the norm and spectral radius of a matrix. Further, we have $Lip(T, O, E) = \inf_{F(\cdot) \in \Psi_3} L_F(T, O, E)$, namely, $Lip(T, O, E)$ is the infimum of all nonlinear operator moduli of T over all SEFs (see Theorem 1). This shows that for any sufficiently small $\varepsilon > 0$, there is a SEF $F_\varepsilon(\cdot)$ such that the subordinated $L_{F_\varepsilon}(T, O, E) \le Lip(T, O, E) + \varepsilon$. Here, $F_\varepsilon(\cdot)$ can be constructed explicitly as (9).

No matter whether system (1) is exponentially stable and E is bounded and T is Lipschitz continuous, we have:

Theorem 1: For system (1), the following quantitative results are stated with respect to the TMS:

(i) The e-EB of trajectories starting from E has the following properties:

$$\eta = Lip(T, O, E) = \inf_{F(\cdot) \in \Psi_3} L_F(T, O, E). \tag{8}$$

(ii) For any sufficiently small $\varepsilon > 0$, the growth coefficient $M(\varepsilon)$ can be regarded approximately as the equivalent ratio between $\|\cdot\|$ and a specific SEF $F_\varepsilon(\cdot)$ whose subordinated $L_{F_\varepsilon}(T, O, E) \le \eta + \varepsilon$. Namely, $M(\varepsilon) = R(\|\cdot\|, F_\varepsilon(\cdot))$. Here, $F_\varepsilon(\cdot)$ can be constructed as that in (9) (in (9), $Lip(T, O, E)$ can be replaced by η).

Proof: We first prove $Lip(T, O, E) = \inf_{F(\cdot) \in \Psi_3} L_F(T, O, E)$. By (5), we have $L_F(T, O, E) \ge Lip(T, O, E)$ for any $F(\cdot) \in \Psi_3$. This implies that $\inf_{F(\cdot) \in \Psi_3} L_F(T, O, E) \ge Lip(T, O, E)$. By (7), for any $\varepsilon > 0$, there is a positive integer N such that $L_{\|\cdot\|}(T^N, O, E) \le (Lip(T, O, E) + \varepsilon)^N$. Denote

$$F_\varepsilon(x) = \sum_{k=1}^{N} (Lip(T, O, E) + \varepsilon)^{k-1} \cdot \left\| T^{N-k}(x) \right\|, \quad \forall x \in E. \tag{9}$$

$F_\varepsilon(\cdot)$ is a SEF of $\|\cdot\|$ in E. For any $x \in E$, we have

$$\left\|T^N(x)\right\| \le L_{\|\cdot\|}(T^N, O, E)\cdot\|x\| \le (Lip(T,O,E)+\varepsilon)^N \cdot \|x\|.$$

Denote $Lip(T,O,E)+\varepsilon$ by L_ε. For any $x \in E$

$$\begin{aligned}F_\varepsilon(T(x)) &= \sum_{k=1}^{N}(Lip(T,O,E)+\varepsilon)^{k-1}\cdot\left\|T^{N-k+1}(x)\right\| \\ &= L_\varepsilon\sum_{k=1}^{N}L_\varepsilon^{k-1}\left\|T^{N-k}(x)\right\| + \left\|T^N(x)\right\| - L_\varepsilon{}^N\|x\| \\ &\le (Lip(T,O,E)+\varepsilon)F_\varepsilon(x).\end{aligned}$$

Thus, with respect to $F_\varepsilon(\cdot)$, we have

$$L_{F_\varepsilon}(T,O,E) = \sup_{x\ne O}\frac{F_\varepsilon(T(x))}{F_\varepsilon(x)} \le Lip(T,O,E)+\varepsilon. \tag{10}$$

Since ε is arbitrary, $\inf_{F(\cdot)\in\Psi_3} L_F(T,O,E) \le Lip(T,O,E)$ holds. Hence, we have $\inf_{F(\cdot)\in\Psi_3} L_F(T,O,E) = Lip(T,O,E)$.

For any given $\varepsilon > 0$, there exists a SEF $F_\varepsilon(\cdot) \in \Psi_3$ such that $L_{F_\varepsilon}(T,O,E) \le Lip(T,O,E)+\varepsilon$. Here, $F_\varepsilon(\cdot)$ can be explicitly constructed as that in (9). Let $C_2(\varepsilon) \ge C_1(\varepsilon) > 0$ represent arbitrary strongly equivalent coefficients between $\|\cdot\|$ and $F_\varepsilon(x)$. Then for any $x(0) \in E$ and any positive integer m, we have

$$\begin{aligned}\|x(m)\| &\le \frac{1}{C_1(\varepsilon)}\cdot F_\varepsilon(x(m)) \le \frac{L_{F_\varepsilon}(T,O,E)^m}{C_1(\varepsilon)}\cdot F_\varepsilon(x(0)) \\ &\le \frac{C_2(\varepsilon)}{C_1(\varepsilon)}\cdot(Lip(T,O,E)+\varepsilon)^m\cdot\|x(0)\|.\end{aligned} \tag{11}$$

Equation (11) holds for arbitrary strongly equivalent coefficients. Hence

$$\|x(m)\| \le R(\|\cdot\|, F_\varepsilon(\cdot))\cdot(Lip(T,O,E)+\varepsilon)^m\cdot\|x(0)\|. \tag{12}$$

Thus, $Lip(T,O,E)+\varepsilon$ is an exponential bound. Since ε can be arbitrary, we have $Lip(T,O,E) \geqslant \eta$.

Suppose that $\beta > 0$ is an arbitrary exponential bound of trajectories starting from E. By (2), there exists a positive constant $M(\beta) > 0$ such that $\left\|T^k x(0)\right\| = \|x(k)\| \le M(\beta)\cdot\beta^k\cdot\|x(0)\|$ holds for any $x(0) \in E$ and any positive integer k. Thus, we have $L_{\|\cdot\|}(T^k, O, E) \le M(\beta)\cdot\beta^k$ for any positive integer k. This implies $Lip(T,O,E) \le \beta$. Since β can be arbitrary exponential bound, we have $Lip(T,O,E) \le \eta$. This implies that $\eta = Lip(T,O,E)$. According to (12), for any $\varepsilon > 0$, $R(\|\cdot\|, F_\varepsilon(\cdot))$ is the growth coefficient corresponding to $\eta+\varepsilon$. The proof is completed. ■

Remark 1:

(i) The e-EB η is the infimum of all nonlinear operator moduli of T over all SEFs. It is an invariant quantity derived from different SEFs or equivalent norms of $\|\cdot\|$ via the formula (6).

(ii) Theorem 1 explains how $M(\varepsilon)$ is changed with $\|\cdot\|$ or ε or T. For example, if ε is changed as $\varepsilon_1 > 0$, then $M(\varepsilon_1) = R(\|\cdot\|, F_{\varepsilon_1}(\cdot))$. If $\|\cdot\|$ is replaced by $\|\cdot\|_*$, then $M(\varepsilon) = R(\|\cdot\|_*, F_\varepsilon(\cdot))$. Here, $\|\cdot\|$ in $F_\varepsilon(\cdot)$ (see (9)) is replaced by $\|\cdot\|_*$ and N might be different. If T is different operator, then $M(\varepsilon) = R(\|\cdot\|, F_\varepsilon(\cdot))$ but η and N in $F_\varepsilon(\cdot)$ are varied with T.

(iii) Based on SEFs, the modulus of T and $Lip(T,O,E)$, some famous results in linear cases can be extended to nonlinear cases.

(iv) For different $\beta > 0$ and positive integer Q, the functions

$$F_{\beta,Q}(x) = \sum_{k=1}^{Q}\beta^{k-1}\left\|T^{Q-k}(x)\right\|, \quad \forall x \in E \tag{13}$$

form a set of SEFs of $\|\cdot\|$ in E. Denote the set by Φ. Then $\Psi_1 \subset \Phi$ and $F_{\beta,1}(x) = \|x\|$ holds for any $x \in E$. Furthermore, we have $\eta = \inf_{F_{\beta,Q}(x)\in\Phi} L_{F_{\beta,Q}}(T,O,E)$.

(v) By [17], we have $\eta \ge \rho(T'(O))$ if T is Frechet differentiable at O. Here, $\rho(T'(O))$ is the spectral radius of $T'(O)$ (the Frechet derivative (or Jacobian matrix in R^n) of T at O).

III. Some Characterizations of Stability Properties

By Theorem 1, system (1) is exponentially stable in E if and only if T is contractive in E with respect to a certain SEF of $\|\cdot\|$ (namely, there is a $F(\cdot) \in \Psi_3$ such that $F(Tx) \le \zeta\cdot F(x)$ holds for a constant $\zeta \in (0,1)$ and any $x \in E$). However, $\sup_{x\ne y}\frac{\|T(x)-T(y)\|}{\|x-y\|} \ge 1$ might hold for any $\|\cdot\| \in \Psi_1$ even if system (1) is exponentially stable in E [15]. Additionally, system (1) might be not exponentially stable in E even if T is contractive in E with respect to a certain topologically equivalent metrics of $\|\cdot\|$ (namely, there is a $d(\cdot,\cdot) \in \Psi_2$ such that $d(Tx,Ty) \le \zeta\cdot d(x,y)$ holds for a constant $\zeta \in (0,1)$ and any $x, y \in E$). Thus, SEFs of $\|\cdot\|$ are equivalent measure functions appropriate for characterizing exponential stability of system (1).

If system (1) is only asymptotically stable rather than exponentially stable in E, then the e-EB $\eta \ge 1$ and T is not contractive in E with respect to any SEF of $\|\cdot\|$. In a special case of system (1) where $X = R^n$, E is an unbounded closed set and T is continuous in E, there is the following equivalent relationship: system (1) is asymptotically stable in E if and only if T is contractive in E with respect to some topologically equivalent metrics of $\|\cdot\|$ (i.e., $\inf_{d(\cdot,\cdot)\in\Psi_2} L_d(T,E) < 1$, here, $L_d(T,E) = \sup_{x\ne y, x,y\in E}\frac{d(T(x),T(y))}{d(x,y)}$). The proof of the statement is as follows:

It is known that system (1) is asymptotically stable in $E \subset R^n$ if and only if it is uniformly asymptotically stable in E (namely, for arbitrarily large $M > 0$ and arbitrarily small $\varepsilon > 0$, there exists a positive integer $Q(M,\varepsilon)$ such that $\|x(k)\| \le \varepsilon$ holds for any $k \ge Q(M,\varepsilon)$ and any $x(0) \in E$ with $\|x(0)\| \le M$) ([3], p. 166). Hence, by Theorem 1.3 in [19] or Theorem 1 in [20], if system (1) is asymptotically stable in E, then T is contractive in E with respect to certain topologically equivalent metric of $\|\cdot\|$. Further, for any sufficiently small $\varepsilon > 0$, there exists a $d_\varepsilon(\cdot,\cdot) \in \Psi_2$ such that $d_\varepsilon(Tx,Ty) \le \varepsilon\cdot d(x,y)$, $forall x, y \in E$ ([19], p. 176). This implies that $\inf_{d(\cdot,\cdot)\in\Psi_2} L_d(T,E) \le \varepsilon$, and for any $x(0) \in E$ and any positive integer k, $d_\varepsilon(x(k),O) \le \varepsilon^k\cdot d_\varepsilon(x(0),O)$ holds. Since ε may be arbitrary, we have $\inf_{d(\cdot,\cdot)\in\Psi_2} L_d(T,E) = 0$. Below, we prove that system (1) is uniformly asymptotically stable in E if T is contractive in E with respect to a certain $d(\cdot,\cdot) \in \Psi_2$. For each pair of positive number M, ε with M arbitrarily large and ε arbitrarily small, denote $B_M = E \cap B(M)$. Here, $B(M) = \{x \in X : \|x\| \le M\}$. Then, B_M is a bounded closed set in R^n, and therefore is a compact set. Since $d(\cdot,\cdot)$ is topologically equivalent to the norm $\|\cdot\|$, B_M is also a compact set in the metric space induced by $d(\cdot,\cdot)$. Denote $diam(B_M) = \sup\{d(x,y) : x, y \in B_M\}$. Then we have $diam(B_M) < \infty$. Since $diam(B_M) < \infty$ and T is contractive in E with respect to $d(\cdot,\cdot)$, it is easy to imply that $T^k(B_M) \to \{O\}$ as $k \to \infty$ in the metric space induced by $d(\cdot,\cdot)$. Thus, there exists a positive integer $Q(M,\varepsilon)$ such that for any integer $k \ge Q(M,\varepsilon)$ and any $x \in B_M$, there holds $T^k(x) \in B(\varepsilon)$, i.e., $\|x(k)\| \le \varepsilon$. Thus, system (1) is uniformly asymptotically stable in E. The proof is completed.

Remark 2: Theorem 1 means that no matter which equivalent norm replaces $\|\cdot\|$ in (2), the exponential bound of trajectories cannot be less than η. However, if system (1) is asymptotically or exponentially stable in the closed $E \subset R^n$, then $\inf_{d(\cdot,\cdot)\in\Psi_2} L_d(T,E) = 0$ holds. It means that for any sufficiently small $\varepsilon \in (0,\eta)$, we can find a $d_\varepsilon(\cdot,\cdot) \in \Psi_2$ such that the exponential bound of trajectories described by $d_\varepsilon(\cdot,\cdot)$ is smaller than ε. This demonstrates that a NDDS might exhibit different TMS or AGBT in Banach spaces and in metric spaces.

IV. Exponential Stability and Asymptotic Stability

Lemma 1: If system (1) is uniformly asymptotically stable in E, then the e-EB of trajectories starting from different bounded subsets (containing O as an interior point) of E is a constant.

Proof: Suppose that $D_1 \subset E$ and $D_2 \subset E$ are two bounded sets containing O as an interior point. Let $W \subset D_1 \cap D_2$ represent a neighborhood of O. Denote the e-EBs of trajectories starting from D_1, D_2, W by η_1, η_2 and η_w, respectively. Then we have $\eta_w \leq \eta_1$ and $\eta_w \leq \eta_2$. Let α represent an exponential bound of trajectories starting from W. By (2), there is $M(\alpha) > 1$ such that $\|x(k)\| \leq M(\alpha) \cdot \alpha^k \cdot \|x(0)\|$ holds for any $x(0) \in W$ and any positive integer k. Since trajectories starting from D_1 converge uniformly to O, there exists a positive integer K such that $T^k x \in W$ for any $x \in D_1$ and any positive integer $k \geq K$. Thus, for any $x(0) \in D_1$ and any positive integer $m \geq K$

$$\|T^m x(0)\| = \|x(m)\| \leq M(\alpha) \cdot \left(\frac{L_{\|\cdot\|}(T, O, E)}{\alpha}\right)^K \cdot \alpha^m \cdot \|x(0)\|. \quad (14)$$

This implies $\eta_1 = \lim_{m\to\infty} L_{\|\cdot\|}(T^m, O, D_1)^{1/m} \leq \alpha$. α can be arbitrary exponential bound, so $\eta_1 \leq \eta_w$. Thus, we have $\eta_1 = \eta_w$. Similarly, $\eta_2 = \eta_w$ can be proved. This implies $\eta_1 = \eta_2$. The proof is completed. ■

The constant in Lemma 1 is called the local essential exponential bound (local e-EB) of trajectories of system (1), denoted by ξ. If T is continuously differentiable in a local neighborhood of O, then by Lemma 1 of [17], we have $\xi = \rho(T'(O))$.

In a special case of system (1) where $X = R^n$, E is bounded and T is continuous in E, by Lemma 1, it is easy to imply that system (1) is exponentially stable in E if and only if system (1) is asymptotically stable in E and locally exponentially stable (namely, exponentially stable in a local neighborhood of O). When E is unbounded, we have:

Theorem 2: In system (1), suppose that $X = R^n$, E is unbounded, T is continuous in E and $\limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} \leq \alpha$ holds for a constant $\alpha \in (0,1)$. Then system (1) is exponentially stable in E if and only if it is asymptotically stable in E and locally exponentially stable. Further, the e-EB $\eta \leq \max\{\alpha, \xi\}$.

Proof: We only need to prove the sufficient condition. Let $\beta \in (0,1)$ denote the exponential bound of trajectories starting from a local neighborhood of O. Denote $L = \max\{\alpha, \beta\}$, $c = 1 - L$. Since $\limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} \leq \alpha < 1$, for any $\varepsilon \in (0, c)$, there exists $b > 0$ such that $\frac{\|T(x)\|}{\|x\|} \leq \alpha + \varepsilon < 1$ holds for any $x \in E$ with $\|x\| \geq b$. Denote $E(2b) = \{x \in E : \|x\| > 2b\}$ and $B(2b) = \{x \in E : \|x\| \leq 2b\}$. Observe that for any $x \in E(2b)$ and any positive integer h, if $T^i(x) \in E(2b)$ hold for all $i = 1, 2, \ldots, h-1$, then we have

$$\left\|T^h(x)\right\| = \left\|T(T^{h-1}(x))\right\| \leq (\alpha+\varepsilon)^h \|x\| \leq (L+\varepsilon)^h \|x\|. \quad (15)$$

In the following we verify the exponential convergence of trajectories starting from $B(2b)$ and $E(2b)$, respectively.

It is known that system (1) is uniformly asymptotically stable in E as well. By Lemma 1, we imply that the e-EB of trajectories starting from $B(2b)$ is not larger than β. This implies that for any positive integer k and any $x(0) \in B(2b)$, there exists $Q > 1$ such that

$$\|x(k)\| \leq Q \cdot (\beta + \varepsilon)^k \cdot \|x(0)\| \leq Q \cdot (L + \varepsilon)^k \cdot \|x(0)\|. \quad (16)$$

Subsequently, we prove that trajectories starting from $E(2b)$ converge exponentially to O. Since system (1) is asymptotically stable in E, we have $\lim_{k\to\infty} T^k(x) = O$ for any $x \in E(2b)$. This implies that for each given $x \in E(2b)$, we can assume that there exists a positive integer m_x such that $T^i(x) \in E(2b)$ for all positive integers $i < m_x$ and $T^{m_x}(x) \in B(2b)$. For each fixed positive integer k, there are two possible cases: either $k > m_x$ or $k \leq m_x$. For any fixed $x \in E(2b)$ and any fixed positive integer k, we consider the two cases respectively below:

(i) If $k > m_x$, then by using (15) and (16), we get

$$\left\|T^k(x)\right\| \leq Q \cdot (L+\varepsilon)^{k-m_x} \cdot (L+\varepsilon)^{m_x} \cdot \|x\| \leq Q \cdot (L+\varepsilon)^k \cdot \|x\|. \quad (17)$$

(ii) If $k \leq m_x$, then $T^i(x) \in E(2b)$ for any positive integer $i \leq k$. Thus, by (15), we have

$$\left\|T^k(x)\right\| \leq (L + \varepsilon)^k \cdot \|x\|. \quad (18)$$

By (17) and (18), for any $x(0) \in E(2b)$ and any positive integer k, we have

$$\|x(k)\| \leq Q \cdot (L + \varepsilon)^k \cdot \|x(0)\|. \quad (19)$$

By (16) and (19), for any $x(0) \in E$ and any positive integer k, we thus have $\|x(k)\| \leq Q \cdot (L + \varepsilon)^k \cdot \|x(0)\|$. Since $L + \varepsilon < 1$, system (1) is exponentially stable in E. Since ε is allowed to be arbitrary and β can be arbitrary exponential bound, by Lemma 1, $\eta \leq= \max\{\alpha, \xi\}$. The proof is completed. ■

Remark 3:

(i) System (1) is locally exponentially stable if there exists a small neighborhood U of O and a constant $\zeta \in (0,1)$ such that $\|Tx\| \leq \zeta \cdot \|x\|$ holds for any $x \in U$. If T is continuously differentiable in a local neighborhood of O, then system (1) is locally exponentially stable if and only if $\rho(T'(O)) < 1$.

(ii) Asymptotic stability and exponential stability of system (1) can be distinguished from each other by a simple condition: local exponential stability of system (1) or $\rho(T'(O)) < 1$.

V. Exponential Bounds of Trajectories

The e-EB η and the local e-EB ξ are intrinsic quantities of exponentially stable system (1). ξ describes the smallest exponential bound of trajectories starting from any bounded set $E \cap B(r)$ of O. Here, $r > 0$ may be arbitrary. η describes the smallest exponential bound of trajectories starting from the whole region E. If E is bounded, then $\eta = \xi$. If E is unbounded, then $\xi \leq \eta$. We have the following results:

(i) For any given sufficiently large $h > 0$ and sufficiently small $\varepsilon > 0$, there exists a $M(h, \varepsilon) > 0$ such that for any $x(0) \in E \cap B(h)$

$$\|x(k)\| \leq M(h, \varepsilon) \cdot (\xi + \varepsilon)^k \cdot \|x(0)\|, \quad k = 0, 1, 2, \ldots. \quad (20)$$

Here, $M(h, \varepsilon)$ might be varied with h.

(ii) For any given $\varepsilon > 0$, there exists a $M(\varepsilon) > 0$ such that for any $x(0) \in E$

$$\|x(k)\| \leq M(\varepsilon) \cdot (\eta + \varepsilon)^k \cdot \|x(0)\|, \quad k = 0, 1, 2, \ldots. \quad (21)$$

Based on Lemma 1 and the proof of Theorem 2, we have:

Corollary 1: Suppose that system (1) is exponentially stable in the unbounded set E. (i) If $\limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} \leq \alpha$ for a constant $\alpha \in (0,1)$, then we have $\xi \leq \eta \leq \max\{\alpha, \xi\}$. (ii) If $\limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} \leq \xi$, then we have $\eta = \xi$.

Remark 4: If T is continuously differentiable in a local neighborhood of O, then $\rho(T'(O)) \leq \eta \leq \max\{\alpha, \rho(T'(O))\}$. Further, if T is bounded in E (namely, $\sup_{x\in E}\|T(x)\| < \infty$) or satisfies $\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|} = 0$, then $\eta = \rho(T'(O))$.

If system (1) is only asymptotically stable rather than exponentially stable in E, then we have $\eta \geq 1$, but do not know whether $\eta = 1$ holds or not. In some special cases of system (1), we have:

Corollary 2: In system (1), suppose that $X = R^n$, T is continuous in E and is continuously differentiable in a local neighborhood of O, system (1) is only asymptotically stable rather than exponentially stable in E. Then we have $\eta = 1$ if E is bounded, or if E is unbounded but $\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|} \leq 1$.

Proof: We only need to prove $\eta \leq 1$ in both cases. Since system (1) is uniformly asymptotically stable in E, we have $\xi = \rho(T'(O))$ and $\rho(T'(O)) \leq 1$ [9]. This implies $\xi \leq 1$. If E is bounded, then by Lemma 1, we have $\eta = \xi \leq 1$. By using the proof similar as one in Theorem 2, we can prove that $\eta \leq 1$ if E is unbounded but $\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|} \leq 1$. The proof is completed. ■

VI. Examples

Example 1: In system (1), if $E = X$ and T is a positive homogeneous operator (namely, $T(\lambda x) = \lambda \cdot T(x)$ for any $x \in X$ and any $\lambda > 0$), then $Lip(T, O, X) = Lip(T, O, B(r))$ for any $r > 0$. Further, $Lip(T, O, X) = \rho(T'(O))$ holds if T is continuously differentiable in a local neighborhood of O.

Example 2: Consider a system in [9] as follows:

$$x(k+1) = T(x(k)), T(x) = \begin{pmatrix} x_1 e^{-x_1^2} + e^{-x_2^2} \\ x_2 e^{-x_2^2} \end{pmatrix}, x(0) \in R^2. \quad (22)$$

The system has the unique equilibrium point O, $\rho(T'(O)) = 1$, and is asymptotically stable rather than exponentially stable in R^2 [9]. Since $\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|} \leq 1$, by Corollary 2, we have $\eta = 1$. Similarly, for the system in Example 2.10 of [9], we can also imply that $\eta = 1$.

Example 3: Consider the discrete-time recurrent neural network

$$x(k+1) = Dx(k) + Pf(Ax(k) + u_1) + u_2, \quad x(0) \in R^n \quad (23)$$

where, $D = diag(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with $0 < |d_i| < 1$, $u_1, u_2 \in R^n$ are two constant external inputs, P and A are two real-valued matrices, $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T$ is the nonlinear activation function and $\limsup_{\|x\|\to\infty}\frac{\|f(x)\|}{\|x\|} = 0$, $f_i(x_i)$ satisfies $\|f_i(x_i) - f_i(0)\| \leq h\|x_i\|$ for a constant $h > 0$ and any $x_i \in R^1$. Equation (23) includes neural network models in [7], [14], [18] and [22], [23] as special cases. According to [3], by changing coordinates, the NDDS whose equilibrium point is not at O can be transformed into one whose equilibrium point is at O. Thus, without loss of generality, we assume that O is the unique equilibrium point of (23). Denote $T(x) = Dx + Pf(Ax + u_1) + u_2$ for any $x \in R^n$. We have $L_{\|\cdot\|}(T, O, R^n) < \infty$ and $\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|} \leq \gamma < 1$. Here, $\gamma = \max\{|d_i| : i = 1, 2, \ldots, n\}$. By Theorems 2 and Corollary 2, we can easily imply the following results:

(a) If the network is globally exponentially stable, then the e-EB $\eta \leq \max\{\gamma, \xi\}$. Further, if the network has small state feedback coefficients in the sense of $\gamma \leq \xi$, then $\eta = \xi$. Here, ξ is the local e-EB (ξ can be replaced by $\rho(T'(O)) = \rho(D + P\cdot f'(u_1)\cdot A)$ if $f(x)$ is continuously differentiable in a local neighborhood of O).

(b) If $f_i(x_i)$ is continuous, then the network is exponentially stable in R^n if and only if it is asymptotically stable in R^n and locally exponentially stable (or $\rho(T'(O)) < 1$ if $f(x)$ is continuously differentiable in a local neighborhood of O).

(c) If $f_i(x_i)$ is continuous, $f(x)$ is continuously differentiable in a local neighborhood of O, and the network is only asymptotically stable rather than exponentially stable in R^n, then $\eta = 1$.

Example 4: Consider a system as follows:

$$x(k+1) = T(x(k)), T(x) = \frac{x}{1+x}, x(0) \neq -1, x(0) \in R^1. \quad (24)$$

The function T is not Lipschitz continuous in R^1, O is the unique equilibrium point and $T'(O) = 1$. For any $x(0) \neq -1$ and any positive integer k, we have $x(k+1) = \frac{x(0)}{1+k\cdot x(0)}$. Let Ω denote the set $\{-\frac{1}{n} : n \in Z^+\}$, where Z^+ represents the set of all positive integers. For any $x(0) \in \Omega$, there exists a positive integer m such that $1 + m\cdot x(0) = 0$, and thus $x(m+1)$ cannot be defined. Hence, in (24), we only need to consider the trajectory motion speed of the trajectories starting from $R^1 - \Omega$.

The system is not locally exponentially stable, and not asymptotically stable in R^1, but uniformly asymptotically stable in $[0, \infty)$. Since $\limsup_{\|x\|\to\infty}\frac{\|T(x)\|}{\|x\|} = 0$ and $|T(x)| \leq |x|$ holds for any $x \in [0, \infty)$, by the proof of Theorem 2, we can imply that the e-EB of the trajectories starting from $(0, \infty)$ is not larger than one (namely, $\eta \leq 1$). Meanwhile, we have $\eta \geq 1$, because the system is not exponentially stable. This implies that the trajectories starting from $(0, \infty)$ converge to O with the e-EB $\eta = 1$.

For any $x(0) \in (-\infty, -1)$, $x(1) \in (1, \infty)$. Hence, the trajectories starting from $(-\infty, -1)$ have the same e-EB as one of the trajectories starting from $(1, \infty)$. In other words, the trajectories starting from $(-\infty, -1)$ also converge to O with the e-EB $\eta = 1$.

For any given sufficiently small $\varepsilon > 0$, there exists a positive integer k_ε such that $1 + k_\varepsilon \cdot x(0) < 0$ holds for any $x(0) \in (-1, -\varepsilon) - \Omega$. This implies that $x(k_\varepsilon + 1) \in (0, \infty)$ for any $x(0) \in (-1, -\varepsilon) - \Omega$. Thus, for any given sufficiently small $\varepsilon \in (0, 1)$, the trajectories starting from $(-1, -\varepsilon) - \Omega$ have the same e-EB as one of the trajectories starting from $(1, \infty)$. Namely, the trajectories starting from $(-1, -\varepsilon) - \Omega$ also converge to O with the e-EB $\eta = 1$. Based on the results above, we can deduce that for any given sufficiently small $\varepsilon > 0$, the trajectories starting from $(0, \infty)$, $(-\infty, -1)$ and $(-1, -\varepsilon) - \Omega$ converge to O with the e-EB $\eta = 1$.

Example 5: Consider the system as follows:

$$x(k+1) = T(x(k)), \quad x(0) \in R^1, \quad T(x) = sign(x). \quad (25)$$

Here, T is not continuous in R^1, the system has three equilibrium points: $x_1^* = -1, x_2^* = 0, x_3^* = 1$, and is not asymptotically stable in R^1. However, x_1^* and x_3^* are exponentially stable in $D_1 = (-\infty, 0)$ and $D_3 = (0, +\infty)$, respectively. In such case, we may analyze how fast the trajectories starting from D_1 (or D_3) converge to x_1^* (or x_3^*). Alternatively, we may also analyze the e-EB of trajectories starting from the whole R^1, with respect to x_1^* or x_3^*. Let $y(k) = x(k) - x_3^*$. Then (25) is transformed into $y(k+1) = G(y(k)), \quad y(0) \in R^1$. Here, $G(y) = sign(y + x_3^*) - x_3^*$, and O is exponentially stable in $(-1, +\infty)$. By Corollary 1, in the transformed system, the trajectories starting from $(-1, +\infty)$ converge to O with the e-EB $\eta = 0$. Thus, in system (25), the trajectories starting from D_1 (or D_3) converge to x_1^* (or x_3^*) with the e-EB $\eta = 0$. For any $x \in R^1$ and any positive integer k, $|T^k(x) - x_1^*| \leq 2|x - x_1^*|$ and $|T^k(x) - x_3^*| \leq 2|x - x_3^*|$ holds. Thus, with respect to x_1^* (or x_3^*), the e-EB of trajectories starting from R^1 equals to 1, and the growth coefficient may be selected as 2. With respect to x_2^*, trajectories starting from R^1 have no bounded e-EB.

Example 6: Consider a system as follows:

$$x(k+1) = T(x(k)), \quad x(0) \in R^1, \quad T(x) = x^2. \quad (26)$$

Here, T is not Lipschitz continuous, the system has two equilibrium points: O and $x = 1$. O is exponentially stably in the region $(-1, 1)$, but $x = 1$ is unstable. The trajectories starting from $(-1, 1)$ will exponentially converge to O with the e-EB $\eta = 0$, but will exponentially diverge from $x = 1$ with the e-EB $\eta = 2$ (Since $\eta \geq T'(1) = 2$ and $|T(x) - 1| \leq 2 \cdot |x - 1|$ for any $x \in (-1, 1)$). For any $M > 0$, trajectories starting from the region $D = (1, M) \cup (-M, -1)$ will diverge from O with the e-EB $\eta = M^2$ (since $|x(k+1)| \leq M^{-1} \cdot (M^2)^k \cdot |x|$ for any positive integer k and any $x \in D$).

VII. Discussions

This technical note discusses the AGBT of system (1) where T might be not Lipschitz continuous (even not continuous), E might be unbounded ($T(E)$ is also unbounded) and the equilibrium point might be unstable. The results in this technical note extend our previous results in [15]–[17] where special cases of system (1) were studied. In [15], *lub Lipschitz constant* $L_{\|\cdot\|}(T, E) = \sup\limits_{x \neq y, x, y \in E} \frac{\|Tx - Ty\|}{\|x - y\|}$ and *Lip constant* $Lip(T, E) = \lim\limits_{n \to \infty} L_{\|\cdot\|}(T^n, E)^{\frac{1}{n}}$ were introduced to describe the exponential stability of a special case of system (1) where E is bounded and T Lipschitz continuous. $L_{\|\cdot\|}(T, E)$ and $Lip(T, E)$ can only be defined for a Lipschitz continuous operator T. Hence, they cannot be used to analyze the AGBT of system (1). Main results in [15] (Theorems 3–4 and partial results of Theorem 2) are special cases of the results in this technical note (where E is assumed to be bounded). In [16], the quantitative relation between the e-EB η and Lyapunov functions of exponentially stable system (1) was studied. However, the method in [16] cannot be used to describe AGBT of system (1) if the system is not exponentially stable. The main results (Theorems 1–2) in [16] are special cases of Theorems 1 and 2 in this technical note. In [17], we discussed a special case of (1) where E is a bounded and closed subset and T continuously differentiable in E (i.e., Lipschitz continuous). Main results in [17] (i.e., Theorems 1–2) are special cases of Theorems 1–2 and Corollary 1 in this technical note.

The activation function $f(x)$ in (23) can be unbounded. Hence, the results in the example 3 improve the main results in [18] where $f(x)$ was assumed to be bounded and continuous. In [7], [22], and [23], sufficient conditions for global asymptotic stability of different neural networks were discussed. By example 3, we can further judge whether these networks are globally exponentially stable or not through computing $\rho(D + P \cdot f'(u_1) \cdot A)$.

For the equivalent relationship between asymptotic stability and uniform asymptotic stability of system (1), we do not know whether the conditions "T is continuous" and "$X = R^n$" are necessary or not. If no, then these conditions can be removed from Theorem 2, Corollary 2 and Example 3. $\limsup\limits_{\|x\| \to \infty} \frac{\|T(x)\|}{\|x\|} \leq \alpha \ (or\ 1)$ is used in the proof of Theorem 2 and Corollaries 1–2, but we do not know whether or not the condition could be removed or replaced by a more weak condition. In the future, we will discuss the related problems.

VIII. Conclusion

This technical note explains the mechanism of how the e-EB of trajectories is quantitatively determined in system (1), and discusses the computation of the e-EB and characterizations of exponential stability and asymptotic stability of system (1). These results extend existing results obtained in the special cases of system (1), and are helpful for quantitative analysis and understanding of AGBT of system (1).

Acknowledgment

The authors wish to thank Dr. C. Moog and the reviewers for their valuable comments and helpful suggestions that have greatly improved the technical note.

References

[1] V. C. Aitken and H. M. Schwartz, "On the exponential stability of discrete-time systems with applications in observer design," *IEEE Trans. Autom. Control*, vol. 39, pp. 1959–1962, 1994.

[2] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. New York: Academic Press, 1970.

[3] S. N. Elaydi, *An Introduction to Difference Equations*. New York: Springer-Verlag, 1995.

[4] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Rev.*, vol. 49, no. 4, pp. 545–592, 2007.

[5] S. E. Tuna, "Growth rate of switched homogeneous systems," *Automatica*, vol. 44, no. 11, pp. 2857–2862, 2008.

[6] T. Ransford, "On pseudospectra and power growth," *SIAM J. Matrix Anal. Appl.*, vol. 29, no. 3, pp. 699–711, 2007.

[7] N. E. Barabanov and D. V. Prokhorov, "A new method for stability analysis of nonlinear discrete-time systems," *IEEE Trans. Autom. Control*, vol. 48, no. 12, pp. 2250–2255, Dec. 2003.

[8] L. F. C. Alberto, T. R. Calliero, and A. C. P. Martins, "An invariance principle for nonlinear discrete autonomous dynamical systems," *IEEE Trans. Autom. Control*, vol. 52, no. 4, pp. 692–697, Apr. 2007.

[9] W. Li and F. Szidarovszky, "An elementary result in the stability theory of time-invariant nonlinear discrete dynamical systems," *Appl. Math. Comput.*, vol. 102, no. 1, pp. 35–49, 1999.

[10] T. Hu and Z. Lin, "Absolute stability analysis of discrete-time systems with composite quadratic Lyapunov function," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 781–797, Jun. 2005.

[11] I. Karafyllis and S. Kotsios, "Necessary and sufficient conditions for robust global asymptotic stabilization of discrete-time systems," *J. Difference Equations Appl.*, vol. 12, no. 7, pp. 741–768, 2006.

[12] Z. P. Jiang and Y. Wang, "A converse Lyapunov theorem for discrete-time systems with disturbances," *Syst. Control Lett.*, vol. 45, pp. 49–58, 2002.

[13] C. M. Kellett and A. R. Teel, "On the robustness of KL-stability for difference inclusions: smooth discrete-time Lyapunov functions," *SIAM J. Control Optim.*, vol. 44, pp. 777–800, 2005.

[14] K. L. Mak, J. G. Peng, Z. Xu, and K. F. C. Yiu, "A new stability criterion for discrete-time neural networks: nonlinear spectral radius," *Chaos, Solitons Fractals*, vol. 31, pp. 424–436, 2007.

[15] L. Wang and Z. Xu, "On characterizations of exponential stability of nonlinear discrete dynamical systems on bounded regions," *IEEE Trans. Autom. Control*, vol. 52, no. 10, pp. 1871–1881, Oct. 2007.

[16] L. Wang, Z. Xu, and Q. Zhao, "Quantitative characterizations of exponential convergent property of nonlinear discrete dynamical systems," *IEEE Trans. Autom. Control*, vol. 52, no. 11, pp. 2129–2134, Nov. 2007.

[17] L. Wang, P. A. Heng, K. S. Leung, and Z. Xu, "Global exponential asymptotic stability in nonlinear discrete dynamic systems," *J. Math. Anal. Appl.*, vol. 258, pp. 349–358, 2001.

[18] L. Wang and Z. Xu, "Sufficient and necessary conditions for global exponential stability of discrete-time recurrent neural networks," *IEEE Trans. Circuits Syst. I*, vol. 53, no. 6, pp. 1373–1380, 2006.

[19] V. I. Opoitsev, "A converse to the principle of contracting maps," *Russian Math. Surveys*, vol. 31, no. 4, pp. 175–204, 1976.

[20] S. Leader, "A topological characterization of Banach contractions," *Pacific J. Math.*, vol. 69, no. 2, pp. 461–466, 1977.

[21] G. Soderlind, "Bounds on nonlinear operators in finite-dimensional Banach spaces," *Numerische Mathematik*, vol. 50, pp. 27–44, 1986.

[22] S. Hu and J. Wang, "Global stability of a class of discrete-time recurrent neural networks," *IEEE Trans. Circuits Syst. I*, vol. 49, no. 8, pp. 1104–1117, 2002.

[23] L. Jin and M. M. Gupta, "Global asymptotic stability of discrete-time analog neural networks," *IEEE Trans. Neural Networks*, vol. 7, no. 6, pp. 1024–1031, 1996.