Derivatives of Key Functions w.r.t. Machine Efficiencies in Two-Machine Geometric Line

Chao-Bo Yan[a,b] **and Lingchen Liu**[a,b]

[a] State Key Laboratory for Manufacturing Systems Engineering,
Xi'an Jiaotong University, Xi'an, Shaanxi 710054, China
[b] School of Automation Science and Engineering,
Faculty of Electronic and Information Engineering,
Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China

1 Analytical Expression of Production Rate

For the two-machine geometric line, the production rate is expressed as [1]

$$PR = e_2[1 - Q(p_1, r_1, p_2, r_2, N)], \tag{1.1}$$

where

$$Q(p_1, r_1, p_2, r_2, N) = \begin{cases} \frac{p_1\beta_2}{(p_1+r_1)(r_1+r_2-r_1r_2)}, & \text{if } N = 1, \\ \frac{p_1\alpha_1\alpha_2\beta_2^2(p_2+r_2)}{A+B+C+D}, & \text{if } N \neq 1, \end{cases} \tag{1.2}$$

and

$$\begin{aligned}
\alpha_1 =& p_1 + p_2 - p_1p_2 - p_2r_1, \\
\alpha_2 =& p_1 + p_2 - p_1p_2 - p_1r_2, \\
\beta_1 =& r_1 + r_2 - r_1r_2 - p_1r_2, \\
\beta_2 =& r_1 + r_2 - r_1r_2 - p_2r_1, \\
\sigma =& \frac{\alpha_2\beta_1}{\alpha_1\beta_2}, \\
A =& p_1r_2\alpha_1\alpha_2\beta_2(p_2 + \beta_2), \\
B =& p_1r_1r_2\alpha_2[\beta_2^2 + p_2(\alpha_1 + \beta_1)(\alpha_2 + 2\beta_2)], \\
C =& \sum_{k=2}^{N-1} p_1p_2r_1r_2(\alpha_2 + \beta_2)^3\sigma^{k-1}, \\
D =& p_2r_1\alpha_1\beta_2[r_2(\alpha_1 + \beta_1) + \alpha_2(p_1 + r_1)]\sigma^{N-1}.
\end{aligned} \tag{1.3}$$

In addition, machine efficiencies are expressed as

$$e_i = \frac{r_i}{p_i + r_i}, \ i = 1, 2. \tag{1.4}$$

Assume p_1 and p_2 are fixed and r_1 and r_2 vary in $(0, 1]$. In the following, some useful derivatives are derived.

2 Partial Derivatives

From (1.4), it follows that

$$r_i = \frac{p_i e_i}{1 - e_i},\ p_i + r_i = \frac{r_i}{e_i} = \frac{p_i}{1 - e_i},\ i = 1, 2, \tag{2.1}$$

and thus,

$$\frac{dr_i}{de_i} = \frac{p_i}{(1 - e_i)^2},\ i = 1, 2. \tag{2.2}$$

In addition, taking the partial derivative of both sides of (1.1) with respect to e_1 and e_2, respectively, we have

$$\begin{aligned} \frac{\partial PR}{\partial e_1} &= -e_2 \frac{\partial Q}{\partial e_1}, \\ \frac{\partial PR}{\partial e_2} &= (1 - Q) - e_2 \frac{\partial Q}{\partial e_2}. \end{aligned} \tag{2.3}$$

Since PR is monotonically increasing in r_1 and r_2 (and thus, e_1 and e_2), respectively [1], we have $\frac{\partial PR}{\partial e_i} > 0$, $i = 1, 2$, which implies that

$$e_2 \frac{\partial Q}{\partial e_1} < 0,\ e_2 \frac{\partial Q}{\partial e_2} - (1 - Q) < 0. \tag{2.4}$$

2.1 Derivatives for $N = 1$

For $N = 1$, the Q-function and the production rate can be simplified as

$$\begin{aligned} Q &= \frac{p_1(r_1 + r_2 - r_1 r_2 - p_2 r_1)}{(p_1 + r_1)(r_1 + r_2 - r_1 r_2)}, \\ PR &= e_1 e_2 \Big(1 + \frac{p_1 p_2}{r_1 + r_2 - r_1 r_2}\Big). \end{aligned} \tag{2.5}$$

Note that the expression of PR can be re-written as

$$\begin{aligned} PR &= e_1 e_2 \Big(1 + \frac{p_1 p_2}{r_1 + r_2 - r_1 r_2}\Big) \\ &= e_1 e_2 \Big(1 + \frac{p_1 p_2}{\frac{p_1 e_1}{1-e_1} + \frac{p_2 e_2}{1-e_2} - \frac{p_1 p_2 e_1 e_2}{(1-e_1)(1-e_2)}}\Big) \\ &= \frac{e_1 e_2 [p_1 e_1 (1 - e_2) + p_2 e_2 (1 - e_1) - p_1 p_2 e_1 e_2 + p_1 p_2 (1 - e_1)(1 - e_2)]}{p_1 e_1 (1 - e_2) + p_2 e_2 (1 - e_1) - p_1 p_2 e_1 e_2}. \end{aligned} \tag{2.6}$$

Let

$$
\begin{aligned}
u =& e_1e_2[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)],\\
v =& p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2,
\end{aligned}
\tag{2.7}
$$

then the production rate can be expressed as $PR = \frac{u}{v}$. As a result,

$$
\begin{aligned}
\frac{\partial u}{\partial e_1} =& e_2[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)]\\
&+e_1e_2[p_1(1-e_2)-p_2e_2-p_1p_2e_2-p_1p_2(1-e_2)],\\
\frac{\partial u}{\partial e_2} =& e_1[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)]\\
&+e_1e_2[-p_1e_1+p_2(1-e_1)-p_1p_2e_1-p_1p_2(1-e_1)],\\
\frac{\partial v}{\partial e_1} =& p_1(1-e_2)-p_2e_2-p_1p_2e_2,\\
\frac{\partial v}{\partial e_2} =& -p_1e_1+p_2(1-e_1)-p_1p_2e_1.
\end{aligned}
\tag{2.8}
$$

Thus, we have

$$
\begin{aligned}
v^2\frac{\partial PR}{\partial e_1} =& \frac{\partial u}{\partial e_1}v-u\frac{\partial v}{\partial e_1}\\
=& \Big\{e_2[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)]\\
&+e_1e_2[p_1(1-e_2)-p_2e_2-p_1p_2e_2-p_1p_2(1-e_2)]\Big\}\\
&\cdot[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]-[p_1(1-e_2)-p_2e_2-p_1p_2e_2]\\
&\cdot e_1e_2[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)]\\
=& e_2[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)]\\
&\cdot[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]-p_1p_2e_1e_2(1-e_2)\\
&\cdot[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]-p_1p_2e_1e_2(1-e_1)(1-e_2)\\
&\cdot[p_1(1-e_2)-p_2e_2-p_1p_2e_2]\\
=& e_2\Big\{p_1p_2^2e_2(1-e_1)(1-e_2)+[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]\\
&\cdot[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1]\Big\}\\
=& e_2\Big\{p_1p_2^2e_2(1-e_1)(1-e_2)+v[v-p_1p_2e_1(1-e_2)]\Big\},
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
v^2\frac{\partial PR}{\partial e_2} =& \frac{\partial u}{\partial e_2}v - u\frac{\partial v}{\partial e_2} \\
=& \Big\{e_1[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)] \\
& + e_1e_2[-p_1e_1+p_2(1-e_1)-p_1p_2e_1-p_1p_2(1-e_1)]\Big\} \\
& \cdot [p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]-[-p_1e_1+p_2(1-e_1)-p_1p_2e_1] \\
& \cdot e_1e_2[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)] \\
=& e_1[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2+p_1p_2(1-e_1)(1-e_2)] \\
& \cdot [p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]-p_1p_2e_1e_2(1-e_1) \\
& \cdot [p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2]-p_1p_2e_1e_2(1-e_1)(1-e_2) \\
& \cdot [-p_1e_1+p_2(1-e_1)-p_1p_2e_1] \\
=& e_1\Big\{p_1^2p_2e_1(1-e_1)(1-e_2)+[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2] \\
& \cdot [p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_2]\Big\} \\
=& e_1\Big\{p_1^2p_2e_1(1-e_1)(1-e_2)+v[v-p_1p_2e_2(1-e_1)]\Big\}.
\end{aligned}
\tag{2.10}
$$

In addition, from (2.6), it follows that

$$
v = p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2 = \frac{p_1p_2e_1e_2(1-e_1)(1-e_2)}{PR-e_1e_2}. \tag{2.11}
$$

Replacing v in (2.9) and (2.10) by the above expression, we have

$$
\begin{aligned}
\frac{\partial PR}{\partial e_1} =& \frac{(PR-e_1e_2)^2+p_1e_1^2(1-e_2)(e_2-PR)}{p_1e_1^2(1-e_1)(1-e_2)}, \\
\frac{\partial PR}{\partial e_2} =& \frac{(PR-e_1e_2)^2+p_2e_2^2(1-e_1)(e_1-PR)}{p_2e_2^2(1-e_1)(1-e_2)},
\end{aligned}
\tag{2.12}
$$

and thus,

$$
\frac{\frac{\partial PR}{\partial e_1}}{\frac{\partial PR}{\partial e_2}} = \frac{p_2e_2^2[(PR-e_1e_2)^2+p_1e_1^2(1-e_2)(e_2-PR)]}{p_1e_1^2[(PR-e_1e_2)^2+p_2e_2^2(1-e_1)(e_1-PR)]}. \tag{2.13}
$$

2.2 First Derivatives for $N > 1$

In the following, the first partial derivatives of α_1, α_2, β_1, β_2, σ, A, B, C, D, and Q with respect to e_1 and e_2 are derived.

The first derivatives of α_1 and α_2 are as follows:

$$\begin{aligned}
\frac{\partial \alpha_1}{\partial e_1} &= \frac{\partial \alpha_1}{\partial r_1}\frac{dr_1}{de_1} = -\frac{p_1 p_2}{(1-e_1)^2},\\
\frac{\partial \alpha_1}{\partial e_2} &= \frac{\partial \alpha_1}{\partial r_2}\frac{dr_2}{de_2} = 0,\\
\frac{\partial \alpha_2}{\partial e_1} &= \frac{\partial \alpha_2}{\partial r_1}\frac{dr_1}{de_1} = 0,\\
\frac{\partial \alpha_2}{\partial e_2} &= \frac{\partial \alpha_2}{\partial r_2}\frac{dr_2}{de_2} = -\frac{p_1 p_2}{(1-e_2)^2}.
\end{aligned} \tag{2.14}$$

The first derivatives of β_1 and β_2 are as follows:

$$\begin{aligned}
\frac{\partial \beta_1}{\partial e_1} &= \frac{\partial \beta_1}{\partial r_1}\frac{dr_1}{de_1} = \frac{p_1}{(1-e_1)^2}\Big(1 - \frac{p_2 e_2}{1-e_2}\Big),\\
\frac{\partial \beta_1}{\partial e_2} &= \frac{\partial \beta_1}{\partial r_2}\frac{dr_2}{de_2} = \frac{p_2}{(1-e_2)^2}\Big(1 - \frac{p_1}{1-e_1}\Big),\\
\frac{\partial \beta_2}{\partial e_1} &= \frac{\partial \beta_2}{\partial r_1}\frac{dr_1}{de_1} = \frac{p_1}{(1-e_1)^2}\Big(1 - \frac{p_2}{1-e_2}\Big),\\
\frac{\partial \beta_2}{\partial e_2} &= \frac{\partial \beta_2}{\partial r_2}\frac{dr_2}{de_2} = \frac{p_2}{(1-e_2)^2}\Big(1 - \frac{p_1 e_1}{1-e_1}\Big).
\end{aligned} \tag{2.15}$$

The first derivatives of σ are as follows:

$$\begin{aligned}
\frac{\partial \sigma}{\partial e_1} &= \frac{1}{\alpha_1\beta_2}\Big(\frac{\partial \alpha_2}{\partial e_1}\beta_1 + \alpha_2\frac{\partial \beta_1}{\partial e_1}\Big) - \frac{\alpha_2\beta_1}{(\alpha_1\beta_2)^2}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\\
&= \frac{1}{\alpha_1\beta_2}\Big[\Big(\frac{\partial \alpha_2}{\partial e_1}\beta_1 + \alpha_2\frac{\partial \beta_1}{\partial e_1}\Big) - \sigma\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big]\\
&= \frac{1}{\alpha_1\beta_2}\Big[\alpha_2\frac{\partial \beta_1}{\partial e_1} - \sigma\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big],\\
\frac{\partial \sigma}{\partial e_2} &= \frac{1}{\alpha_1\beta_2}\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_1 + \alpha_2\frac{\partial \beta_1}{\partial e_2}\Big) - \frac{\alpha_2\beta_1}{(\alpha_1\beta_2)^2}\Big(\frac{\partial \alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_2}\Big)\\
&= \frac{1}{\alpha_1\beta_2}\Big[\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_1 + \alpha_2\frac{\partial \beta_1}{\partial e_2}\Big) - \sigma\Big(\frac{\partial \alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_2}\Big)\Big]\\
&= \frac{1}{\alpha_1\beta_2}\Big[\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_1 + \alpha_2\frac{\partial \beta_1}{\partial e_2}\Big) - \sigma\alpha_1\frac{\partial \beta_2}{\partial e_2}\Big].
\end{aligned} \tag{2.16}$$

The first derivatives of A are as follows:

$$
\begin{aligned}
\frac{\partial A}{\partial e_1} =& p_1 r_2\Big[\frac{\partial \alpha_1}{\partial e_1}\alpha_2\beta_2(p_2+\beta_2) + \alpha_1\frac{\partial \alpha_2}{\partial e_1}\beta_2(p_2+\beta_2) + \alpha_1\alpha_2\frac{\partial \beta_2}{\partial e_1}(p_2+\beta_2) + \alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_1}\Big] \\
=& p_1 r_2\Big[\frac{\partial \alpha_1}{\partial e_1}\alpha_2\beta_2(p_2+\beta_2) + \alpha_1\alpha_2\frac{\partial \beta_2}{\partial e_1}(p_2+2\beta_2)\Big] \\
=& \frac{p_1 p_2 e_2}{1-e_2}\alpha_2\Big[\frac{\partial \alpha_1}{\partial e_1}\beta_2(p_2+\beta_2) + \alpha_1\frac{\partial \beta_2}{\partial e_1}(p_2+2\beta_2)\Big], \\
\frac{\partial A}{\partial e_2} =& p_1 r_2\Big[\frac{\partial \alpha_1}{\partial e_2}\alpha_2\beta_2(p_2+\beta_2) + \alpha_1\frac{\partial \alpha_2}{\partial e_2}\beta_2(p_2+\beta_2) + \alpha_1\alpha_2\frac{\partial \beta_2}{\partial e_2}(p_2+\beta_2) + \alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_2}\Big] \\
&+ p_1\frac{\partial r_2}{\partial e_2}\alpha_1\alpha_2\beta_2(p_2+\beta_2) \\
=& p_1 r_2\Big[\alpha_1\frac{\partial \alpha_2}{\partial e_2}\beta_2(p_2+\beta_2) + \alpha_1\alpha_2\frac{\partial \beta_2}{\partial e_2}(p_2+2\beta_2)\Big] + p_1\frac{\partial r_2}{\partial e_2}\alpha_1\alpha_2\beta_2(p_2+\beta_2) \\
=& \frac{p_1 p_2 e_2}{1-e_2}\alpha_1\Big[\frac{\partial \alpha_2}{\partial e_2}\beta_2(p_2+\beta_2) + \alpha_2\frac{\partial \beta_2}{\partial e_2}(p_2+2\beta_2)\Big] + \frac{p_1 p_2}{(1-e_2)^2}\alpha_1\alpha_2\beta_2(p_2+\beta_2).
\end{aligned}
\tag{2.17}
$$

The first derivatives of B are as follows:

$$\begin{aligned}
\frac{\partial B}{\partial e_1} =& p_1\frac{\partial r_1}{\partial e_1}r_2\alpha_2[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] + p_1r_1r_2\frac{\partial \alpha_2}{\partial e_1}[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)]\\
&+ p_1r_1r_2\alpha_2\Big[2\beta_2\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1}+\frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_1}+2\frac{\partial \beta_2}{\partial e_1}\Big)\Big]\\
=& p_1\frac{\partial r_1}{\partial e_1}r_2\alpha_2[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)]\\
&+ p_1r_1r_2\alpha_2\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1}+\frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2)\Big\}\\
=& \frac{p_1^2p_2e_2\alpha_2}{(1-e_1)^2(1-e_2)}[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)]\\
&+ \frac{p_1^2p_2e_1e_2\alpha_2}{(1-e_1)(1-e_2)}\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1}+\frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2)\Big\},\\
\frac{\partial B}{\partial e_2} =& p_1r_1\frac{\partial r_2}{\partial e_2}\alpha_2[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] + p_1r_1r_2\frac{\partial \alpha_2}{\partial e_2}[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \qquad (2.18)\\
&+ p_1r_1r_2\alpha_2\Big[2\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\Big(\frac{\partial \alpha_1}{\partial e_2}+\frac{\partial \beta_1}{\partial e_2}\Big)(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2}+2\frac{\partial \beta_2}{\partial e_2}\Big)\Big]\\
=& p_1r_1\frac{\partial r_2}{\partial e_2}\alpha_2[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] + p_1r_1r_2\frac{\partial \alpha_2}{\partial e_2}[\beta_2^2 + 2p_2(\alpha_1+\beta_1)(\alpha_2+\beta_2)]\\
&+ p_1r_1r_2\alpha_2\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+2\beta_2)\Big\}\\
=& \frac{p_1^2p_2e_1}{(1-e_1)(1-e_2)^2}\,\alpha_2[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)]\\
&+ \frac{p_1^2p_2e_1e_2}{(1-e_1)(1-e_2)}\frac{\partial \alpha_2}{\partial e_2}[\beta_2^2 + 2p_2(\alpha_1+\beta_1)(\alpha_2+\beta_2)]\\
&+ \frac{p_1^2p_2e_1e_2}{(1-e_1)(1-e_2)}\alpha_2\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+2\beta_2)\Big\}.
\end{aligned}$$

The first derivatives of C are as follows:

$$
\begin{aligned}
\frac{\partial C}{\partial e_1} &= \sum_{k=2}^{N-1} p_1 p_2 r_2 \Big[\frac{dr_1}{de_1}(\alpha_2+\beta_2)^3\sigma^{k-1} + 3r_1(\alpha_2+\beta_2)^2\Big(\frac{\partial \alpha_2}{\partial e_1} + \frac{\partial \beta_2}{\partial e_1}\Big)\sigma^{k-1} \\
&\quad + (k-1)r_1(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial \sigma}{\partial e_1}\Big] \\
&= \sum_{k=2}^{N-1} p_1 p_2 r_2 \frac{dr_1}{de_1}(\alpha_2+\beta_2)^3\sigma^{k-1} + \sum_{k=2}^{N-1} 3p_1p_2r_1r_2(\alpha_2+\beta_2)^2\frac{\partial \beta_2}{\partial e_1}\sigma^{k-1} \\
&\quad + \sum_{k=2}^{N-1}(k-1)p_1p_2r_1r_2(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial \sigma}{\partial e_1} \\
&= \sum_{k=2}^{N-1} \frac{p_1^2p_2^2e_2}{(1-e_1)^2(1-e_2)}(\alpha_2+\beta_2)^3\sigma^{k-1} + \sum_{k=2}^{N-1} \frac{3p_1^2p_2^2e_1e_2}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)^2\frac{\partial \beta_2}{\partial e_1}\sigma^{k-1} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1)p_1^2p_2^2e_1e_2}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial \sigma}{\partial e_1}, \\
\frac{\partial C}{\partial e_2} &= \sum_{k=2}^{N-1} p_1 p_2 r_1 \Big[\frac{dr_2}{de_2}(\alpha_2+\beta_2)^3\sigma^{k-1} + 3r_2(\alpha_2+\beta_2)^2\Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\sigma^{k-1} \\
&\quad + (k-1)r_2(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial \sigma}{\partial e_2}\Big] \\
&= \sum_{k=2}^{N-1} p_1 p_2 r_1 \frac{dr_2}{de_2}(\alpha_2+\beta_2)^3\sigma^{k-1} + \sum_{k=2}^{N-1} 3p_1p_2r_1r_2(\alpha_2+\beta_2)^2\Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\sigma^{k-1} \\
&\quad + \sum_{k=2}^{N-1}(k-1)p_1p_2r_1r_2(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial \sigma}{\partial e_2} \\
&= \sum_{k=2}^{N-1} \frac{p_1^2p_2^2e_1}{(1-e_1)(1-e_2)^2}(\alpha_2+\beta_2)^3\sigma^{k-1} + \sum_{k=2}^{N-1} \frac{3p_1^2p_2^2e_1e_2}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)^2\Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\sigma^{k-1} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1)p_1^2p_2^2e_1e_2}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial \sigma}{\partial e_2}.
\end{aligned}
\tag{2.19}
$$

The first derivatives of D are as follows:

$$
\begin{aligned}
\frac{\partial D}{\partial e_1} =& p_2\Big[\frac{dr_1}{de_1}\alpha_1\beta_2 + r_1\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big][r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-1}\\
&+ p_2r_1\alpha_1\beta_2\Big[r_2\Big(\frac{\partial \alpha_1}{\partial e_1}+\frac{\partial \beta_1}{\partial e_1}\Big)+\frac{\partial \alpha_2}{\partial e_1}(p_1+r_1)+\alpha_2\frac{dr_1}{de_1}\Big]\sigma^{N-1}\\
&+(N-1)p_2r_1\alpha_1\beta_2[r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\\
=& p_2\Big[\frac{dr_1}{de_1}\alpha_1\beta_2 + r_1\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big][r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-1}\\
&+ p_2r_1\alpha_1\beta_2\Big[r_2\Big(\frac{\partial \alpha_1}{\partial e_1}+\frac{\partial \beta_1}{\partial e_1}\Big)+\alpha_2\frac{dr_1}{de_1}\Big]\sigma^{N-1}\\
&+(N-1)p_2r_1\alpha_1\beta_2[r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\\
=& \frac{p_1p_2}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\sigma^{N-1}\\
&+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial \alpha_1}{\partial e_1}+\frac{\partial \beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]\sigma^{N-1}\\
&+\frac{(N-1)p_1p_2e_1\alpha_1\beta_2}{1-e_1}\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\sigma^{N-2}\frac{\partial\sigma}{\partial e_1},\\
\frac{\partial D}{\partial e_2} =& p_2r_1\Big(\frac{\partial \alpha_1}{\partial e_2}\beta_2+\alpha_1\frac{\partial \beta_2}{\partial e_2}\Big)[r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-1}\\
&+p_2r_1\alpha_1\beta_2\Big[\frac{dr_2}{de_2}(\alpha_1+\beta_1)+r_2\Big(\frac{\partial \alpha_1}{\partial e_2}+\frac{\partial \beta_1}{\partial e_2}\Big)+\frac{\partial \alpha_2}{\partial e_2}(p_1+r_1)\Big]\sigma^{N-1}\\
&+(N-1)p_2r_1\alpha_1\beta_2[r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-2}\frac{\partial\sigma}{\partial e_2}\\
=& p_2r_1\alpha_1\frac{\partial \beta_2}{\partial e_2}[r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-1}\\
&+p_2r_1\alpha_1\beta_2\Big[\frac{dr_2}{de_2}(\alpha_1+\beta_1)+r_2\frac{\partial \beta_1}{\partial e_2}+\frac{\partial \alpha_2}{\partial e_2}(p_1+r_1)\Big]\sigma^{N-1}\\
&+(N-1)p_2r_1\alpha_1\beta_2[r_2(\alpha_1+\beta_1)+\alpha_2(p_1+r_1)]\sigma^{N-2}\frac{\partial\sigma}{\partial e_2}\\
=& \frac{p_1p_2e_1\alpha_1}{1-e_1}\frac{\partial \beta_2}{\partial e_2}\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\sigma^{N-1}\\
&+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1)+\frac{p_2e_2}{1-e_2}\frac{\partial \beta_1}{\partial e_2}+\frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big]\sigma^{N-1}\\
&+\frac{(N-1)p_1p_2e_1\alpha_1\beta_2}{1-e_1}\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\sigma^{N-2}\frac{\partial\sigma}{\partial e_2}.
\end{aligned}
\tag{2.20}
$$

The first derivatives of Q are as follows:

$$\begin{aligned}
\frac{\partial Q}{\partial e_1} =&\frac{p_1(p_2+r_2)\Big[(\frac{\partial \alpha_1}{\partial e_1}\alpha_2+\alpha_1\frac{\partial \alpha_2}{\partial e_1})\beta_2^2+2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_1}\Big]}{A+B+C+D}-\frac{p_1\alpha_1\alpha_2\beta_2^2(p_2+r_2)(\frac{\partial A}{\partial e_1}+\frac{\partial B}{\partial e_1}+\frac{\partial C}{\partial e_1}+\frac{\partial D}{\partial e_1})}{(A+B+C+D)^2}\\
=&\frac{p_1(p_2+r_2)(\frac{\partial \alpha_1}{\partial e_1}\alpha_2\beta_2^2+2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_1})}{A+B+C+D}-\frac{p_1\alpha_1\alpha_2\beta_2^2(p_2+r_2)(\frac{\partial A}{\partial e_1}+\frac{\partial B}{\partial e_1}+\frac{\partial C}{\partial e_1}+\frac{\partial D}{\partial e_1})}{(A+B+C+D)^2}\\
=&\frac{p_1p_2\alpha_2\beta_2(\frac{\partial \alpha_1}{\partial e_1}\beta_2+2\alpha_1\frac{\partial \beta_2}{\partial e_1})}{(1-e_2)(A+B+C+D)}-\frac{p_1p_2\alpha_1\alpha_2\beta_2^2(\frac{\partial A}{\partial e_1}+\frac{\partial B}{\partial e_1}+\frac{\partial C}{\partial e_1}+\frac{\partial D}{\partial e_1})}{(1-e_2)(A+B+C+D)^2},\\
\frac{\partial Q}{\partial e_2} =&\frac{p_1\Big[(\frac{\partial \alpha_1}{\partial e_2}\alpha_2+\alpha_1\frac{\partial \alpha_2}{\partial e_2})\beta_2^2+2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_2}\Big](p_2+r_2)+p_1\alpha_1\alpha_2\beta_2^2\frac{dr_2}{de_2}}{A+B+C+D}\\
&-\frac{p_1\alpha_1\alpha_2\beta_2^2(p_2+r_2)(\frac{\partial A}{\partial e_2}+\frac{\partial B}{\partial e_2}+\frac{\partial C}{\partial e_2}+\frac{\partial D}{\partial e_2})}{(A+B+C+D)^2}\\
=&\frac{p_1(\alpha_1\frac{\partial \alpha_2}{\partial e_2}\beta_2^2+2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_2})(p_2+r_2)+p_1\alpha_1\alpha_2\beta_2^2\frac{dr_2}{de_2}}{A+B+C+D}-\frac{p_1\alpha_1\alpha_2\beta_2^2(p_2+r_2)(\frac{\partial A}{\partial e_2}+\frac{\partial B}{\partial e_2}+\frac{\partial C}{\partial e_2}+\frac{\partial D}{\partial e_2})}{(A+B+C+D)^2}\\
=&\frac{p_1p_2\alpha_1\beta_2\Big[(1-e_2)(\frac{\partial \alpha_2}{\partial e_2}\beta_2+2\alpha_2\frac{\partial \beta_2}{\partial e_2})+\alpha_2\beta_2\Big]}{(1-e_2)^2(A+B+C+D)}-\frac{p_1p_2\alpha_1\alpha_2\beta_2^2(\frac{\partial A}{\partial e_2}+\frac{\partial B}{\partial e_2}+\frac{\partial C}{\partial e_2}+\frac{\partial D}{\partial e_2})}{(1-e_2)(A+B+C+D)^2}.
\end{aligned} \tag{2.21}$$

2.3 Second Derivatives for $N > 1$

In the following, the second partial derivatives of $\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma, A, B, C, D$, and Q with respect to e_1 and e_2 are derived.

The second derivatives of α_1 and α_2 are as follows:

$$\begin{aligned}
\frac{\partial^2 \alpha_1}{\partial e_1^2} =&-\frac{2p_1p_2}{(1-e_1)^3},\\
\frac{\partial^2 \alpha_1}{\partial e_1\partial e_2} =&0,\\
\frac{\partial^2 \alpha_1}{\partial e_2^2} =&0,\\
\frac{\partial^2 \alpha_2}{\partial e_1^2} =&0,\\
\frac{\partial^2 \alpha_2}{\partial e_1\partial e_2} =&0,\\
\frac{\partial^2 \alpha_2}{\partial e_2^2} =&-\frac{2p_1p_2}{(1-e_2)^3}.
\end{aligned} \tag{2.22}$$

The second derivatives of β_1 and β_2 are as follows:

$$\begin{aligned}
\frac{\partial^2 \beta_1}{\partial e_1^2} &= \frac{2p_1}{(1-e_1)^3}\Big(1 - \frac{p_2 e_2}{1-e_2}\Big),\\
\frac{\partial^2 \beta_1}{\partial e_1 \partial e_2} &= -\frac{p_1 p_2}{(1-e_1)^2(1-e_2)^2},\\
\frac{\partial^2 \beta_1}{\partial e_2^2} &= \frac{2p_2}{(1-e_2)^3}\Big(1 - \frac{p_1}{1-e_1}\Big),\\
\frac{\partial^2 \beta_2}{\partial e_1^2} &= \frac{2p_1}{(1-e_1)^3}\Big(1 - \frac{p_2}{1-e_2}\Big),\\
\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2} &= -\frac{p_1 p_2}{(1-e_1)^2(1-e_2)^2},\\
\frac{\partial^2 \beta_2}{\partial e_2^2} &= \frac{2p_2}{(1-e_2)^3}\Big(1 - \frac{p_1 e_1}{1-e_1}\Big).
\end{aligned} \tag{2.23}$$

The second derivatives of σ are as follows:

$$
\begin{aligned}
\frac{\partial^2\sigma}{\partial e_1^2} =& \frac{1}{\alpha_1\beta_2}\Big[\frac{\partial\alpha_2}{\partial e_1}\frac{\partial\beta_1}{\partial e_1} + \alpha_2\frac{\partial^2\beta_1}{\partial e_1^2} - \frac{\partial\sigma}{\partial e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big) \\
&- \sigma\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2 + \frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1} + \frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1} + \alpha_1\frac{\partial^2\beta_2}{\partial e_1^2}\Big)\Big] \\
&- \frac{\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}}{\alpha_1^2\beta_2^2}\Big[\alpha_2\frac{\partial\beta_1}{\partial e_1} - \sigma\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big] \\
=& \frac{1}{\alpha_1\beta_2}\Big[\alpha_2\frac{\partial^2\beta_1}{\partial e_1^2} - \frac{\partial\sigma}{\partial e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big) - \sigma\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2 + 2\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1} + \alpha_1\frac{\partial^2\beta_2}{\partial e_1^2}\Big)\Big] \\
&- \frac{1}{\alpha_1^2\beta_2^2}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big[\alpha_2\frac{\partial\beta_1}{\partial e_1} - \sigma\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big], \\
\frac{\partial^2\sigma}{\partial e_1\partial e_2} =& \frac{1}{\alpha_1\beta_2}\Big[\frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_1}{\partial e_1} + \alpha_2\frac{\partial^2\beta_1}{\partial e_1\partial e_2} - \frac{\partial\sigma}{\partial e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big) \\
&- \sigma\Big(\frac{\partial^2\alpha_1}{\partial e_1\partial e_2}\beta_2 + \frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_2} + \frac{\partial\alpha_1}{\partial e_2}\frac{\partial\beta_2}{\partial e_1} + \alpha_1\frac{\partial^2\beta_2}{\partial e_1\partial e_2}\Big)\Big] \\
&- \frac{\frac{\partial\alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_2}}{\alpha_1^2\beta_2^2}\Big[\alpha_2\frac{\partial\beta_1}{\partial e_1} - \sigma\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big] \\
=& \frac{1}{\alpha_1\beta_2}\Big[\frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_1}{\partial e_1} + \alpha_2\frac{\partial^2\beta_1}{\partial e_1\partial e_2} - \frac{\partial\sigma}{\partial e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big) - \sigma\Big(\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_2} + \alpha_1\frac{\partial^2\beta_2}{\partial e_1\partial e_2}\Big)\Big] \\
&- \frac{1}{\alpha_1\beta_2^2}\frac{\partial\beta_2}{\partial e_2}\Big[\alpha_2\frac{\partial\beta_1}{\partial e_1} - \sigma\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big], \\
\frac{\partial^2\sigma}{\partial e_2^2} =& \frac{1}{\alpha_1\beta_2}\Big[\frac{\partial^2\alpha_2}{\partial e_2^2}\beta_1 + \frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_1}{\partial e_2} + \frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_1}{\partial e_2} + \alpha_2\frac{\partial^2\beta_1}{\partial e_2^2} - \frac{\partial\sigma}{\partial e_2}\alpha_1\frac{\partial\beta_2}{\partial e_2} \\
&- \sigma\Big(\frac{\partial\alpha_1}{\partial e_2}\frac{\partial\beta_2}{\partial e_2} + \alpha_1\frac{\partial^2\beta_2}{\partial e_2^2}\Big)\Big] - \frac{\frac{\partial\alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial\beta_2}{\partial e_2}}{\alpha_1^2\beta_2^2}\Big[\Big(\frac{\partial\alpha_2}{\partial e_2}\beta_1 + \alpha_2\frac{\partial\beta_1}{\partial e_2}\Big) - \sigma\alpha_1\frac{\partial\beta_2}{\partial e_2}\Big] \\
=& \frac{1}{\alpha_1\beta_2}\Big(\frac{\partial^2\alpha_2}{\partial e_2^2}\beta_1 + 2\frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_1}{\partial e_2} + \alpha_2\frac{\partial^2\beta_1}{\partial e_2^2} - \frac{\partial\sigma}{\partial e_2}\alpha_1\frac{\partial\beta_2}{\partial e_2} - \sigma\alpha_1\frac{\partial^2\beta_2}{\partial e_2^2}\Big) \\
&- \frac{1}{\alpha_1\beta_2^2}\frac{\partial\beta_2}{\partial e_2}\Big[\Big(\frac{\partial\alpha_2}{\partial e_2}\beta_1 + \alpha_2\frac{\partial\beta_1}{\partial e_2}\Big) - \sigma\alpha_1\frac{\partial\beta_2}{\partial e_2}\Big].
\end{aligned}
\tag{2.24}
$$

The second derivatives of A are as follows:

$$
\begin{aligned}
\frac{\partial^2 A}{\partial e_1^2} =& \frac{p_1p_2e_2}{1-e_2}\frac{\partial\alpha_2}{\partial e_1}\Big[\frac{\partial\alpha_1}{\partial e_1}\beta_2(p_2+\beta_2)+\alpha_1\frac{\partial\beta_2}{\partial e_1}(p_2+2\beta_2)\Big] \\
&+\frac{p_1p_2e_2}{1-e_2}\alpha_2\Big[\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2(p_2+\beta_2)+\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}(p_2+\beta_2)+\frac{\partial\alpha_1}{\partial e_1}\beta_2\frac{\partial\beta_2}{\partial e_1} \\
&+\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}(p_2+2\beta_2)+\alpha_1\frac{\partial^2\beta_2}{\partial e_1^2}(p_2+2\beta_2)+2\alpha_1\frac{\partial\beta_2}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}\Big] \\
=& \frac{p_1p_2e_2}{1-e_2}\alpha_2\Big[\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2(p_2+\beta_2)+\Big(2\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}+\alpha_1\frac{\partial^2\beta_2}{\partial e_1^2}\Big)(p_2+2\beta_2)+2\alpha_1\Big(\frac{\partial\beta_2}{\partial e_1}\Big)^2\Big], \\
\frac{\partial^2 A}{\partial e_1\partial e_2} =& \Big[\frac{p_1p_2}{(1-e_2)^2}\alpha_2+\frac{p_1p_2e_2}{1-e_2}\frac{\partial\alpha_2}{\partial e_2}\Big]\Big[\frac{\partial\alpha_1}{\partial e_1}\beta_2(p_2+\beta_2)+\alpha_1\frac{\partial\beta_2}{\partial e_1}(p_2+2\beta_2)\Big] \\
&+\frac{p_1p_2e_2}{1-e_2}\alpha_2\Big[\frac{\partial^2\alpha_1}{\partial e_1\partial e_2}\beta_2(p_2+\beta_2)+\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_2}(p_2+\beta_2)+\frac{\partial\alpha_1}{\partial e_1}\beta_2\frac{\partial\beta_2}{\partial e_2} \\
&+\frac{\partial\alpha_1}{\partial e_2}\frac{\partial\beta_2}{\partial e_1}(p_2+2\beta_2)+\alpha_1\frac{\partial^2\beta_2}{\partial e_1\partial e_2}(p_2+2\beta_2)+2\alpha_1\frac{\partial\beta_2}{\partial e_1}\frac{\partial\beta_2}{\partial e_2}\Big] \\
=& \frac{p_1p_2}{(1-e_2)^2}\Big[\alpha_2+e_2(1-e_2)\frac{\partial\alpha_2}{\partial e_2}\Big]\Big[\frac{\partial\alpha_1}{\partial e_1}\beta_2(p_2+\beta_2)+\alpha_1\frac{\partial\beta_2}{\partial e_1}(p_2+2\beta_2)\Big] \\
&+\frac{p_1p_2e_2}{1-e_2}\alpha_2\Big[\Big(\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_2}+\alpha_1\frac{\partial^2\beta_2}{\partial e_1\partial e_2}\Big)(p_2+2\beta_2)+2\alpha_1\frac{\partial\beta_2}{\partial e_1}\frac{\partial\beta_2}{\partial e_2}\Big], \\
\frac{\partial^2 A}{\partial e_2^2} =& \Big[\frac{p_1p_2}{(1-e_2)^2}\alpha_1+\frac{p_1p_2e_2}{1-e_2}\frac{\partial\alpha_1}{\partial e_2}\Big]\Big[\frac{\partial\alpha_2}{\partial e_2}\beta_2(p_2+\beta_2)+\alpha_2\frac{\partial\beta_2}{\partial e_2}(p_2+2\beta_2)\Big] \\
&+\frac{p_1p_2e_2}{1-e_2}\alpha_1\Big[\frac{\partial^2\alpha_2}{\partial e_2^2}\beta_2(p_2+\beta_2)+\frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_2}{\partial e_2}(p_2+\beta_2)+\frac{\partial\alpha_2}{\partial e_2}\beta_2\frac{\partial\beta_2}{\partial e_2} \\
&+\frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_2}{\partial e_2}(p_2+2\beta_2)+\alpha_2\frac{\partial^2\beta_2}{\partial e_2^2}(p_2+2\beta_2)+2\alpha_2\frac{\partial\beta_2}{\partial e_2}\frac{\partial\beta_2}{\partial e_2}\Big] \\
&+\frac{2p_1p_2}{(1-e_2)^3}\alpha_1\alpha_2\beta_2(p_2+\beta_2)+\frac{p_1p_2}{(1-e_2)^2}\Big[\Big(\frac{\partial\alpha_1}{\partial e_2}\alpha_2+\alpha_1\frac{\partial\alpha_2}{\partial e_2}\Big)\beta_2(p_2+\beta_2) \\
&+\alpha_1\alpha_2\frac{\partial\beta_2}{\partial e_2}(p_2+\beta_2)+\alpha_1\alpha_2\beta_2\frac{\partial\beta_2}{\partial e_2}\Big] \\
=& \frac{2p_1p_2\alpha_1}{(1-e_2)^2}\Big[\frac{\partial\alpha_2}{\partial e_2}\beta_2(p_2+\beta_2)+\alpha_2\frac{\partial\beta_2}{\partial e_2}(p_2+2\beta_2)\Big]+\frac{p_1p_2e_2}{1-e_2}\alpha_1\Big[\frac{\partial^2\alpha_2}{\partial e_2^2}\beta_2(p_2+\beta_2) \\
&+\Big(2\frac{\partial\alpha_2}{\partial e_2}\frac{\partial\beta_2}{\partial e_2}+\alpha_2\frac{\partial^2\beta_2}{\partial e_2^2}\Big)(p_2+2\beta_2)+2\alpha_2\Big(\frac{\partial\beta_2}{\partial e_2}\Big)^2\Big]+\frac{2p_1p_2}{(1-e_2)^3}\alpha_1\alpha_2\beta_2(p_2+\beta_2).
\end{aligned}
\tag{2.25}
$$

The second derivatives of B are as follows:

$$
\begin{aligned}
\frac{\partial^2 B}{\partial e_1^2} =& \frac{p_1^2 p_2 e_2}{1-e_2}\Big[\frac{2\alpha_2}{(1-e_1)^3} + \frac{1}{(1-e_1)^2}\frac{\partial \alpha_2}{\partial e_1}\Big][\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \\
&+ \frac{p_1^2 p_2 e_2 \alpha_2}{(1-e_1)^2(1-e_2)}\Big[2\beta_2\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_1} + 2\frac{\partial \beta_2}{\partial e_1}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_2}{1-e_2}\Big[\frac{\alpha_2}{(1-e_1)^2} + \frac{e_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_1}\Big]\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2)\Big\} \\
&+ \frac{p_1^2 p_2 e_1 e_2 \alpha_2}{(1-e_1)(1-e_2)}\Big\{2\Big[p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big) + \frac{\partial \beta_2}{\partial e_1}\Big]\frac{\partial \beta_2}{\partial e_1} + 2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial^2 \beta_2}{\partial e_1^2} \\
&+ p_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1^2} + \frac{\partial^2 \beta_1}{\partial e_1^2}\Big)(\alpha_2+2\beta_2) + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)\Big(\frac{\partial \alpha_2}{\partial e_1} + 2\frac{\partial \beta_2}{\partial e_1}\Big)\Big\} \\
=& \frac{2p_1^2 p_2 e_2 \alpha_2}{(1-e_1)^3(1-e_2)}[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \\
&+ \frac{2p_1^2 p_2 e_2 \alpha_2}{(1-e_1)^2(1-e_2)}\Big[2\beta_2\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2) + 2p_2(\alpha_1+\beta_1)\frac{\partial \beta_2}{\partial e_1}\Big] \\
&+ \frac{p_1^2 p_2 e_1 e_2 \alpha_2}{(1-e_1)(1-e_2)}\Big\{2\Big[2p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big) + \frac{\partial \beta_2}{\partial e_1}\Big]\frac{\partial \beta_2}{\partial e_1} + 2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial^2 \beta_2}{\partial e_1^2} \\
&+ p_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1^2} + \frac{\partial^2 \beta_1}{\partial e_1^2}\Big)(\alpha_2+2\beta_2)\Big\}, \\
\frac{\partial^2 B}{\partial e_1 \partial e_2} =& \frac{p_1^2 p_2}{(1-e_1)^2}\Big[\frac{\alpha_2}{(1-e_2)^2} + \frac{e_2}{1-e_2}\frac{\partial \alpha_2}{\partial e_2}\Big][\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \\
&+ \frac{p_1^2 p_2 e_2 \alpha_2}{(1-e_1)^2(1-e_2)}\Big[2\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big)(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{\alpha_2}{(1-e_2)^2} + \frac{e_2}{1-e_2}\frac{\partial \alpha_2}{\partial e_2}\Big]\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2)\Big\} \\
&+ \frac{p_1^2 p_2 e_1 e_2 \alpha_2}{(1-e_1)(1-e_2)}\Big\{2\Big[p_2\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big) + \frac{\partial \beta_2}{\partial e_2}\Big]\frac{\partial \beta_2}{\partial e_1} + 2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2} \\
&+ p_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1 \partial e_2} + \frac{\partial^2 \beta_1}{\partial e_1 \partial e_2}\Big)(\alpha_2+2\beta_2) + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big\} \\
=& \frac{p_1^2 p_2}{(1-e_1)^2}\Big[\frac{\alpha_2}{(1-e_2)^2} + \frac{e_2}{1-e_2}\frac{\partial \alpha_2}{\partial e_2}\Big][\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \\
&+ \frac{p_1^2 p_2 e_2 \alpha_2}{(1-e_1)^2(1-e_2)}\Big[2\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{\alpha_2}{(1-e_2)^2} + \frac{e_2}{1-e_2}\frac{\partial \alpha_2}{\partial e_2}\Big]\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_1} + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)(\alpha_2+2\beta_2)\Big\} \\
&+ \frac{p_1^2 p_2 e_1 e_2 \alpha_2}{(1-e_1)(1-e_2)}\Big\{2\Big(p_2\frac{\partial \beta_1}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\frac{\partial \beta_2}{\partial e_1} + 2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2} \\
&+ p_2\frac{\partial^2 \beta_1}{\partial e_1 \partial e_2}(\alpha_2+2\beta_2) + p_2\Big(\frac{\partial \alpha_1}{\partial e_1} + \frac{\partial \beta_1}{\partial e_1}\Big)\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big\},
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
\frac{\partial^2 B}{\partial e_2^2} =& \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{2\alpha_2}{(1-e_2)^3} + \frac{1}{(1-e_2)^2}\frac{\partial \alpha_2}{\partial e_2}\Big][\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \\
&+ \frac{p_1^2 p_2 e_1 \alpha_2}{(1-e_1)(1-e_2)^2}\Big[2\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big)(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{1}{(1-e_2)^2}\frac{\partial \alpha_2}{\partial e_2} + \frac{e_2}{1-e_2}\frac{\partial^2 \alpha_2}{\partial e_2^2}\Big][\beta_2^2 + 2p_2(\alpha_1+\beta_1)(\alpha_2+\beta_2)] \\
&+ \frac{2p_1^2 p_2 e_1 e_2}{(1-e_1)(1-e_2)}\frac{\partial \alpha_2}{\partial e_2}\Big[\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big)(\alpha_2+\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{\alpha_2}{(1-e_2)^2} + \frac{e_2}{1-e_2}\frac{\partial \alpha_2}{\partial e_2}\Big]\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+2\beta_2)\Big\} \\
&+ \frac{p_1^2 p_2 e_1 e_2 \alpha_2}{(1-e_1)(1-e_2)}\Big\{2\Big[p_2\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big) + \frac{\partial \beta_2}{\partial e_2}\Big]\frac{\partial \beta_2}{\partial e_2} + 2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial^2 \beta_2}{\partial e_2^2} \\
&+ p_2\frac{\partial^2 \beta_1}{\partial e_2^2}(\alpha_2+2\beta_2) + p_2\frac{\partial \beta_1}{\partial e_2}\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big\} \\
=& \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{2\alpha_2}{(1-e_2)^3} + \frac{1}{(1-e_2)^2}\frac{\partial \alpha_2}{\partial e_2}\Big][\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)] \\
&+ \frac{p_1^2 p_2 e_1 \alpha_2}{(1-e_1)(1-e_2)^2}\Big[2\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+2\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{1}{(1-e_2)^2}\frac{\partial \alpha_2}{\partial e_2} + \frac{e_2}{1-e_2}\frac{\partial^2 \alpha_2}{\partial e_2^2}\Big][\beta_2^2 + 2p_2(\alpha_1+\beta_1)(\alpha_2+\beta_2)] \\
&+ \frac{2p_1^2 p_2 e_1 e_2}{(1-e_1)(1-e_2)}\frac{\partial \alpha_2}{\partial e_2}\Big[\beta_2\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+\beta_2) + p_2(\alpha_1+\beta_1)\Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\Big] \\
&+ \frac{p_1^2 p_2 e_1}{1-e_1}\Big[\frac{\alpha_2}{(1-e_2)^2} + \frac{e_2}{1-e_2}\frac{\partial \alpha_2}{\partial e_2}\Big]\Big\{2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial \beta_2}{\partial e_2} + p_2\frac{\partial \beta_1}{\partial e_2}(\alpha_2+2\beta_2)\Big\} \\
&+ \frac{p_1^2 p_2 e_1 e_2 \alpha_2}{(1-e_1)(1-e_2)}\Big\{2\Big(p_2\frac{\partial \beta_1}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2}\Big)\frac{\partial \beta_2}{\partial e_2} + 2[p_2(\alpha_1+\beta_1)+\beta_2]\frac{\partial^2 \beta_2}{\partial e_2^2} \\
&+ p_2\frac{\partial^2 \beta_1}{\partial e_2^2}(\alpha_2+2\beta_2) + p_2\frac{\partial \beta_1}{\partial e_2}\Big(\frac{\partial \alpha_2}{\partial e_2} + 2\frac{\partial \beta_2}{\partial e_2}\Big)\Big\}.
\end{aligned}
\tag{2.27}
$$

The second derivatives of C are as follows:

$$
\begin{aligned}
\frac{\partial^2 C}{\partial e_1^2} &= \sum_{k=2}^{N-1} \frac{p_1^2 p_2^2 e_2}{1-e_2} \Big\{ \frac{2}{(1-e_1)^3}(\alpha_2+\beta_2)^3 \sigma^{k-1} \\
&\quad + \frac{1}{(1-e_1)^2}\Big[3(\alpha_2+\beta_2)^2 \Big(\frac{\partial \alpha_2}{\partial e_1} + \frac{\partial \beta_2}{\partial e_1}\Big)\sigma^{k-1} + (\alpha_2+\beta_2)^3 (k-1)\sigma^{k-2}\frac{\partial \sigma}{\partial e_1}\Big]\Big\} \\
&\quad + \sum_{k=2}^{N-1} \frac{3p_1^2 p_2^2 e_2}{1-e_2} \Big\{ \frac{1}{(1-e_1)^2}(\alpha_2+\beta_2)^2 \frac{\partial \beta_2}{\partial e_1}\sigma^{k-1} + \frac{e_1}{1-e_1}\Big[2(\alpha_2+\beta_2)\Big(\frac{\partial \alpha_2}{\partial e_1} + \frac{\partial \beta_2}{\partial e_1}\Big)\frac{\partial \beta_2}{\partial e_1}\sigma^{k-1} \\
&\quad + (\alpha_2+\beta_2)^2 \frac{\partial^2 \beta_2}{\partial e_1^2}\sigma^{k-1} + (\alpha_2+\beta_2)^2 \frac{\partial \beta_2}{\partial e_1}(k-1)\sigma^{k-2}\frac{\partial \sigma}{\partial e_1}\Big]\Big\} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1)p_1^2 p_2^2 e_2}{1-e_2} \Big\{ \frac{1}{(1-e_1)^2}(\alpha_2+\beta_2)^3 \sigma^{k-2}\frac{\partial \sigma}{\partial e_1} + \frac{e_1}{1-e_1} \\
&\quad \cdot \Big[3(\alpha_2+\beta_2)^2 \Big(\frac{\partial \alpha_2}{\partial e_1} + \frac{\partial \beta_2}{\partial e_1}\Big)\sigma^{k-2}\frac{\partial \sigma}{\partial e_1} + (\alpha_2+\beta_2)^3 (k-2)\sigma^{k-3}\Big(\frac{\partial \sigma}{\partial e_1}\Big)^2 + (\alpha_2+\beta_2)^3 \sigma^{k-2}\frac{\partial^2 \sigma}{\partial e_1^2}\Big]\Big\} \\
&= \sum_{k=2}^{N-1} \frac{p_1^2 p_2^2 e_2}{(1-e_1)^2(1-e_2)}(\alpha_2+\beta_2)^2 \sigma^{k-2}\Big[\frac{2}{1-e_1}(\alpha_2+\beta_2)\sigma + 3\frac{\partial \beta_2}{\partial e_1}\sigma + (k-1)(\alpha_2+\beta_2)\frac{\partial \sigma}{\partial e_1}\Big] \\
&\quad + \sum_{k=2}^{N-1} \frac{3p_1^2 p_2^2 e_2}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)\sigma^{k-2}\Big\{ \frac{1}{1-e_1}(\alpha_2+\beta_2)\frac{\partial \beta_2}{\partial e_1}\sigma + 2e_1\Big(\frac{\partial \beta_2}{\partial e_1}\Big)^2\sigma \\
&\quad + e_1(\alpha_2+\beta_2)\Big[\frac{\partial^2 \beta_2}{\partial e_1^2}\sigma + (k-1)\frac{\partial \beta_2}{\partial e_1}\frac{\partial \sigma}{\partial e_1}\Big]\Big\} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1)p_1^2 p_2^2 e_2}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)^2 \sigma^{k-3}\Big\{ \frac{1}{1-e_1}(\alpha_2+\beta_2)\sigma\frac{\partial \sigma}{\partial e_1} + 3e_1\frac{\partial \beta_2}{\partial e_1}\sigma\frac{\partial \sigma}{\partial e_1} \\
&\quad + e_1(\alpha_2+\beta_2)\Big[(k-2)\Big(\frac{\partial \sigma}{\partial e_1}\Big)^2 + \sigma\frac{\partial^2 \sigma}{\partial e_1^2}\Big]\Big\},
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
\frac{\partial^2 C}{\partial e_1 \partial e_2} &= \sum_{k=2}^{N-1} \frac{p_1^2 p_2^2}{(1-e_1)^2} \bigg\{ \frac{1}{(1-e_2)^2} (\alpha_2 + \beta_2)^3 \sigma^{k-1} \\
&\quad + \frac{e_2}{1-e_2} \Big[3(\alpha_2+\beta_2)^2 \Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2} \Big) \sigma^{k-1} + (\alpha_2+\beta_2)^3 (k-1) \sigma^{k-2} \frac{\partial \sigma}{\partial e_2} \Big] \bigg\} \\
&\quad + \sum_{k=2}^{N-1} \frac{3 p_1^2 p_2^2 e_1}{1-e_1} \bigg\{ \frac{1}{(1-e_2)^2} (\alpha_2+\beta_2)^2 \frac{\partial \beta_2}{\partial e_1} \sigma^{k-1} + \frac{e_2}{1-e_2} \Big[2(\alpha_2+\beta_2) \Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2} \Big) \frac{\partial \beta_2}{\partial e_1} \sigma^{k-1} \\
&\quad + (\alpha_2+\beta_2)^2 \frac{\partial^2 \beta_2}{\partial e_1 \partial e_2} \sigma^{k-1} + (\alpha_2+\beta_2)^2 \frac{\partial \beta_2}{\partial e_1} (k-1) \sigma^{k-2} \frac{\partial \sigma}{\partial e_2} \Big] \bigg\} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1) p_1^2 p_2^2 e_1}{1-e_1} \bigg\{ \frac{1}{(1-e_2)^2} (\alpha_2+\beta_2)^3 \sigma^{k-2} \frac{\partial \sigma}{\partial e_1} + \frac{e_2}{1-e_2} \Big[3(\alpha_2+\beta_2)^2 \Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2} \Big) \sigma^{k-2} \frac{\partial \sigma}{\partial e_1} \\
&\quad + (\alpha_2+\beta_2)^3 (k-2) \sigma^{k-3} \frac{\partial \sigma}{\partial e_2} \frac{\partial \sigma}{\partial e_1} + (\alpha_2+\beta_2)^3 \sigma^{k-2} \frac{\partial^2 \sigma}{\partial e_1 \partial e_2} \Big] \bigg\} \\
&= \sum_{k=2}^{N-1} \frac{p_1^2 p_2^2}{(1-e_1)^2 (1-e_2)} (\alpha_2+\beta_2)^2 \sigma^{k-2} \Big[\frac{1}{1-e_2} (\alpha_2+\beta_2) \sigma \\
&\quad + 3 e_2 \Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2} \Big) \sigma + (k-1) e_2 (\alpha_2+\beta_2) \frac{\partial \sigma}{\partial e_2} \Big] \\
&\quad + \sum_{k=2}^{N-1} \frac{3 p_1^2 p_2^2 e_1}{(1-e_1)(1-e_2)} (\alpha_2+\beta_2) \sigma^{k-2} \bigg\{ \frac{1}{1-e_2} (\alpha_2+\beta_2) \frac{\partial \beta_2}{\partial e_1} \sigma + 2 e_2 \Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2} \Big) \frac{\partial \beta_2}{\partial e_1} \sigma \\
&\quad + e_2 (\alpha_2+\beta_2) \Big[\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2} \sigma + (k-1) \frac{\partial \beta_2}{\partial e_1} \frac{\partial \sigma}{\partial e_2} \Big] \bigg\} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1) p_1^2 p_2^2 e_1}{(1-e_1)(1-e_2)} (\alpha_2+\beta_2)^2 \sigma^{k-3} \bigg\{ \frac{1}{1-e_2} (\alpha_2+\beta_2) \sigma \frac{\partial \sigma}{\partial e_1} + 3 e_2 \Big(\frac{\partial \alpha_2}{\partial e_2} + \frac{\partial \beta_2}{\partial e_2} \Big) \sigma \frac{\partial \sigma}{\partial e_1} \\
&\quad + e_2 (\alpha_2+\beta_2) \Big[(k-2) \frac{\partial \sigma}{\partial e_1} \frac{\partial \sigma}{\partial e_2} + \sigma \frac{\partial^2 \sigma}{\partial e_1 \partial e_2} \Big] \bigg\},
\end{aligned}
\tag{2.29}
$$

$$
\begin{aligned}
\frac{\partial^2 C}{\partial e_2^2} &= \sum_{k=2}^{N-1} \frac{p_1^2 p_2^2 e_1}{1-e_1}\bigg\{\frac{2}{(1-e_2)^3}(\alpha_2+\beta_2)^3\sigma^{k-1} \\
&\quad + \frac{1}{(1-e_2)^2}\bigg[3(\alpha_2+\beta_2)^2\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\sigma^{k-1} + (\alpha_2+\beta_2)^3(k-1)\sigma^{k-2}\frac{\partial\sigma}{\partial e_2}\bigg]\bigg\} \\
&\quad + \sum_{k=2}^{N-1} \frac{3p_1^2 p_2^2 e_1}{1-e_1}\bigg\{\frac{1}{(1-e_2)^2}(\alpha_2+\beta_2)^2\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\sigma^{k-1} + \frac{e_2}{1-e_2}\bigg[2(\alpha_2+\beta_2)\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)^2\sigma^{k-1} \\
&\quad + (\alpha_2+\beta_2)^2\Big(\frac{\partial^2\alpha_2}{\partial e_2^2}+\frac{\partial^2\beta_2}{\partial e_2^2}\Big)\sigma^{k-1} + (\alpha_2+\beta_2)^2\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)(k-1)\sigma^{k-2}\frac{\partial\sigma}{\partial e_2}\bigg]\bigg\} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1)p_1^2 p_2^2 e_1}{1-e_1}\bigg\{\frac{1}{(1-e_2)^2}(\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial\sigma}{\partial e_2} + \frac{e_2}{1-e_2}\bigg[3(\alpha_2+\beta_2)^2\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\sigma^{k-2}\frac{\partial\sigma}{\partial e_2} \\
&\quad + (\alpha_2+\beta_2)^3(k-2)\sigma^{k-3}\Big(\frac{\partial\sigma}{\partial e_2}\Big)^2 + (\alpha_2+\beta_2)^3\sigma^{k-2}\frac{\partial^2\sigma}{\partial e_2^2}\bigg]\bigg\} \\
&= \sum_{k=2}^{N-1} \frac{p_1^2 p_2^2 e_1}{(1-e_1)(1-e_2)^2}(\alpha_2+\beta_2)^2\sigma^{k-2}\bigg[\frac{2(\alpha_2+\beta_2)\sigma}{1-e_2} + 3\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\sigma + (k-1)(\alpha_2+\beta_2)\frac{\partial\sigma}{\partial e_2}\bigg] \\
&\quad + \sum_{k=2}^{N-1} \frac{3p_1^2 p_2^2 e_1}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)\sigma^{k-2}\bigg\{\frac{1}{1-e_2}(\alpha_2+\beta_2)\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\sigma \\
&\quad + 2e_2\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)^2\sigma + e_2(\alpha_2+\beta_2)\bigg[\Big(\frac{\partial^2\alpha_2}{\partial e_2^2}+\frac{\partial^2\beta_2}{\partial e_2^2}\Big)\sigma + (k-1)\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\frac{\partial\sigma}{\partial e_2}\bigg]\bigg\} \\
&\quad + \sum_{k=2}^{N-1} \frac{(k-1)p_1^2 p_2^2 e_1}{(1-e_1)(1-e_2)}(\alpha_2+\beta_2)^2\sigma^{k-3}\bigg\{\frac{1}{1-e_2}(\alpha_2+\beta_2)\sigma\frac{\partial\sigma}{\partial e_2} \\
&\quad + 3e_2\Big(\frac{\partial\alpha_2}{\partial e_2}+\frac{\partial\beta_2}{\partial e_2}\Big)\sigma\frac{\partial\sigma}{\partial e_2} + e_2(\alpha_2+\beta_2)\bigg[(k-2)\Big(\frac{\partial\sigma}{\partial e_2}\Big)^2 + \sigma\frac{\partial^2\sigma}{\partial e_2^2}\bigg]\bigg\}.
\end{aligned}
\tag{2.30}
$$

The second derivatives of D are as follows:

$$
\begin{aligned}
\frac{\partial^2 D}{\partial e_1^2} =& p_1p_2\Big[\frac{\sigma^{N-1}}{(1-e_1)^2}+\frac{(N-1)\sigma^{N-2}}{1-e_1}\frac{\partial\sigma}{\partial e_1}\Big]\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{p_1p_2\sigma^{N-1}}{1-e_1}\Big[\frac{\alpha_1\beta_2}{(1-e_1)^2}+\frac{1}{1-e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)+\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\\
&+e_1\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2+\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}+\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}+\alpha_1\frac{\partial\beta_2^2}{\partial e_1^2}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{p_1p_2\sigma^{N-1}}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}+\frac{p_1}{1-e_1}\frac{\partial\alpha_2}{\partial e_1}\Big]\\
&+p_1p_2\Big[\frac{\alpha_1\beta_2}{(1-e_1)^2}\sigma^{N-1}+\frac{e_1}{1-e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\sigma^{N-1}+\frac{e_1\alpha_1\beta_2}{1-e_1}(N-1)\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}+\frac{\partial^2\beta_1}{\partial e_1^2}\Big)+\frac{2p_1\alpha_2}{(1-e_1)^3}\\
&+\frac{p_1}{(1-e_1)^2}\frac{\partial\alpha_2}{\partial e_1}\Big]+(N-1)p_1p_2\Big[\frac{\alpha_1\beta_2}{(1-e_1)^2}\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}+\frac{e_1}{1-e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\\
&+\frac{e_1\alpha_1\beta_2}{1-e_1}(N-2)\sigma^{N-3}\Big(\frac{\partial\sigma}{\partial e_1}\Big)^2+\frac{e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-2}\frac{\partial^2\sigma}{\partial e_1^2}\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{(N-1)p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}+\frac{p_1}{1-e_1}\frac{\partial\alpha_2}{\partial e_1}\Big]\\
=&\frac{p_1p_2\sigma^{N-2}}{1-e_1}\Big[\frac{\sigma}{1-e_1}+(N-1)\frac{\partial\sigma}{\partial e_1}\Big]\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{p_1p_2\sigma^{N-1}}{1-e_1}\Big[\frac{\alpha_1\beta_2}{(1-e_1)^2}+\frac{2-e_1}{1-e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)+e_1\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2+2\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}+\alpha_1\frac{\partial\beta_2^2}{\partial e_1^2}\Big)\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]+\frac{p_1p_2\sigma^{N-1}}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)\\
&+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]+\frac{p_1p_2\sigma^{N-2}}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}\sigma+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\sigma+(N-1)e_1\alpha_1\beta_2\frac{\partial\sigma}{\partial e_1}\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}+\frac{\partial^2\beta_1}{\partial e_1^2}\Big)+\frac{2p_1\alpha_2}{(1-e_1)^3}\Big]\\
&+\frac{(N-1)p_1p_2}{1-e_1}\sigma^{N-3}\Big[\frac{\alpha_1\beta_2}{1-e_1}\sigma\frac{\partial\sigma}{\partial e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\sigma\frac{\partial\sigma}{\partial e_1}\\
&+(N-2)e_1\alpha_1\beta_2\Big(\frac{\partial\sigma}{\partial e_1}\Big)^2+e_1\alpha_1\beta_2\sigma\frac{\partial^2\sigma}{\partial e_1^2}\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{(N-1)p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]\\
=&\frac{p_1p_2\sigma^{N-1}}{1-e_1}\Big[\frac{2\alpha_1\beta_2}{(1-e_1)^2}+\frac{2}{1-e_1}\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)+e_1\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}\beta_2+2\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_1}+\alpha_1\frac{\partial\beta_2^2}{\partial e_1^2}\Big)\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]+\frac{2p_1p_2\sigma^{N-2}}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}\sigma+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\sigma+(N-1)e_1\alpha_1\beta_2\frac{\partial\sigma}{\partial e_1}\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial^2\alpha_1}{\partial e_1^2}+\frac{\partial^2\beta_1}{\partial e_1^2}\Big)+\frac{2p_1\alpha_2}{(1-e_1)^3}\Big]\\
&+\frac{(N-1)p_1p_2}{1-e_1}\sigma^{N-3}\Big[\frac{2\alpha_1\beta_2}{1-e_1}\sigma\frac{\partial\sigma}{\partial e_1}+2e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\sigma\frac{\partial\sigma}{\partial e_1}+(N-2)e_1\alpha_1\beta_2\Big(\frac{\partial\sigma}{\partial e_1}\Big)^2\\
&+e_1\alpha_1\beta_2\sigma\frac{\partial^2\sigma}{\partial e_1^2}\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big],
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
\frac{\partial^2 D}{\partial e_1\partial e_2} =&\frac{p_1p_2}{1-e_1}\Big[\frac{1}{1-e_1}\Big(\frac{\partial\alpha_1}{\partial e_2}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_2}\Big)+e_1\Big(\frac{\partial^2\alpha_1}{\partial e_1\partial e_2}\beta_2+\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_2}+\frac{\partial\alpha_1}{\partial e_2}\frac{\partial\beta_2}{\partial e_1}+\alpha_1\frac{\partial^2\beta_2}{\partial e_1\partial e_2}\Big)\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\sigma^{N-1}+\frac{p_1p_2}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\\
&\cdot\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1)+\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_2}+\frac{\partial\beta_1}{\partial e_2}\Big)+\frac{p_1}{1-e_1}\frac{\partial\alpha_2}{\partial e_2}\Big]\sigma^{N-1}\\
&+\frac{p_1p_2}{1-e_1}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big](N-1)\sigma^{N-2}\frac{\partial\sigma}{\partial e_2}\\
&+\frac{p_1p_2e_1}{1-e_1}\Big[\Big(\frac{\partial\alpha_1}{\partial e_2}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_2}\Big)\sigma^{N-1}+\alpha_1\beta_2(N-1)\sigma^{N-2}\frac{\partial\sigma}{\partial e_2}\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]\\
&+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{p_2}{(1-e_2)^2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_2e_2}{1-e_2}\Big(\frac{\partial^2\alpha_1}{\partial e_1\partial e_2}+\frac{\partial^2\beta_1}{\partial e_1\partial e_2}\Big)+\frac{p_1}{(1-e_1)^2}\frac{\partial\alpha_2}{\partial e_2}\Big]\\
&+\frac{(N-1)p_1p_2e_1}{1-e_1}\Big\{\Big(\frac{\partial\alpha_1}{\partial e_2}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_2}\Big)\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\\
&+\alpha_1\beta_2\Big[(N-2)\sigma^{N-3}\frac{\partial\sigma}{\partial e_2}\frac{\partial\sigma}{\partial e_1}+\sigma^{N-2}\frac{\partial^2\sigma}{\partial e_1\partial e_2}\Big]\Big\}\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{(N-1)p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1)+\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_2}+\frac{\partial\beta_1}{\partial e_2}\Big)+\frac{p_1}{1-e_1}\frac{\partial\alpha_2}{\partial e_2}\Big]\\
=&\frac{p_1p_2}{1-e_1}\sigma^{N-1}\Big[\frac{\alpha_1}{1-e_1}\frac{\partial\beta_2}{\partial e_2}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\frac{\partial\beta_2}{\partial e_2}+\alpha_1\frac{\partial^2\beta_2}{\partial e_1\partial e_2}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{p_1p_2}{1-e_1}\sigma^{N-1}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2(\alpha_1+\beta_1)}{(1-e_2)^2}+\frac{p_2e_2}{1-e_2}\frac{\partial\beta_1}{\partial e_2}+\frac{p_1}{1-e_1}\frac{\partial\alpha_2}{\partial e_2}\Big]\\
&+\frac{p_1p_2}{1-e_1}(N-1)\sigma^{N-2}\frac{\partial\sigma}{\partial e_2}\Big[\frac{\alpha_1\beta_2}{1-e_1}+e_1\Big(\frac{\partial\alpha_1}{\partial e_1}\beta_2+\alpha_1\frac{\partial\beta_2}{\partial e_1}\Big)\Big]\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{p_1p_2e_1\alpha_1}{1-e_1}\sigma^{N-2}\Big[\frac{\partial\beta_2}{\partial e_2}\sigma+(N-1)\beta_2\frac{\partial\sigma}{\partial e_2}\Big]\Big[\frac{p_2e_2}{1-e_2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_1\alpha_2}{(1-e_1)^2}\Big]\\
&+\frac{p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{p_2}{(1-e_2)^2}\Big(\frac{\partial\alpha_1}{\partial e_1}+\frac{\partial\beta_1}{\partial e_1}\Big)+\frac{p_2e_2}{1-e_2}\frac{\partial^2\beta_1}{\partial e_1\partial e_2}+\frac{p_1}{(1-e_1)^2}\frac{\partial\alpha_2}{\partial e_2}\Big]\\
&+\frac{(N-1)p_1p_2e_1\alpha_1}{1-e_1}\sigma^{N-3}\Big[\frac{\partial\beta_2}{\partial e_2}\sigma\frac{\partial\sigma}{\partial e_1}+(N-2)\beta_2\frac{\partial\sigma}{\partial e_1}\frac{\partial\sigma}{\partial e_2}+\beta_2\sigma\frac{\partial^2\sigma}{\partial e_1\partial e_2}\Big]\\
&\cdot\Big[\frac{p_2e_2}{1-e_2}(\alpha_1+\beta_1)+\frac{p_1\alpha_2}{1-e_1}\Big]\\
&+\frac{(N-1)p_1p_2e_1\alpha_1\beta_2}{1-e_1}\sigma^{N-2}\frac{\partial\sigma}{\partial e_1}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1)+\frac{p_2e_2}{1-e_2}\frac{\partial\beta_1}{\partial e_2}+\frac{p_1}{1-e_1}\frac{\partial\alpha_2}{\partial e_2}\Big],
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
\frac{\partial^2 D}{\partial e_2^2} =& \frac{p_1 p_2 e_1}{1-e_1}\Big[\Big(\frac{\partial \alpha_1}{\partial e_2}\frac{\partial \beta_2}{\partial e_2} + \alpha_1 \frac{\partial^2 \beta_2}{\partial e_2^2}\Big)\sigma^{N-1} + \alpha_1 \frac{\partial \beta_2}{\partial e_2}(N-1)\sigma^{N-2}\frac{\partial \sigma}{\partial e_2}\Big]\Big[\frac{p_2 e_2}{1-e_2}(\alpha_1+\beta_1) + \frac{p_1 \alpha_2}{1-e_1}\Big] \\
&+ \frac{p_1 p_2 e_1 \alpha_1}{1-e_1}\frac{\partial \beta_2}{\partial e_2}\sigma^{N-1}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) + \frac{p_2 e_2}{1-e_2}\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big) + \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] \\
&+ \frac{p_1 p_2 e_1}{1-e_1}\Big[\Big(\frac{\partial \alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_2}\Big)\sigma^{N-1} + \alpha_1\beta_2(N-1)\sigma^{N-2}\frac{\partial \sigma}{\partial e_2}\Big]\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) \\
&+ \frac{p_2 e_2}{1-e_2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] + \frac{p_1 p_2 e_1 \alpha_1 \beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{2p_2}{(1-e_2)^3}(\alpha_1+\beta_1) + \frac{p_2}{(1-e_2)^2}\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big) \\
&+ \frac{p_2}{(1-e_2)^2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_2 e_2}{1-e_2}\frac{\partial^2 \beta_1}{\partial e_2^2} + \frac{p_1}{1-e_1}\frac{\partial^2 \alpha_2}{\partial e_2^2}\Big] + \frac{(N-1)p_1 p_2 e_1}{1-e_1}\Big\{\Big(\frac{\partial \alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_2}\Big)\sigma^{N-2}\frac{\partial \sigma}{\partial e_2} \\
&+ \alpha_1\beta_2\Big[(N-2)\sigma^{N-3}\Big(\frac{\partial \sigma}{\partial e_2}\Big)^2 + \sigma^{N-2}\frac{\partial^2 \sigma}{\partial e_2^2}\Big]\Big\}\Big[\frac{p_2 e_2}{1-e_2}(\alpha_1+\beta_1) + \frac{p_1\alpha_2}{1-e_1}\Big] \\
&+ \frac{(N-1)p_1 p_2 e_1 \alpha_1 \beta_2}{1-e_1}\sigma^{N-2}\frac{\partial \sigma}{\partial e_2}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) + \frac{p_2 e_2}{1-e_2}\Big(\frac{\partial \alpha_1}{\partial e_2} + \frac{\partial \beta_1}{\partial e_2}\Big) + \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] \\
=& \frac{p_1 p_2 e_1}{1-e_1}\Big[\alpha_1\frac{\partial^2 \beta_2}{\partial e_2^2}\sigma^{N-1} + (N-1)\alpha_1\frac{\partial \beta_2}{\partial e_2}\sigma^{N-2}\frac{\partial \sigma}{\partial e_2}\Big]\Big[\frac{p_2 e_2}{1-e_2}(\alpha_1+\beta_1) + \frac{p_1\alpha_2}{1-e_1}\Big] \\
&+ \frac{p_1 p_2 e_1 \alpha_1}{1-e_1}\frac{\partial \beta_2}{\partial e_2}\sigma^{N-1}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) + \frac{p_2 e_2}{1-e_2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] \\
&+ \frac{p_1 p_2 e_1}{1-e_1}\Big[\alpha_1\frac{\partial \beta_2}{\partial e_2}\sigma^{N-1} + (N-1)\alpha_1\beta_2\sigma^{N-2}\frac{\partial \sigma}{\partial e_2}\Big]\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) + \frac{p_2 e_2}{1-e_2}\frac{\partial \beta_1}{\partial e_2} \\
&+ \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] + \frac{p_1 p_2 e_1 \alpha_1 \beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{2p_2}{(1-e_2)^3}(\alpha_1+\beta_1) + \frac{p_2}{(1-e_2)^2}\frac{\partial \beta_1}{\partial e_2} \\
&+ \frac{p_2}{(1-e_2)^2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_2 e_2}{1-e_2}\frac{\partial^2 \beta_1}{\partial e_2^2} + \frac{p_1}{1-e_1}\frac{\partial^2 \alpha_2}{\partial e_2^2}\Big] + \frac{(N-1)p_1 p_2 e_1}{1-e_1}\Big\{\alpha_1\frac{\partial \beta_2}{\partial e_2}\sigma^{N-2}\frac{\partial \sigma}{\partial e_2} \\
&+ \alpha_1\beta_2\Big[(N-2)\sigma^{N-3}\Big(\frac{\partial \sigma}{\partial e_2}\Big)^2 + \sigma^{N-2}\frac{\partial^2 \sigma}{\partial e_2^2}\Big]\Big\}\Big[\frac{p_2 e_2}{1-e_2}(\alpha_1+\beta_1) + \frac{p_1\alpha_2}{1-e_1}\Big] \\
&+ \frac{(N-1)p_1 p_2 e_1 \alpha_1 \beta_2}{1-e_1}\sigma^{N-2}\frac{\partial \sigma}{\partial e_2}\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) + \frac{p_2 e_2}{1-e_2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] \\
=& \frac{p_1 p_2 e_1 \alpha_1}{1-e_1}\sigma^{N-2}\Big[\frac{\partial^2 \beta_2}{\partial e_2^2}\sigma + (N-1)\frac{\partial \beta_2}{\partial e_2}\frac{\partial \sigma}{\partial e_2}\Big]\Big[\frac{p_2 e_2}{1-e_2}(\alpha_1+\beta_1) + \frac{p_1\alpha_2}{1-e_1}\Big] \\
&+ \frac{2p_1 p_2 e_1 \alpha_1}{1-e_1}\sigma^{N-2}\Big[\frac{\partial \beta_2}{\partial e_2}\sigma + (N-1)\beta_2\frac{\partial \sigma}{\partial e_2}\Big]\Big[\frac{p_2}{(1-e_2)^2}(\alpha_1+\beta_1) + \frac{p_2 e_2}{1-e_2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_1}{1-e_1}\frac{\partial \alpha_2}{\partial e_2}\Big] \\
&+ \frac{p_1 p_2 e_1 \alpha_1 \beta_2}{1-e_1}\sigma^{N-1}\Big[\frac{2p_2}{(1-e_2)^3}(\alpha_1+\beta_1) + \frac{2p_2}{(1-e_2)^2}\frac{\partial \beta_1}{\partial e_2} + \frac{p_2 e_2}{1-e_2}\frac{\partial^2 \beta_1}{\partial e_2^2} + \frac{p_1}{1-e_1}\frac{\partial^2 \alpha_2}{\partial e_2^2}\Big] \\
&+ \frac{(N-1)p_1 p_2 e_1 \alpha_1}{1-e_1}\sigma^{N-3}\Big\{\frac{\partial \beta_2}{\partial e_2}\sigma\frac{\partial \sigma}{\partial e_2} + \Big[(N-2)\beta_2\Big(\frac{\partial \sigma}{\partial e_2}\Big)^2 + \beta_2\sigma\frac{\partial^2 \sigma}{\partial e_2^2}\Big]\Big\} \\
&\cdot\Big[\frac{p_2 e_2}{1-e_2}(\alpha_1+\beta_1) + \frac{p_1\alpha_2}{1-e_1}\Big].
\end{aligned}
\tag{2.33}
$$

The second derivatives of Q are as follows:

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial e_1^2} =& \frac{p_1 p_2}{(1-e_2)(A+B+C+D)}\Big[\Big(\frac{\partial \alpha_2}{\partial e_1}\beta_2 + \alpha_2\frac{\partial \beta_2}{\partial e_1}\Big)\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) \\
&+ \alpha_2\beta_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1^2}\beta_2 + \frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_1} + 2\frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_1} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1^2}\Big)\Big] \\
&- \frac{p_1 p_2 \alpha_2 \beta_2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&- \frac{p_1 p_2}{(1-e_2)(A+B+C+D)^2}\Big\{\Big[\Big(\frac{\partial \alpha_1}{\partial e_1}\alpha_2 + \alpha_1\frac{\partial \alpha_2}{\partial e_1}\Big)\beta_2^2 + 2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_1}\Big] \\
&\cdot\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) + \alpha_1\alpha_2\beta_2^2\Big(\frac{\partial^2 A}{\partial e_1^2} + \frac{\partial^2 B}{\partial e_1^2} + \frac{\partial^2 C}{\partial e_1^2} + \frac{\partial^2 D}{\partial e_1^2}\Big)\Big\} \\
&+ \frac{2 p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^3}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big)^2 \\
=& \frac{p_1 p_2 \alpha_2}{(1-e_2)(A+B+C+D)}\Big[\frac{\partial \beta_2}{\partial e_1}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) + \beta_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1^2}\beta_2 + 3\frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_1} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1^2}\Big)\Big] \\
&- \frac{p_1 p_2 \alpha_2 \beta_2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&- \frac{p_1 p_2 \alpha_2 \beta_2}{(1-e_2)(A+B+C+D)^2}\Big[\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&+ \alpha_1\beta_2\Big(\frac{\partial^2 A}{\partial e_1^2} + \frac{\partial^2 B}{\partial e_1^2} + \frac{\partial^2 C}{\partial e_1^2} + \frac{\partial^2 D}{\partial e_1^2}\Big)\Big] + \frac{2 p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^3}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big)^2 \\
=& \frac{p_1 p_2 \alpha_2}{(1-e_2)(A+B+C+D)}\Big[2\alpha_1\Big(\frac{\partial \beta_2}{\partial e_1}\Big)^2 + \beta_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1^2}\beta_2 + 4\frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_1} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1^2}\Big)\Big] \\
&- \frac{2 p_1 p_2 \alpha_2 \beta_2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&- \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial^2 A}{\partial e_1^2} + \frac{\partial^2 B}{\partial e_1^2} + \frac{\partial^2 C}{\partial e_1^2} + \frac{\partial^2 D}{\partial e_1^2}\Big) \\
&+ \frac{2 p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^3}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big)^2 \\
=& \frac{p_1 p_2 \alpha_2}{(1-e_2)(A+B+C+D)}\Big[2\alpha_1\Big(\frac{\partial \beta_2}{\partial e_1}\Big)^2 + \beta_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1^2}\beta_2 + 4\frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_1} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1^2}\Big)\Big] \\
&- \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial^2 A}{\partial e_1^2} + \frac{\partial^2 B}{\partial e_1^2} + \frac{\partial^2 C}{\partial e_1^2} + \frac{\partial^2 D}{\partial e_1^2}\Big) \\
&- \frac{2}{A+B+C+D}\frac{\partial Q}{\partial e_1}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big),
\end{aligned} \tag{2.34}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial e_1 \partial e_2} =& \frac{p_1 p_2}{(1-e_2)(A+B+C+D)}\Big[\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + \alpha_2\frac{\partial \beta_2}{\partial e_2}\Big)\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) \\
&+ \alpha_2\beta_2\Big(\frac{\partial^2 \alpha_1}{\partial e_1 \partial e_2}\beta_2 + \frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_2} + 2\frac{\partial \alpha_1}{\partial e_2}\frac{\partial \beta_2}{\partial e_1} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2}\Big)\Big] \\
&- \frac{p_1 p_2 \alpha_2 \beta_2}{(1-e_2)^2(A+B+C+D)^2}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) \\
&\cdot\Big[-(A+B+C+D) + (1-e_2)\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big)\Big] \\
&- \frac{p_1 p_2}{(1-e_2)(A+B+C+D)^2}\Big\{\Big[\Big(\frac{\partial \alpha_1}{\partial e_2}\alpha_2 + \alpha_1\frac{\partial \alpha_2}{\partial e_2}\Big)\beta_2^2 + 2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_2}\Big] \\
&\cdot\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) + \alpha_1\alpha_2\beta_2^2\Big(\frac{\partial^2 A}{\partial e_1 \partial e_2} + \frac{\partial^2 B}{\partial e_1 \partial e_2} + \frac{\partial^2 C}{\partial e_1 \partial e_2} + \frac{\partial^2 D}{\partial e_1 \partial e_2}\Big)\Big\} \\
&+ \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)^2(A+B+C+D)^4}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&\cdot\Big[-(A+B+C+D)^2 + 2(1-e_2)(A+B+C+D)\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big)\Big] \\
=& \frac{p_1 p_2}{(1-e_2)(A+B+C+D)}\Big[\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + \alpha_2\frac{\partial \beta_2}{\partial e_2}\Big)\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) \\
&+ \alpha_2\beta_2\Big(\frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_2} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2}\Big)\Big] + \frac{p_1 p_2 \alpha_2 \beta_2}{(1-e_2)^2(A+B+C+D)}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) \\
&- \frac{p_1 p_2 \alpha_2 \beta_2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big)\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&- \frac{p_1 p_2 \alpha_1 \beta_2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big)\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&- \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial^2 A}{\partial e_1 \partial e_2} + \frac{\partial^2 B}{\partial e_1 \partial e_2} + \frac{\partial^2 C}{\partial e_1 \partial e_2} + \frac{\partial^2 D}{\partial e_1 \partial e_2}\Big) \\
&- \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)^2(A+B+C+D)^2}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big) \\
&+ \frac{2 p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^3}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big)\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
=& \frac{p_1 p_2}{(1-e_2)(A+B+C+D)}\Big[\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + \alpha_2\frac{\partial \beta_2}{\partial e_2}\Big)\Big(\frac{\partial \alpha_1}{\partial e_1}\beta_2 + 2\alpha_1\frac{\partial \beta_2}{\partial e_1}\Big) \\
&+ \alpha_2\beta_2\Big(\frac{\partial \alpha_1}{\partial e_1}\frac{\partial \beta_2}{\partial e_2} + 2\alpha_1\frac{\partial^2 \beta_2}{\partial e_1 \partial e_2}\Big)\Big] - \frac{1}{A+B+C+D}\frac{\partial Q}{\partial e_1}\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&- \frac{1}{A+B+C+D}\Big(\frac{\partial A}{\partial e_1} + \frac{\partial B}{\partial e_1} + \frac{\partial C}{\partial e_1} + \frac{\partial D}{\partial e_1}\Big)\Big[\frac{\partial Q}{\partial e_2} - \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)^2(A+B+C+D)}\Big] \\
&+ \frac{1}{1-e_2}\frac{\partial Q}{\partial e_1} - \frac{p_1 p_2 \alpha_1 \alpha_2 \beta_2^2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial^2 A}{\partial e_1 \partial e_2} + \frac{\partial^2 B}{\partial e_1 \partial e_2} + \frac{\partial^2 C}{\partial e_1 \partial e_2} + \frac{\partial^2 D}{\partial e_1 \partial e_2}\Big),
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial e_2^2} =& \frac{p_1 p_2}{(1-e_2)^2(A+B+C+D)}\Big\{\Big(\frac{\partial \alpha_1}{\partial e_2}\beta_2 + \alpha_1\frac{\partial \beta_2}{\partial e_2}\Big)\Big[(1-e_2)\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big) + \alpha_2\beta_2\Big] \\
&+ \alpha_1\beta_2\Big[-\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big) + (1-e_2)\Big(\frac{\partial^2 \alpha_2}{\partial e_2^2}\beta_2 + \frac{\partial \alpha_2}{\partial e_2}\frac{\partial \beta_2}{\partial e_2} + 2\frac{\partial \alpha_2}{\partial e_2}\frac{\partial \beta_2}{\partial e_2} + 2\alpha_2\frac{\partial^2 \beta_2}{\partial e_2^2}\Big) \\
&+ \frac{\partial \alpha_2}{\partial e_2}\beta_2 + \alpha_2\frac{\partial \beta_2}{\partial e_2}\Big]\Big\} - \frac{p_1 p_2\alpha_1\beta_2}{(1-e_2)^4(A+B+C+D)^2}\Big[(1-e_2)\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big) + \alpha_2\beta_2\Big] \\
&\cdot\Big[-2(1-e_2)(A+B+C+D) + (1-e_2)^2\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big)\Big] \\
&- \frac{p_1 p_2}{(1-e_2)(A+B+C+D)^2}\Big\{\Big[\Big(\frac{\partial \alpha_1}{\partial e_2}\alpha_2 + \alpha_1\frac{\partial \alpha_2}{\partial e_2}\Big)\beta_2^2 + 2\alpha_1\alpha_2\beta_2\frac{\partial \beta_2}{\partial e_2}\Big]\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&+ \alpha_1\alpha_2\beta_2^2\Big(\frac{\partial^2 A}{\partial e_2^2} + \frac{\partial^2 B}{\partial e_2^2} + \frac{\partial^2 C}{\partial e_2^2} + \frac{\partial^2 D}{\partial e_2^2}\Big)\Big\} + \frac{p_1 p_2\alpha_1\alpha_2\beta_2^2}{(1-e_2)^2(A+B+C+D)^4}\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&\cdot\Big[-(A+B+C+D)^2 + 2(1-e_2)(A+B+C+D)\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big)\Big] \\
=& \frac{p_1 p_2\alpha_1}{(1-e_2)(A+B+C+D)}\Big[2\alpha_2\Big(\frac{\partial \beta_2}{\partial e_2}\Big)^2 + \beta_2\Big(\frac{\partial^2 \alpha_2}{\partial e_2^2}\beta_2 + 4\frac{\partial \alpha_2}{\partial e_2}\frac{\partial \beta_2}{\partial e_2} + 2\alpha_2\frac{\partial^2 \beta_2}{\partial e_2^2}\Big)\Big] \\
&+ \frac{2p_1 p_2\alpha_1\beta_2}{(1-e_2)^3(A+B+C+D)}\Big[(1-e_2)\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big) + \alpha_2\beta_2\Big] \\
&- \frac{p_1 p_2\alpha_1\beta_2}{(1-e_2)^2(A+B+C+D)^2}\Big[(1-e_2)\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big) + \alpha_2\beta_2\Big]\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&- \frac{p_1 p_2\alpha_1\beta_2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big)\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&- \frac{p_1 p_2\alpha_1\alpha_2\beta_2^2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial^2 A}{\partial e_2^2} + \frac{\partial^2 B}{\partial e_2^2} + \frac{\partial^2 C}{\partial e_2^2} + \frac{\partial^2 D}{\partial e_2^2}\Big) \\
&- \frac{p_1 p_2\alpha_1\alpha_2\beta_2^2}{(1-e_2)^2(A+B+C+D)^2}\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&+ \frac{2p_1 p_2\alpha_1\alpha_2\beta_2^2}{(1-e_2)(A+B+C+D)^3}\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big)^2 \\
=& \frac{p_1 p_2\alpha_1}{(1-e_2)(A+B+C+D)}\Big[2\alpha_2\Big(\frac{\partial \beta_2}{\partial e_2}\Big)^2 + \beta_2\Big(\frac{\partial^2 \alpha_2}{\partial e_2^2}\beta_2 + 4\frac{\partial \alpha_2}{\partial e_2}\frac{\partial \beta_2}{\partial e_2} + 2\alpha_2\frac{\partial^2 \beta_2}{\partial e_2^2}\Big)\Big] \\
&+ \frac{2p_1 p_2\alpha_1\beta_2}{(1-e_2)^3(A+B+C+D)}\Big[(1-e_2)\Big(\frac{\partial \alpha_2}{\partial e_2}\beta_2 + 2\alpha_2\frac{\partial \beta_2}{\partial e_2}\Big) + \alpha_2\beta_2\Big] \\
&- \frac{2}{A+B+C+D}\frac{\partial Q}{\partial e_2}\Big(\frac{\partial A}{\partial e_2} + \frac{\partial B}{\partial e_2} + \frac{\partial C}{\partial e_2} + \frac{\partial D}{\partial e_2}\Big) \\
&- \frac{p_1 p_2\alpha_1\alpha_2\beta_2^2}{(1-e_2)(A+B+C+D)^2}\Big(\frac{\partial^2 A}{\partial e_2^2} + \frac{\partial^2 B}{\partial e_2^2} + \frac{\partial^2 C}{\partial e_2^2} + \frac{\partial^2 D}{\partial e_2^2}\Big).
\end{aligned}
\tag{2.36}
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