APPDIEX

A. Proof of Theorem 1

The state transfer equations of fast timescale and slow timescale are known as (2) and (8), respectively.

The assumption of the relationship between state variables of slow timescale and fast timescale in the kth slow timescale is $X[k]=x_{nk}$. Accordingly, the relationship in the k+1th slow timescale can be written as $X[k+1]=x_{nk+n}$.

When the slow timescale state transfers from $X[k]$ to $X[k+1]$, the fast timescale state transfers from x_{nk} to x_{nk+n}, with n times fast timescale transfer. According to the fast timescale state transfer equations, this transfer process can be described as

$$\begin{aligned} x_{nk+1} &= ax_{nk} + bu_{nk} + cx_{nk}u_{nk} \\ x_{nk+2} &= ax_{nk+1} + bu_{nk+1} + cx_{nk+1}u_{nk+1} \\ &\vdots \\ x_{nk+n} &= ax_{nk+n-1} + bu_{nk+n-1} + cx_{nk+n-1}u_{nk+n-1} \end{aligned} \tag{1}$$

According to (1), we obtain

$$\begin{aligned} x_{nk+n} &= a^n x_{nk} + \\ & b\left\{\begin{aligned} & a^{n-1}u_{nk} + a^{n-2}u_{nk+1} + \cdots + u_{nk+n-1} \\ & + a^{n-2}cu_{nk}\sum_{j=1}^{n-1}u_{nk+j} + a^{n-3}cu_{nk+1}\sum_{j=2}^{n-1}u_{nk+j} + \cdots + cu_{nk+n-2}u_{nk+n-1} \\ & + a^{n-3}c^2u_{nk}\sum_{i\neq j,i,j=1}^{n-1}u_{nk+i}u_{nk+j} + a^{n-4}c^2u_{nk+1}\sum_{i\neq j,i,j=2}^{n-1}u_{nk+i}u_{nk+j} \\ & + \cdots + c^2u_{nk+n-3}u_{nk+n-2}u_{nk+n-1} \\ & + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right\} \\ & + cx_{nk}\left\{a^{n-1}\sum_{i=0}^{n-1}u_{nk+i} + a^{n-2}c\sum_{i\neq j,i,j=0}^{n-1}u_{nk+i}u_{nk+j} + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1}\right\} \end{aligned} \tag{2}$$

Combine $X[k]=x_{nk}$ and $X[k+1]=x_{nk+n}$ with (56), subtract $AX[k]$ from $X[k+1]$, we can obtain the state parameters is same as (9) and the control variable is same as (10).

B. Proof of Theorem 2

Proof by Contradiction. Suppose that there exist two same $U[k]$, making slow timescale state transfer equation as follows.

$$X[k+1] = A'X[k] + B'U[k] + C'X[k]U[k] \tag{3}$$

Thus, (3) has the similar state mapping relation with the slow timescale state equation (8), that is, the same $X[k]$ will transfer to the same $X[k+1]$ under the control effect of $U[k]$ or $U_B[k]$ and $U_C[k]$, respectively:

$$\begin{aligned} X[k+1] &= AX[k] + BU_B[k] + CU_C[k] \\ X[k+1] &= A'X[k] + B'U[k] + C'U[k] \end{aligned} \tag{4}$$

Subtract

$$\begin{aligned} 0 = &(A-A')X[k] + BU_B[k] - B'U[k] \\ &+ X[k]\left(CU_C[k] - C'U[k]\right) \end{aligned} \tag{5}$$

Combine $U_B[k]$ and $U_C[k]$ with (5)

$$\begin{aligned} 0 = &(A-A')X[k] \\ &+ B\left(\begin{aligned} & a^{n-1}u_{nk} + a^{n-2}u_{nk+1} + \cdots + u_{nk+n-1} \\ & + a^{n-2}cu_{nk}\sum_{j=1}^{n-1}u_{nk+j} + \cdots + cu_{nk+n-2}u_{nk+n-1} \\ & + a^{n-3}c^2u_{nk}\sum_{i\neq j,i,j=1}^{n-1}u_{nk+i}u_{nk+j} + \cdots + c^2u_{nk+n-3}u_{nk+n-2}u_{nk+n-1} \\ & + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right) \\ &- B'U[k] \\ &+ CX[k]\left(\begin{aligned} & a^{n-1}\sum_{i=0}^{n-1}u_{nk+i} \\ & + a^{n-2}c\sum_{i\neq j,i,j=0}^{n-1}u_{nk+i}u_{nk+j} + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right) \\ &- C'U[k] \end{aligned} \tag{6}$$

The values of $X[k]$, u and $U[k]$ are arbitrary within limits, and the right-hand side of (6) must be identically equal to 0, so each term in (6) must be equal to 0.

$$\left\{\begin{aligned} & A - A' = 0 \\ & B\left(\begin{aligned} & a^{n-1}u_{nk} + a^{n-2}u_{nk+1} + \cdots + u_{nk+n-1} \\ & + a^{n-2}cu_{nk}\sum_{j=1}^{n-1}u_{nk+j} + \cdots + cu_{nk+n-2}u_{nk+n-1} \\ & + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right) - B'U[k] = 0 \\ & C\left(\begin{aligned} & a^{n-1}\sum_{i=0}^{n-1}u_{nk+i} \\ & + a^{n-2}c\sum_{i\neq j,i,j=0}^{n-1}u_{nk+i}u_{nk+j} \\ & + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right) - C'U[k] = 0 \end{aligned}\right. \tag{7}$$

That is,

$$\left\{\begin{aligned} & A = A' \\ & U[k] = \frac{B}{B'}\left(\begin{aligned} & a^{n-1}u_{nk} + a^{n-2}u_{nk+1} + \cdots + u_{nk+n-1} \\ & + a^{n-2}cu_{nk}\sum_{j=1}^{n-1}u_{nk+j} + \cdots + cu_{nk+n-2}u_{nk+n-1} \\ & + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right) \\ & U[k] = \frac{C}{C'}\left(\begin{aligned} & a^{n-1}\sum_{i=0}^{n-1}u_{nk+i} \\ & + a^{n-2}c\sum_{i\neq j,i,j=0}^{n-1}u_{nk+i}u_{nk+j} \\ & + \cdots + c^{n-1}u_{nk}\cdots u_{nk+n-1} \end{aligned}\right) \end{aligned}\right. \tag{8}$$

Since these coefficients are constants, the bottom two equations contradict each other for any $a \neq 1$. Thus, the proof is completed.