Cellular Production Lines with Asymptotically Reliable Bernoulli Machines: Lead Time Analysis and Control *

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Abstract

Cellular lines are production systems consisting of cells comprised of machines performing similar operations. These systems are notorious for having excessively long lead time (LT), often orders of magnitude longer than the total processing time by all machines in the system. The main goal of this paper is to provide a plausible explanation of this phenomenon and offer methods for its alleviation. To accomplish this, the paper develops a technique for performance evaluation of cellular lines with asymptotically reliable Bernoulli machines and uses it to analyze and control LT. It shows, in particular, that robustness of LT is a decreasing function of the number of machines in a cell. Thus, in cells with many machines, small variations in either release rates or machine efficiencies may lead to dramatic increases of LT, if operating points are not selected appropriately. A method for selecting appropriate operating points and, thus, controlling release rates in the open-loop regime is provided. In addition, feedback control of raw material release is considered and shown to be effective for improving lead time performance for any operating point.

Keywords: Production systems, Cellular lines, Lead time, Asymptotically reliable Bernoulli machines, Open- and closed-loop release control.

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1 Introduction

This paper considers production lines consisting of cells comprised of machines performing similar operations. Such a production line with M cells, $c_1, c_2, \ldots, c_M$, each comprised of C_i, $i = 1, 2, \ldots, M$, machines and $M - 1$ buffers, $b_1, b_2, \ldots, b_{M-1}$, is shown in Figure 1.1. These lines are referred to as *cellular*. Cellular lines are often encountered in machining operations (e.g., turning, boring, milling, drilling, and grinding cells in automotive transmission plants). Under a fixed dispatch policy, re-entrant lines in semiconductor manufacturing also can be viewed as cellular (e.g., lithography, etching, and deposition tool groups). Cellular lines are sometimes referred to as series-parallel production systems.

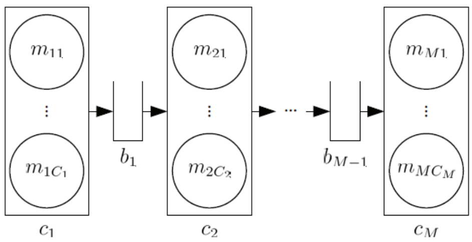


Figure 1.1: Cellular production line

In practice, performance of cellular lines is often marred by excessively long lead time (LT), i.e., the average time a part spends in the system, being processed and waiting for processing. In some cases, LT has been observed to be orders of magnitude longer than the total processing time by all machines in the system. (As an anecdotal evidence, the first author encountered gears in a machining department of an automotive transmission plant, which rusted through while waiting for processing.) Therefore, performance analysis of cellular lines and, in particular, of LT and, based on this analysis, insights and recommendations for management and control are important theoretical and practical problems.

Unlike production systems with a single machine per stage, which have been analyzed extensively in the literature (see, for instance, monographs [1]-[8]), quantitative analysis of cellular lines received relatively little attention. Related articles can be classified into two groups: those that aggregate parallel machines into a single machine (see [9]-[16]) and those that develop exact methods for performance analysis of two-cell systems (see [17]-[21]). The aggregation approach provides a relatively low accuracy of performance measures. The exact methods are typically computationally intensive. Also, while [9]-[21] present methods for performance evaluation, none considers raw material release mechanisms and the resulting *LT* as a function of system parameters. In our previous work [22], we introduced the problem of *LT* analysis and control for the usual, serial lines. In the current paper, we investigate the cellular systems.

The development is based on several simplifying assumptions. First, the machines are assumed to obey the *Bernoulli reliability model*. According to this model, a machine is up during a cycle time with probability p and down with probability $1 - p$. This implies that the number of machines in a cell, which are up during a cycle time, is distributed according to a binomial distribution and, therefore, such lines may also be called *binomial*. The parameter p is referred to as the *machine efficiency*.

The second assumption postulates that the machines are *asymptotically reliable*, i.e.,

$$p_{ij} = 1 - \varepsilon k_{ij},\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, C_i, \tag{1.1}$$

where p_{ij} is the efficiency of machine m_{ij} (i.e., the j-th machine in the i-th cell), $0 < \varepsilon \ll 1$ is a small parameter, and k_{ij} is a number of order one.

The third assumption, introduced to make the results more transparent, postulates that the number of machines in each cell is the same, i.e.,

$$C_i = C,\ \forall i = 1, 2, \ldots, M, \tag{1.2}$$

and, in addition, the machines in a cell are of equal efficiency, i.e.,

$$p_{ij} = p_i,\ i = 1, 2, \ldots, M,\ j = 1, 2, \ldots, C. \tag{1.3}$$

This implies, of course, that $k_{ij} = k_i$, $\forall j$.

Next, it is assumed that the capacity, N_i, of all buffers, $b_1, b_2, \ldots, b_{M-1}$, is infinite, i.e.,

$$N_i = \infty,\ i = 1, 2, \ldots, M - 1. \tag{1.4}$$

Finally, the last assumption refers to the *release mechanism* of raw material. We model the release mechanism by another cell, c_0, also consisting of C identical asymptotically reliable Bernoulli machines, each defined by

$$p_0 = 1 - \varepsilon k_0, \tag{1.5}$$

where, as before, $0 < \varepsilon \ll 1$ and k_0 is of order one. While the efficiencies, p_i, $i = 1, 2, \dots, M$, of the producing machines are fixed, the efficiency, p_0, of the release machines, is free and viewed as a design parameter, to be selected so that the lead time, LT, takes the desired value. The value of p_0 is called the raw material *release rate*.

A cellular line with such a release mechanism is shown in Figure 1.2, where the release cell, c_0, and the raw material buffer, b_0, are indicated in gray.

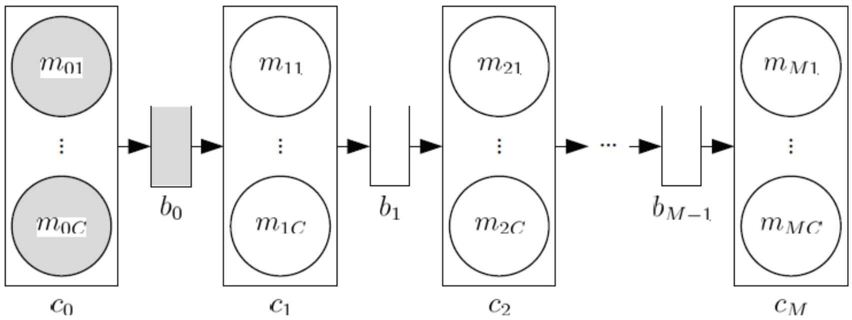


Figure 1.2: Cellular line with a raw material release cell

Under these assumptions, this paper develops a method for performance evaluation of cellular lines, investigates structural and quantitative properties of LT, and offers open- and closed-loop methods for LT control by selecting an appropriate p_0 or an equivalent hourly release rate. More precisely, the paper presents the following novel results:

- An analytical method for performance evaluation of cellular lines with asymptotically reliable Bernoulli machines.

- An expression for LT as a function of the producing machine efficiency and raw material release rate.

- An expression for LT as a function of the so-called *relative load* on the system, $\rho = \frac{p_0}{\min\limits_{1 \leq i \leq M} p_i}$, and shows that this function has a “knee”, beyond which LT grows extremely fast; the position of this knee is quantified analytically.

- Robustness properties of LT with respect to p_0 (i.e., when the release rate is not exactly as expected) and with respect to p_i (i.e., when the producing machine efficiency is not exactly the one nominally assumed).

- An expression for the release rate that ensures the desired LT, while maximizing the production rate.

- A method for calculating deterministic, *hourly*, raw material release rate, which results in the performance being close to the desired LT.

- A feedback control law for raw material release, which ensures the desired LT even when the operating point is beyond the knee and the machine efficiency is different from nominally assumed.

In spite of the simplifying assumptions, under which these results are obtained, we believe they provide useful insights into the lead time behavior of cellular lines.

The outline of this paper is as follows: In Section 2, a formal model of a cellular line with asymptotically reliable Bernoulli machines and a raw material release mechanism is introduced. Section 3 develops a method for its performance evaluation. Structural and quantitative analyses of LT in systems with identical machines are carried out in Section 4. Some of these results are extended to systems with non-identical machines in Section 5. Deterministic, hourly, release rate is investigated in Section 6, and feedback control of LT is discussed in Section 7. The conclusions and topic for future work are given in Section 8. All proofs are included in the Appendices.

2 Modeling and Problems Addressed

The following model is considered throughout this paper:

(i) The system consists of M producing cells, $c_1, c_2, \dots, c_M$; a release cell, c_0; $M-1$ work-in-process buffers, $b_1, b_2, \dots, b_{M-1}$; and a raw material buffer, b_0 (see Figure 1.2).

(ii) Each cell c_i, $i = 0, 1, \ldots, M$, is comprised of C machines, $m_{i1}, m_{i2}, \ldots, m_{iC}$.

(iii) Each machine, m_{ij}, $i = 0, 1, \ldots, M$, $j = 1, 2, \ldots, C$, is asymptotically reliable and obeys the Bernoulli reliability model with the efficiency

$$p_{ij} = p_i = 1 - \varepsilon k_i, \tag{2.1}$$

where $0 < \varepsilon \ll 1$ and k_i is a number of order one. All machines have identical cycle time, τ. The time is slotted with the slot duration τ.

(iv) All buffers are of infinite capacity.

(v) Machines in the release cell, c_0, are never starved. Machines in the producing cells, $c_1, c_2, \ldots, c_M$, may be either starved or non-starved according to the following convention: Let $h_i(n)$, $i = 0, 1, \ldots, M-1$, be the number of parts in buffer b_i at the beginning of time slot $n \in \{1, 2, \ldots\}$ and $C_{i+1}(n)$ be the number of machines in cell c_{i+1} that are up during the time slot n. Then,

- if $h_i(n) \geqslant C_{i+1}(n)$, none of the machines in cell c_{i+1} is starved;
- if $0 < h_i(n) < C_{i+1}(n)$, $C_{i+1}(n) - h_i(n)$ machines are starved;
- if $h_i(n) = 0$, the whole cell c_{i+1} is starved.

Note: In general, raw material release mechanism could be modeled as a single machine "producing" parts according to a binomial distribution. However, we find it more convenient to model the release as adopted in this paper, i.e., as a cell with Bernoulli machines. Nevertheless, considering hourly release (see Sections 6 and 7), we remove this assumption and provide recommendations for releasing a deterministic amount of raw material, which leads to *LT* being close to the desired.

Given the model defined by assumptions (i)-(v), the problems addressed in this paper are:

- Develop a method for evaluating the production rate (*PR*), work-in-process (*WIP*), and lead time (*LT*).
- Analyze *LT* as a function of system parameters and, in particular, investigate robustness of *LT* with respect to p_0 and p_i.

- Based on these analyses, provide a plausible explanation for poor lead time performance of cellular lines.
- Offer a method for calculating raw material release rates that ensure the desired LT, while maximizing PR.
- Develop a feedback control policy for raw material release that leads to the desired LT.

Solutions of these problems are given in Sections 3-7.

3 Performance Evaluation

In this section, we develop a method for performance evaluation of cellular lines defined by assumptions (i)-(v). We begin with the simplest case of a single producing cell and a release cell and then extend the results to multiple producing cells.

3.1 Single producing cell

Theorem 3.1 *Under assumptions (i)-(v) with $M = 1$ and $p_0 < p$,*

- *the production rate of the cellular line is given by*

$$PR = Cp_0; \tag{3.1}$$

- *there exists ε_0 such that for all $0 < \varepsilon \leqslant \varepsilon_0$, the expressions*

$$\widehat{WIP} = C - 1 + \frac{p_0^C(1 - p_0^C)}{p^C - p_0^C}, \tag{3.2}$$

$$\widehat{LT} = \frac{C - 1}{Cp_0} + \frac{p_0^{C-1}(1 - p_0^C)}{C(p^C - p_0^C)} \tag{3.3}$$

provide estimates of WIP and LT with the accuracy $O(\varepsilon^2)$, i.e.,

$$\begin{aligned} \left|\widehat{WIP} - WIP\right| &= O(\varepsilon^2), \\ \left|\widehat{LT} - LT\right| &= O(\varepsilon^2), \end{aligned} \tag{3.4}$$

where LT and $\widehat{LT}$ are in units of the machine cycle time.

Proof: See the Appendix.

Note that when $C = 1$, estimates (3.2) and (3.3) reduce to the exact values of *WIP* and *LT* derived in [22] for the usual serial lines. Also, as can be surmised from the proof of Theorem 3.1, estimates of *WIP* and *LT* can be derived with accuracy higher than $O(\varepsilon^2)$ by keeping additional terms in Taylor expansions of the transition probabilities.

It is of interest to evaluate ε_0, which would result in practically acceptable accuracy of estimates (3.2) and (3.3). This could be carried out analytically, but the results are typically very conservative. Therefore, we carry this out by simulating the systems at hand, evaluating *WIP* and *LT*, and comparing them with $\widehat{WIP}$ and $\widehat{LT}$. Each simulation runs for $100,000$ time slots, including $50,000$ time slots of warm-up period; 20 simulation runs are carried out for each set of system parameters. The accuracy of $\widehat{WIP}$ and $\widehat{LT}$ is evaluated by $\epsilon_{WIP} = \frac{\widehat{WIP}-WIP}{WIP} \times 100\%$ and $\epsilon_{LT} = \frac{\widehat{LT}-LT}{LT} \times 100\%$. The results presented in Table 3.1 show that the accuracy is quite high for $p \in [0.9, 0.99]$ and $\rho \in [0.85, 0.99]$, where $\rho = \frac{p_0}{p}$. Thus, we conclude that ε_0 can be selected as 0.1.

3.2 Multiple producing cells

For performance analysis of cellular lines with $M > 1$ producing cells, we use a generalization of the recursive aggregation procedure developed in [7] for serial (non-cellular) production lines with Bernoulli machines. According to this procedure, pairs of consecutive cells are recursively aggregated into a single cell, using the so-called backward and forward aggregations. As a result, the cellular line with M producing cells and a release cell is represented by M cellular lines, each comprised of two cells with *aggregated* machines. Then, using expressions (3.2) and (3.3) with p replaced by the efficiency of the aggregated machines (which, for the model (i)-(v), turn out to be $p_1, p_2, \ldots, p_M$), we introduce the following estimates of WIP_i and LT for cellular lines defined by

Table 3.1: Accuracy of estimates (3.2) and (3.3)

C	p	ρ	Simulation		Estimate		Error	
			WIP	LT	$\widehat{WIP}$	$\widehat{LT}$	ϵ_{WIP}	ϵ_{LT}
2	0.9	0.85	2.05	1.34	2.08	1.36	1.34%	1.33%
		0.90	2.48	1.53	2.47	1.52	−0.52%	−0.48%
		0.95	3.55	2.08	3.49	2.04	−1.77%	−1.74%
		0.99	11.30	6.34	11.15	6.26	−1.28%	−1.25%
	0.95	0.85	1.88	1.16	1.91	1.18	1.48%	1.53%
		0.90	2.14	1.25	2.15	1.26	0.15%	0.20%
		0.95	2.72	1.51	2.72	1.51	0	0
		0.99	6.84	3.64	6.69	3.55	−2.20%	−2.21%
	0.99	0.85	1.74	1.03	1.76	1.05	1.38%	1.40%
		0.90	1.87	1.05	1.88	1.05	0.68%	0
		0.95	2.07	1.10	2.07	1.10	0	0
		0.99	2.96	1.51	2.94	1.50	−0.74%	−0.75%
5	0.9	0.85	4.24	1.11	4.59	1.20	8.23%	8.18%
		0.90	4.80	1.18	4.94	1.22	2.93%	2.94%
		0.95	6.01	1.41	5.86	1.37	−2.47%	−2.48%
		0.99	14.24	3.20	12.51	2.81	−12.13%	−12.13%
	0.95	0.85	4.24	1.05	4.52	1.12	6.65%	6.63%
		0.90	4.64	1.09	4.78	1.12	3.11%	3.10%
		0.95	5.37	1.19	5.37	1.19	0	0
		0.99	9.52	2.02	9.13	1.94	−4.10%	−4.11%
	0.99	0.85	4.25	1.01	4.46	1.06	5.00%	5.01%
		0.90	4.53	1.02	4.63	1.04	2.32%	2.31%
		0.95	4.87	1.04	4.90	1.04	0.62%	0
		0.99	5.89	1.20	5.86	1.19	−0.56%	−0.55%

assumptions (i)-(v) with $M > 1$ cells:

$$\widehat{WIP}_i = C - 1 + \frac{p_0^C(1-p_0^C)}{p_{i+1}^C - p_0^C}, \quad i = 0, 1, \ldots, M-1, \tag{3.5}$$

$$\widehat{LT} = M\frac{C-1}{Cp_0} + \sum_{i=1}^{M} \frac{p_0^{C-1}(1-p_0^C)}{C(p_i^C - p_0^C)}, \tag{3.6}$$

where, as before, $\widehat{LT}$ is in units of the machine cycle time.

As for the production rate, PR, of such lines, its exact value remains the same as in the case of $M = 1$ (see the Appendix), i.e.,

$$PR = Cp_0. \tag{3.7}$$

The accuracy of estimates (3.5) and (3.6) has been evaluated numerically, using the simulation procedure described in Subsection 3.1. The results for cellular lines with ten cells having identical and non-identical machines are shown in Tables 3.2 (with $\rho = \frac{p_0}{p}$) and 3.3 (with $\rho_{max} = \frac{p_0}{\min\limits_{1 \leqslant i \leqslant M} p_i}$), respectively. Lines L_1-L_3 involved in Table 3.3 are given by

$$\begin{aligned} L_1 &: \ p = [0.99, 0.93, 0.96, 0.95, 0.99, 0.97, 0.95, 0.91, 0.98, 0.95]; \\ L_2 &: \ p = [0.91, 0.94, 0.98, 0.91, 0.92, 0.92, 0.92, 0.96, 0.93, 0.92]; \\ L_3 &: \ p = [0.96, 0.98, 0.99, 0.97, 0.92, 0.94, 0.99, 0.99, 0.94, 0.99], \end{aligned} \tag{3.8}$$

where the p_i's have been selected randomly and equiprobably from the interval $[0.9, 0.99]$. As it follows from these tables, expressions (3.5) and (3.6) provide relatively precise estimates of WIP and LT when $\varepsilon_0 = 0.1$.

Table 3.2: Accuracy of estimates (3.5) and (3.6) (identical machine case)

C	p	ρ	Simulation		Estimate		Error	
			WIP	LT	$\widehat{WIP}$	$\widehat{LT}$	ϵ_{WIP}	ϵ_{LT}
2	0.9	0.85	20.34	13.30	20.80	13.59	2.27%	2.23%
		0.90	24.56	15.16	24.66	15.22	0.40%	0.39%
		0.95	35.35	20.68	34.90	20.41	−1.29%	−1.29%
		0.99	116.69	65.48	111.52	62.58	−4.43%	−4.43%
	0.95	0.85	18.67	11.56	19.06	11.80	2.10%	2.10%
		0.90	21.32	12.46	21.47	12.55	0.68%	0.73%
		0.95	27.26	15.10	27.17	15.05	−0.34%	−0.34%
		0.99	68.78	36.56	66.87	35.55	−2.78%	−2.76%
	0.99	0.85	17.34	10.30	17.60	10.46	1.51%	1.49%
		0.90	18.66	10.47	18.79	10.54	0.71%	0.69%
		0.95	20.65	10.98	20.69	11.00	0.21%	0.19%
		0.99	29.30	14.95	29.41	15.00	0.38%	0.36%
5	0.9	0.85	41.95	10.97	45.89	12.00	9.40%	9.38%
		0.90	47.46	11.72	49.39	12.20	4.06%	4.07%
		0.95	59.88	14.00	58.58	13.70	−2.17%	−2.14%
		0.99	142.67	32.02	125.08	28.08	−12.33%	−12.32%
	0.95	0.85	42.14	10.44	45.24	11.20	7.36%	7.37%
		0.90	46.13	10.79	47.83	11.19	3.68%	3.74%
		0.95	53.49	11.85	53.73	11.91	0.45%	0.47%
		0.99	94.55	20.11	91.25	19.41	−3.49%	−3.50%
	0.99	0.85	42.43	10.09	44.61	10.60	5.15%	5.10%
		0.90	45.21	10.15	46.32	10.40	2.45%	2.42%
		0.95	48.69	10.35	49.03	10.43	0.71%	0.74%
		0.99	58.65	11.97	58.55	11.95	−0.17%	−0.18%

Table 3.3: Accuracy of estimates (3.5) and (3.6) (non-identical machine case)

C	line	ρ_{max}	Simulation		Estimate		Error	
			WIP	LT	$\widehat{WIP}$	$\widehat{LT}$	ϵ_{WIP}	ϵ_{LT}
2	L_1	0.85	17.16	11.10	17.69	11.43	3.05%	3.04%
		0.90	18.99	11.59	19.29	11.78	1.58%	1.57%
		0.95	22.15	12.81	22.16	12.82	0.06%	0.09%
		0.99	33.31	18.49	32.61	18.10	−2.09%	−2.11%
	L_2	0.85	18.63	12.05	19.12	12.36	2.61%	2.60%
		0.90	21.37	13.05	21.65	13.22	1.34%	1.28%
		0.95	27.43	15.86	27.26	15.77	−0.59%	−0.58%
		0.99	54.67	30.34	53.55	29.72	−2.04%	−2.05%
	L_3	0.85	17.02	10.89	17.50	11.19	2.83%	2.81%
		0.90	18.70	11.29	18.99	11.47	1.53%	1.52%
		0.95	21.52	12.31	21.62	12.37	0.44%	0.46%
		0.99	31.01	17.02	31.27	17.17	0.83%	0.84%
5	L_1	0.85	39.81	10.29	43.89	11.35	10.26%	10.27%
		0.90	42.87	10.47	45.57	11.13	6.30%	6.25%
		0.95	47.34	10.95	48.59	11.24	2.65%	2.68%
		0.99	59.50	13.21	58.39	12.96	−1.87%	−1.87%
	L_2	0.85	40.88	10.57	44.86	11.60	9.72%	9.73%
		0.90	44.92	10.97	47.33	11.56	5.36%	5.33%
		0.95	52.20	12.07	52.72	12.20	1.00%	1.01%
		0.99	80.29	17.83	76.42	16.97	−4.82%	−4.82%
	L_3	0.85	40.05	10.24	43.86	11.22	9.51%	9.52%
		0.90	43.06	10.40	45.48	10.99	5.62%	5.63%
		0.95	47.20	10.80	48.34	11.06	2.43%	2.43%
		0.99	58.30	12.80	57.59	12.65	−1.21%	−1.21%

Based on the results of this section, we conclude that (3.5) and (3.6) provide sufficiently precise estimates of WIP and LT for cellular lines defined by assumptions (i)-(v) if

$$p_i \in [0.9, 0.99],\ i = 1, 2, \ldots, M, \tag{3.9}$$

$$p_0 \in [0.85 \min_{1\leqslant i\leqslant M} p_i,\ 0.99 \min_{1\leqslant i\leqslant M} p_i]. \tag{3.10}$$

The analyses presented below are carried out for cellular lines with parameters defined by (3.9) and (3.10).

4 Analysis and Open-Loop Control of Lead Time in Cellular Lines with Identical Producing Machines

In cellular lines with producing machines having identical efficiency, structural and qualitative properties of LT become especially transparent and instructive. Therefore, in the current section we address this case and in the next one provide a generalization to non-identical machines.

4.1 Structural properties

In cellular lines with $p_i = p$, $i = 1, 2, \ldots, M$, expression (3.6) becomes

$$\widehat{LT} = M\Big[\frac{C-1}{Cp_0} + \frac{p_0^{C-1}(1-p_0^C)}{C(p^C - p_0^C)}\Big]. \tag{4.1}$$

To characterize its structural properties, we use the *relative workload*, ρ, introduced in Section 3, i.e.,

$$\rho = \frac{p_0}{p}, \tag{4.2}$$

and the *relative lead time* defined as

$$\widehat{lt} := \frac{\widehat{LT}}{M}, \tag{4.3}$$

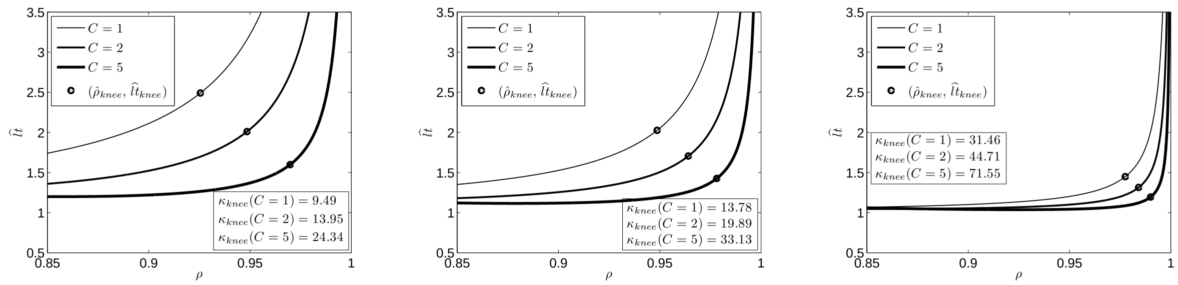
i.e., the lead time in units of the smallest possible lead time (which is M cycle times). In terms of these variables, (4.1) becomes

$$\widehat{lt} = \frac{(C-1)p^{-1}}{C\rho} + \frac{(p\rho)^{C-1}(p^{-C} - \rho^C)}{C(1-\rho^C)}. \tag{4.4}$$

Figure 4.1 illustrates the behavior of $\widehat{lt}$ as a function of $\rho \in [0.85, 1)$ for various p and C, where the curves for $C = 1$ are shown for comparison purposes. As one can see, all curves in this figure have a "knee" beyond which $\widehat{lt}$ grows extremely fast. It is of interest to quantify the position of the knee, ρ_{knee}, on the $\widehat{lt}(\rho)$-curve. To accomplish this, consider the $(\rho, \widehat{lt})$-plane, where a unit interval of ρ-axis corresponds to $A > 1$ units of $\widehat{lt}$-axis (in Figure 4.1, $A = 20$). Introduce the *scaling ratio*, $\alpha = \frac{1}{A}$, and recall that the *curvature*, κ, of a twice differentiable function, $f(x)$, is given by (see

[23])

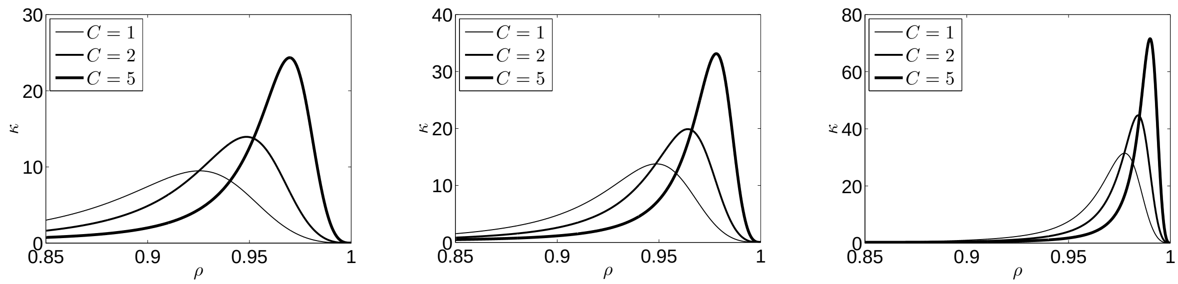
$$\kappa(f(x)) = \frac{\left|f''_{xx}\right|}{(1 + f'^2_x)^{\frac{3}{2}}}. \tag{4.5}$$



(a) $p = 0.9$ (b) $p = 0.95$ (c) $p = 0.99$

Figure 4.1: Behavior of $\widehat{lt}$ as a function of ρ for various p and C

The curvature of function $\alpha\widehat{lt}(\rho)$ calculated using expression (4.5) is illustrated in Figure 4.2 for several values of p and C. The unique maximum of these curves leads to the following:



(a) $p = 0.9$ (b) $p = 0.95$ (c) $p = 0.99$

Figure 4.2: Curvature of $\alpha\widehat{lt}(\rho)$ as a function of ρ for various p and C

Definition 4.1 *The knee, $\hat{\rho}_{knee}$, of $\widehat{lt}$ on the $(\rho, \widehat{lt})$-plane with the scaling ratio α is a point on* $[0.85, 1)$*, at which the curvature of $\alpha\widehat{lt}(\rho)$ reaches its maximum.*

The pairs $(\hat{\rho}_{knee}, \widehat{lt}(\rho_{knee}))$ are indicated in Figure 4.1 by black dots. Thus, releasing raw material with the rate $p_0 < p\hat{\rho}_{knee}$ results in $\widehat{lt}$ below the knee. Clearly, larger p_0 results in larger production rate (see (3.7)). Therefore, it would seem desirable to operate the system close to or at the "knee". Robustness of such an operation is investigated next.

4.2 Robustness

In this subsection, we evaluate robustness of LT with respect to both release rate, p_0, and producing machine efficiency, p.

4.2.1 Robustness with respect to p_0

Assume that a cellular line *nominally* operates at the knee of $\widehat{lt}$, i.e.,

$$p_0 = p\hat{\rho}_{knee}, \quad \widehat{lt} = \widehat{lt}(\hat{\rho}_{knee}), \quad PR = Cp_0. \tag{4.6}$$

Assume that *in reality* the release rate is larger than p_0, i.e.,

$$p_{0,real} > p_0. \tag{4.7}$$

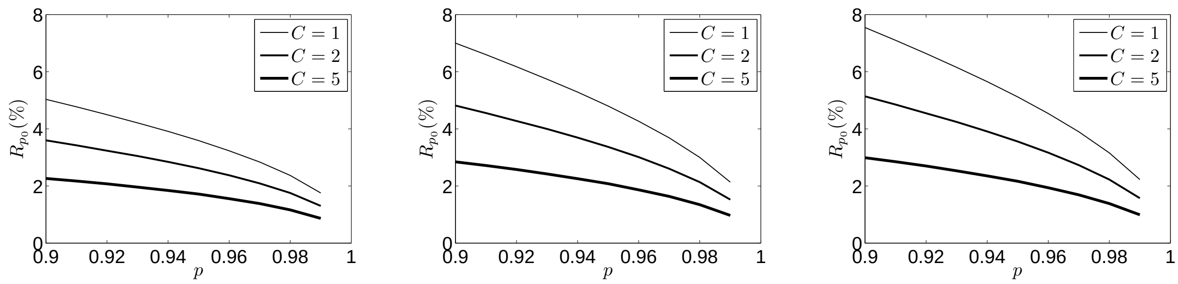
This will, obviously, lead to $\widehat{lt} > \widehat{lt}(\hat{\rho}_{knee})$. Assume, finally, that the largest tolerable $\widehat{lt}$ (e.g., due to perishable nature of the goods produced) is

$$\overline{lt} = K\widehat{lt}(\hat{\rho}_{knee}), \tag{4.8}$$

where $K > 1$. In other words, lt must be no larger than K times $\widehat{lt}_{knee} := \widehat{lt}(\hat{\rho}_{knee})$. Denote the release rate that results in $\overline{lt}$ as $\bar{p}_{0,real}$. Then, *robustness of* $\widehat{lt}$ *with respect to* p_0 can be quantified as

$$R_{p_0} = \frac{\bar{p}_{0,real} - p_0}{p_0} \times 100\%. \tag{4.9}$$

We have investigated R_{p_0} numerically solving equations (4.4) and (4.8) for $M = 10$. The results are shown in Figure 4.3. From this figure we conclude that robustness of $\widehat{lt}$ with respect to p_0 decreases as a function of C: for $C = 5$, this robustness is roughly twice below that for $C = 1$.



(a) $K = 2$ (b) $K = 5$ (c) $K = 10$

Figure 4.3: Robustness of $\widehat{lt}$ with respect to p_0 as a function of p for various K and C

4.2.2 Robustness with respect to p

Assume again that the cellular line nominally operates at $\hat{\rho}_{knee}$ with nominal p and p_0. In reality, however, the producing machines are less efficient than nominally assumed, i.e.,

$$p_{real} < p. \tag{4.10}$$

In this case,

$$\rho_{real} = \frac{p_0}{p_{real}} > \hat{\rho}_{knee}. \tag{4.11}$$

This again will result in $\widehat{lt}(\rho_{real}) > \widehat{lt}(\hat{\rho}_{knee})$. Assume, as before, that the largest $\widehat{lt}$ that can be tolerated is given by

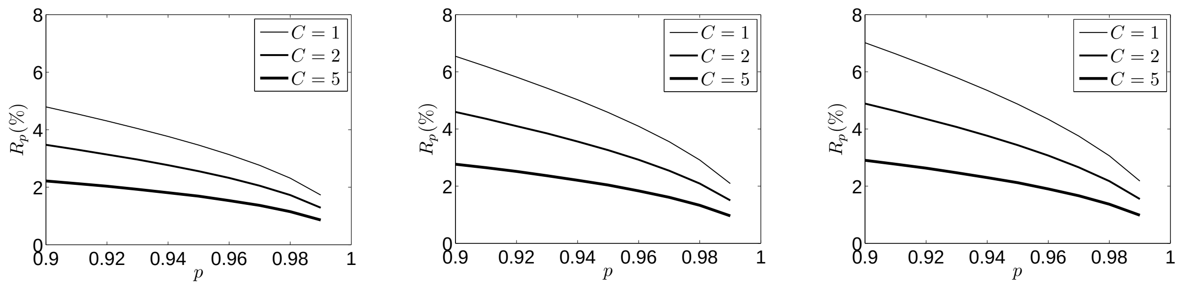
$$\overline{lt}(\rho_{real}) = K\widehat{lt}(\hat{\rho}_{knee}), \tag{4.12}$$

where $K > 1$. Denote the producing machine efficiency that results in $\overline{lt}(\rho_{real})$ as $\bar{p}_{real}$. Then, *robustness of $\widehat{lt}$ with respect to p* can be quantified as

$$R_p = \frac{\bar{p}_{real} - p}{p} \times 100\%. \tag{4.13}$$

The values of R_p have been investigated numerically using expressions (4.4) and (4.12) for cellular lines defined by assumptions (i)-(v) with $M = 10$. The results, shown in Figure 4.4, lead to the conclusion similar to the one derived above: robustness of $\widehat{lt}$ with respect to p for $C = 5$ is roughly twice lower than for $C = 1$.

Thus, the results of this subsection indicate that *a plausible reason for long lead time in cellular*



(a) $K = 2$ (b) $K = 5$ (c) $K = 10$

Figure 4.4: Robustness of $\widehat{lt}$ with respect to p as a function of p for various K and C

lines is their low robustness at or close to the knee.

4.3 Optimal release rate as a function of the desired lead time

Since the operation at the knee is undesirable, a question arises: At which point on the $\widehat{lt}(\rho)$-curve should a cellular line operate? This question can be answered by solving the following constrained optimization problem: Find the release rate $\hat{p}_0^*$ such that $\widehat{lt}$ takes the desired value and PR is maximized. A solution of this open-loop control problem is provided by the following:

Theorem 4.1 *Consider the cellular line defined by assumptions (i)-(v) with $p \in [0.9, 0.99]$ and assume that the desired lead time is lt_d. Consider the following polynomial equation with respect to $\hat{p}_0$:*

$$\hat{p}_0^{2C} - Clt_d\hat{p}_0^{C+1} + (C-2)\hat{p}_0^C + Clt_d p^C \hat{p}_0 - (C-1)p^C = 0. \tag{4.14}$$

Then,

- *this equation has at most two solutions on $(0, 1)$;*
- *if it has no solutions on $(0, 1)$, lt_d is viewed as infeasible;*
- *if it has a single solution in the interval (3.10), this solution is the optimal release rate, $\hat{p}_0^*$;*
- *if it has both solutions in the interval (3.10), the largest one is the optimal release rate, $\hat{p}_0^*$.*

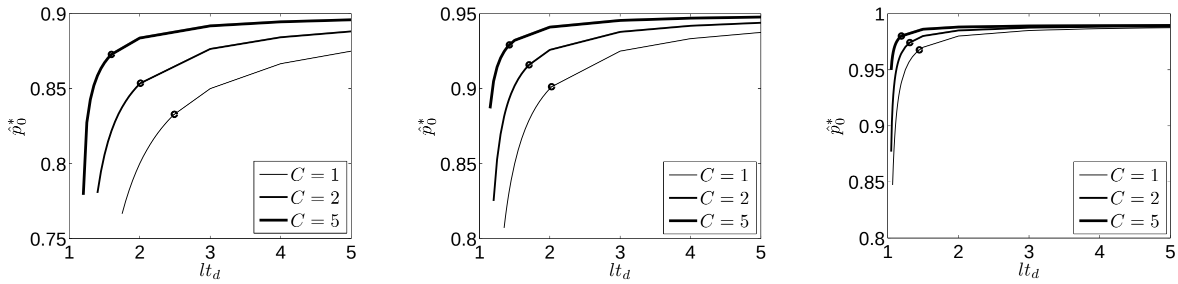
Proof: See the Appendix.

The behavior of $\hat{p}_0^*$ as a function of lt_d is illustrated in Figure 4.5, with black dots indicating $\hat{p}_0^*$ at the knee. From this figure we conclude:

- For $lt_d < \widehat{lt}_{knee}$, the optimal release rate, $\hat{p}_0^*$ (and, therefore, *PR*), is a rapidly increasing function of lt_d.

- For $lt_d > \widehat{lt}_{knee}$, $\hat{p}_0^*$ (and, therefore, *PR*) is practically constant (especially in the case of efficient producing machines and large C).

Thus, *operating beyond the knee is not only unnecessary (since PR is almost constant), but counter-productive as well (since $\widehat{WIP}$ grows without bounds).*



(a) $p = 0.9$ (b) $p = 0.95$ (c) $p = 0.99$

Figure 4.5: Optimal release rate, $\hat{p}_0^*$, as a function of lt_d for various p and C

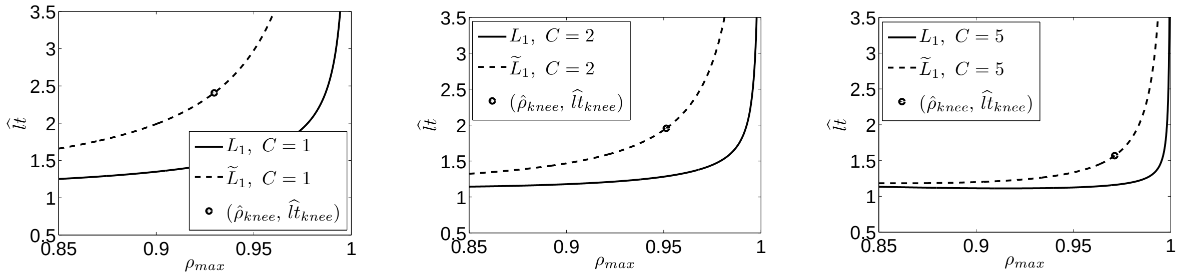
5 Analysis and Open-Loop Control of Lead Time in Cellular Lines with Non-identical Producing Machines

5.1 Structural properties

To illustrate structural behavior of cellular lines with non-identical machines, we use the modified relative workload, ρ_{max}:

$$\rho_{max} = \frac{p_0}{\min\limits_{1 \leqslant i \leqslant M} p_i}, \tag{5.1}$$

where, as before, p_0 is the release rate and p_i, $i = 1, 2, \ldots, M$, is the producing machine efficiency. The relative lead time, $\widehat{lt}$, remains the same as in (4.3), with $\widehat{LT}$ given by (3.6). Then, keeping p_i's constant, we plot $\widehat{lt}$ as a function of ρ_{max}. The results for line L_1 of (3.8) are shown in Figure 5.1 by the solid curves; the results for lines L_2 and L_3 are similar and omitted due to space limitations. Clearly, $\widehat{lt}(\rho_{max})$ has a knee similar to that in the identical machine case. Thus, all structural properties investigated in Subsection 4.1 remain the same for the non-identical machine case as well.



(a) $C = 1$ (b) $C = 2$ (c) $C = 5$

Figure 5.1: Behavior of $\widehat{lt}$ for line L_1 as a function of ρ_{max}

5.2 Robustness

For non-identical machines, robustness of $\widehat{lt}$ with respect to p_0 and p_i can be investigated in the same manner as in Subsection 4.2. However, a different approach is possible as well. Specifically, along with a cellular line with non-identical producing machines, consider an auxiliary line with identical producing machines defined by

$$p_{min} = \min_{1 \leqslant i \leqslant M} p_i. \tag{5.2}$$

For this line, results of Subsection 4.2 are applicable, and its $\widehat{lt}(\rho)$-characteristic is plotted in Figure 5.1 by broken lines, with black dots indicating the knee points. As one can see, these curves provide an upper bound of $\widehat{lt}(\rho)$, and, thus, can be used for quantifying robustness of the latter. Based on the above, we conclude that robustness of $\widehat{lt}$ in the case of non-identical machines remains low.

5.3 Optimal release rate as a function of the desired lead time

Theorem 5.2 *Consider the cellular line defined by assumptions (i)-(v) with* $p_i \in [0.9, 0.99]$, $\forall i$, *and assume that the desired lead time is* LT_d. *Consider the following polynomial equation of* *order* $(M+1)C$ *with respect to* $\hat{p}_0$*:*

$$\begin{aligned}&\hat{p}_0^C(1-\hat{p}_0^C)\sum_{i=1}^{M}\prod_{j=1,j\neq i}^{M}(p_j^C-\hat{p}_0^C)-C\hat{p}_0LT_d\prod_{i=1}^{M}(p_i^C-\hat{p}_0^C)\\&+(C-1)M\prod_{i=1}^{M}(p_i^C-\hat{p}_0^C)=0.\end{aligned}\tag{5.3}$$

Then,

- *this equation has at most two solutions on* $(0,1)$*;*
- *if it has no solutions on* $(0,1)$*,* LT_d *is viewed as infeasible;*
- *if it has a single solution in the interval (3.10), this solution is the optimal release rate,* $\hat{p}_0^*$*;*
- *if it has both solutions in the interval (3.10), the largest one is the optimal release rate,* $\hat{p}_0^*$*.*

Proof: See the Appendix.

6 Deterministic Raw Material Release

In some cases, random raw material release may be inconvenient for practical implementations. In such situations, results of Sections 4 and 5 can be used to define strategies for deterministic, e.g., hourly, release.

To model the hourly release, let the desired lead time be defined in minutes and denoted as $\mathscr{LT}_d$. Then the desired lead time in units of cycle time is given by

$$LT_d = \frac{\mathscr{LT}_d}{\tau},\tag{6.1}$$

where the cycle time, τ, is also in minutes. For example, if $\mathscr{LT}_d = 120$ min, $LT_d = 120$ if $\tau = 1$ min and $LT_d = 1200$ if $\tau = 0.1$ min. Given LT_d defined by (6.1), the corresponding release rate per cycle, $\hat{p}_0^*$, can be calculated using either (4.14) or (5.3). Then, the hourly release, $\mathscr{R}^*$, is defined as

$$\mathscr{R}^* = \lfloor HC\hat{p}_0^*(LT_d)\rfloor,\tag{6.2}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x, and H is the number of cycle times in an hour, i.e., $H = \frac{60}{\tau}$. Releasing each hour the amount of raw material defined by (6.2), leads to the following inequalities:

$$\widehat{LT}(\hat{p}_0^*) < \widehat{LT}(\mathscr{R}^*) < \widehat{LT}(\hat{p}_0^*) + H, \tag{6.3}$$

where $\widehat{LT}(\hat{p}_0^*)$ and $\widehat{LT}(\mathscr{R}^*)$ are the lead time estimates for per-cycle and per-hour release, respectively. Multiplying these inequalities by τ gives

$$\widehat{\mathscr{LT}}_d < \widehat{\mathscr{LT}}(\mathscr{R}^*) < \widehat{\mathscr{LT}}_d + 60. \tag{6.4}$$

The tightness of this bound has been evaluated by simulating cellular lines with ten cells under the hourly release for various τ and $\mathscr{LT}_d$. For each case, the simulation runs $5,000,000$ time slots (including $2,500,000$ time slots of warm-up period) and for 20 repetitions. The results are shown in Tables 6.1 and 6.2 for identical machine case and non-identical machine case, respectively. These results indicate that, if $\mathscr{LT}_d < 60$ min, $\widehat{\mathscr{LT}}(\mathscr{R}^*)$ can be evaluated using the following empirical formula:

$$\widehat{\mathscr{LT}}(\mathscr{R}^*) \approx \frac{\mathscr{LT}_d + 60}{2}. \tag{6.5}$$

Thus, hourly release leads to a relatively insignificant lead time deterioration.

Table 6.1: Lead time, $\mathscr{LT}(\mathscr{R}^*)$, under hourly release for cellular lines with identical machines

(a) $p = 0.9$

C	ρ	p_0	lt_d	τ	$\mathscr{LT}_d$	$\mathscr{LT}(\mathscr{R}^*)$
				0.5	8.97	35.62
	0.86	0.774	1.794	1	17.94	42.86
				2	35.87	57.39
				0.5	10.05	35.62
1	0.89	0.801	2.010	1	20.10	42.86
				2	40.20	57.40
				0.5	11.94	35.62
	0.92	0.828	2.389	1	23.89	42.86
				2	47.78	57.39
				0.5	6.91	34.13
	0.86	0.774	1.382	1	13.82	40.08
				2	27.63	51.76
				0.5	7.38	34.13
2	0.89	0.801	1.477	1	14.77	40.08
				2	29.53	51.75
				0.5	8.25	34.13
	0.92	0.828	1.650	1	16.50	40.09
				2	33.00	51.75
				0.5	6.00	32.80
	0.86	0.774	1.199	1	11.99	38.32
				2	23.99	48.79
				0.5	6.05	32.80
5	0.89	0.801	1.210	1	12.10	38.32
				2	24.21	48.79
				0.5	6.26	32.80
	0.92	0.828	1.251	1	12.51	38.32
				2	25.0	48.79

(b) $p = 0.95$

C	ρ	p_0	lt_d	τ	$\mathscr{LT}_d$	$\mathscr{LT}(\mathscr{R}^*)$
				0.5	6.88	32.25
	0.86	0.817	1.376	1	13.76	38.14
				2	27.52	49.24
				0.5	7.39	32.25
1	0.89	0.846	1.478	1	14.78	38.14
				2	29.57	49.24
				0.5	8.29	32.25
	0.92	0.874	1.658	1	16.58	38.14
				2	33.16	49.24
				0.5	5.95	31.45
	0.86	0.817	1.190	1	11.90	36.87
				2	23.80	47.41
				0.5	6.17	31.45
2	0.89	0.846	1.234	1	12.34	36.87
				2	24.68	47.40
				0.5	6.58	31.45
	0.92	0.874	1.316	1	13.16	36.87
				2	26.33	47.41
				0.5	5.59	30.66
	0.86	0.817	1.117	1	11.17	35.92
				2	22.35	46.04
				0.5	5.58	30.66
5	0.89	0.846	1.116	1	11.16	35.92
				2	22.32	46.04
				0.5	5.66	30.66
	0.92	0.874	1.132	1	11.32	35.92
				2	22.64	46.04

(c) $p = 0.99$

C	ρ	p_0	lt_d	τ	$\mathscr{LT}_d$	$\mathscr{LT}(\mathscr{R}^*)$
				0.5	5.36	29.10
	0.86	0.851	1.072	1	10.72	34.13
				2	21.44	43.92
				0.5	5.46	29.10
1	0.89	0.881	1.092	1	10.92	34.13
				2	21.84	43.92
				0.5	5.63	29.10
	0.92	0.911	1.126	1	11.26	34.13
				2	22.53	43.92
				0.5	5.23	28.97
	0.86	0.851	1.046	1	10.46	33.84
				2	20.92	43.61
				0.5	5.26	28.97
2	0.89	0.881	1.051	1	10.51	33.84
				2	21.02	43.61
				0.5	5.32	28.97
	0.92	0.911	1.065	1	10.65	33.84
				2	21.29	43.61
				0.5	5.27	28.77
	0.86	0.851	1.055	1	10.55	33.71
				2	21.10	43.43
				0.5	5.21	28.77
5	0.89	0.881	1.043	1	10.43	33.71
				2	20.85	43.43
				0.5	5.18	28.77
	0.92	0.911	1.037	1	10.37	33.71
				2	20.74	43.43

Table 6.2: Lead time, $\mathscr{LT}(\mathscr{R}^*)$, under hourly release for cellular lines with non-identical machines

(a) L_1

C	ρ_{max}	p_0	lt_d	τ	$\mathscr{LT}_d$	$\mathscr{LT}(\mathscr{R}^*)$
				0.5	6.59	32.33
	0.86	0.783	1.318	1	13.18	37.74
				2	26.36	48.57
				0.5	7.02	33.46
1	0.89	0.810	1.405	1	14.05	38.93
				2	28.10	50.03
				0.5	7.78	34.32
	0.92	0.837	1.557	1	15.57	40.24
				2	31.13	51.99
				0.5	5.81	31.71
	0.86	0.783	1.162	1	11.62	36.98
				2	23.24	47.06
				0.5	5.99	32.69
2	0.89	0.810	1.198	1	11.98	38.12
				2	23.97	48.24
				0.5	6.34	33.52
	0.92	0.837	1.267	1	12.67	38.97
				2	25.35	49.54
				0.5	5.53	31.13
	0.86	0.783	1.105	1	11.05	36.20
				2	22.11	46.21
				0.5	5.51	32.01
5	0.89	0.810	1.102	1	11.02	37.09
				2	22.03	47.11
				0.5	5.57	32.95
	0.92	0.837	1.113	1	11.13	38.09
				2	22.26	48.01

(b) L_2

C	ρ_{max}	p_0	lt_d	τ	$\mathscr{LT}_d$	$\mathscr{LT}(\mathscr{R}^*)$
				0.5	7.67	34.03
	0.86	0.783	1.533	1	15.33	40.02
				2	30.67	52.15
				0.5	8.39	35.24
1	0.89	0.810	1.679	1	16.79	41.47
				2	33.58	54.51
				0.5	9.67	36.26
	0.92	0.837	1.934	1	19.34	43.56
				2	38.67	58.37
				0.5	6.31	32.98
	0.86	0.783	1.262	1	12.62	38.60
				2	25.25	49.15
				0.5	6.63	33.99
2	0.89	0.810	1.326	1	13.26	39.80
				2	26.51	50.58
				0.5	7.21	34.86
	0.92	0.837	1.443	1	14.43	40.82
				2	28.85	52.65
				0.5	5.74	32.00
	0.86	0.783	1.148	1	11.48	37.29
				2	22.97	47.54
				0.5	5.76	32.91
5	0.89	0.810	1.152	1	11.52	38.21
				2	23.04	48.48
				0.5	5.89	33.86
	0.92	0.837	1.177	1	11.77	39.24
				2	23.55	49.46

(c) L_3

C	ρ_{max}	p_0	lt_d	τ	$\mathscr{LT}_d$	$\mathscr{LT}(\mathscr{R}^*)$
				0.5	6.24	32.00
	0.86	0.791	1.249	1	12.49	37.51
				2	24.97	47.44
				0.5	6.58	33.12
1	0.89	0.819	1.316	1	13.16	38.68
				2	26.33	48.74
				0.5	7.18	33.96
	0.92	0.846	1.435	1	14.35	39.29
				2	28.70	50.32
				0.5	5.65	31.52
	0.86	0.791	1.129	1	11.29	36.68
				2	22.59	46.89
				0.5	5.78	32.49
2	0.89	0.819	1.157	1	11.57	37.80
				2	23.14	48.05
				0.5	6.05	33.45
	0.92	0.846	1.210	1	12.10	38.64
				2	24.21	48.66
				0.5	5.46	31.06
	0.86	0.791	1.091	1	10.91	36.13
				2	21.82	45.95
				0.5	5.42	31.99
5	0.89	0.819	1.085	1	10.85	37.01
				2	21.70	46.84
				0.5	5.46	32.86
	0.92	0.846	1.092	1	10.92	37.89
				2	21.83	47.73

7 Closed-Loop Control

7.1 Scenario

As stated in Theorems 4.1 and 5.2, releasing raw material with rate $\hat{p}_0^*$ ensures the desired lead time for a given p (in the identical machine case) or given $p_1, p_2, \ldots, p_M$ (in the non-identical machine case). However, in reality, machine efficiency may be different from nominally assumed for $\hat{p}_0^*$ calculation. To model this situation, suppose that in the nominally identical machine case, the real machine efficiency is as follows:

$$p_i^{real} = \begin{cases} [0.8, 0.95],\ i = 1, 2, \ldots, M, & \text{for } p = 0.9, \\ [0.85, 0.99],\ i = 1, 2, \ldots, M, & \text{for } p = 0.95, \\ [0.9, 0.99],\ i = 1, 2, \ldots, M, & \text{for } p = 0.99. \end{cases} \tag{7.1}$$

For the non-identical machine case, we assume that

$$p_i^{real} = [0.8, 0.95],\ i = 1, 2, \ldots, M. \tag{7.2}$$

Under these conditions, releasing raw material according to $\hat{p}_0^*$ may result in LT^{real}, which is dramatically different from LT_d, especially if it turns out that $\hat{p}_0^* > \min_{1 \leqslant i \leqslant M} p_i^{real}$.

To avoid this undesirable behavior, closed-loop control of raw material release can be used. In this section, we introduce a simple feedback release control law and investigate its performance for cellular lines with identical and non-identical machines.

7.2 Control law

Although a number of control policies could be considered, we use here a relay-type law given by

$$p_0(n+1) = \begin{cases} \hat{p}_0^*, & \text{if } WIP(n) \leqslant C\hat{p}_0^* LT_d, \\ 0, & \text{otherwise}, \end{cases} \tag{7.3}$$
$$n = 0, 1, \ldots,$$

where n is the index of time slot, $p_0(n+1)$ is the release rate in time slot $n+1$, $\hat{p}_0^*$ is the optimal release rate calculated according to Theorems 4.1 or 5.2 (using the nominal machine efficiencies), and $WIP(n)$ is the work-in-process in the system at time slot n. In other words, according to (7.3), raw material is released with the nominal rate $\hat{p}_0^*$, if the total work-in-process in the system is not above that defined by the release rate that guarantees LT_d; otherwise no release takes place. Note that the information on $WIP(n)$ can be maintained by counting parts released into and from the system.

For the case of hourly release, (7.3) is modified as follows:

$$\mathscr{R}(s+1) = \begin{cases} \left\lfloor HC\hat{p}_0^* \right\rfloor, & \text{if } WIP(s) \leqslant C\hat{p}_0^* LT_d, \\ 0, & \text{otherwise}, \end{cases} \tag{7.4}$$
$$s = 0, 1, \ldots,$$

where s is the index of hour, $\mathscr{R}(s+1)$ is the number of parts released into the system at the beginning of hour $s+1$, H is the number of time slots in an hour, $\lfloor x \rfloor$, as before, is the largest integer not greater than x, and $WIP(s)$ is the number of parts in the system at the end of hour s.

The performance of these laws, for both identical and non-identical machines, has been analyzed by simulations. The results are described below.

7.3 Performance

7.3.1 Identical machine case

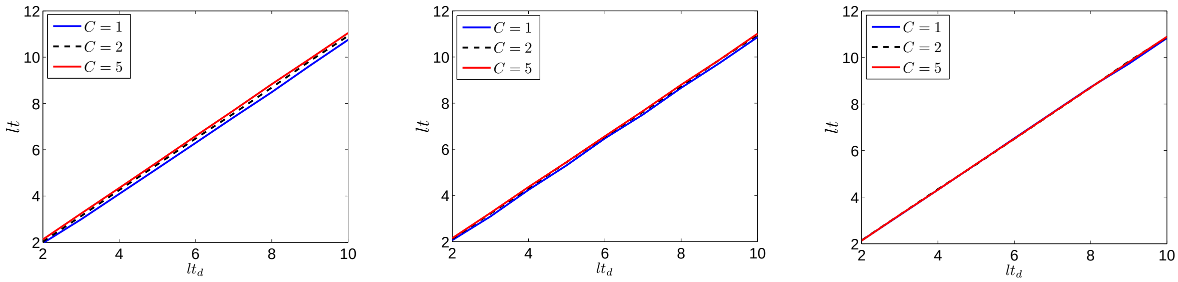
We investigated control law (7.3) for cellular lines with 10 producing cells. The values of p and lt_d have been selected from the following sets

$$p = \{0.9, 0.95, 0.99\}, \; lt_d = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}, \tag{7.5}$$

and the actual machine efficiencies have been determined by selecting p_i randomly and equiprobably from intervals (7.1). For each pair (p, lt_d) from (7.5), 100 production lines have been generated and the relative lead times analyzed by simulation. For each production line, the simulation runs

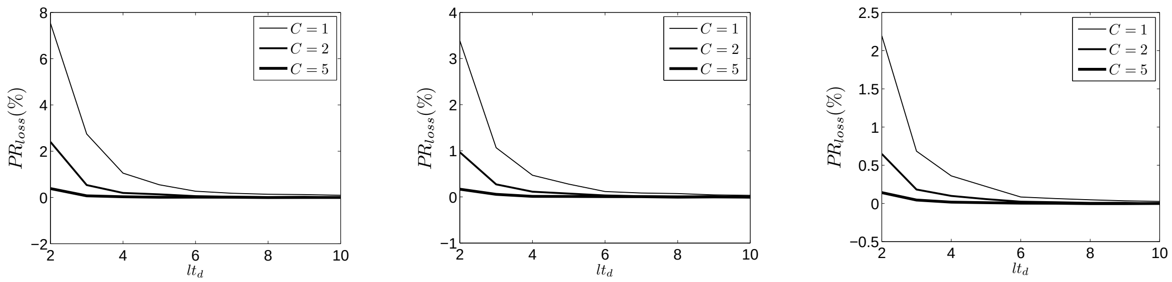
50, 000 time slots (including 25, 000 time slots of warm-up period) and for 20 repetitions.

For control law (7.3), the average relative lead times and production losses due to feedback release are shown in Figs. 7.1 and 7.2, respectively. As one can see, the relative lead time, lt, is close to the desired lead time, lt_d, i.e., the feedback enforces the desired lead time performance. The production losses, quantified by $PR_{loss} = \frac{PR_{FB}-PR_{noFB}}{PR_{noFB}} \times 100\%$ (where PR_{FB} and PR_{noFB} are the production rates of the system with and without feedback control, respectively), is a decreasing function of lt_d; for $lt_d \geqslant 6$, $PR_{loss} \approx 0$. As for control law (7.4), the average relative lead times and production losses due to feedback release are shown in Figs. 7.3 and 7.4, from which we observe that the relative lead time may be up to 2.5 times of the desired lead time for small lt_d and close to it for large lt_d; the production loss is also decreasing and goes to zero as lt_d increases.



(a) $p = 0.9$ (b) $p = 0.95$ (c) $p = 0.99$

Figure 7.1: Relative lt of per-cycle release with feedback control (identical machine case)

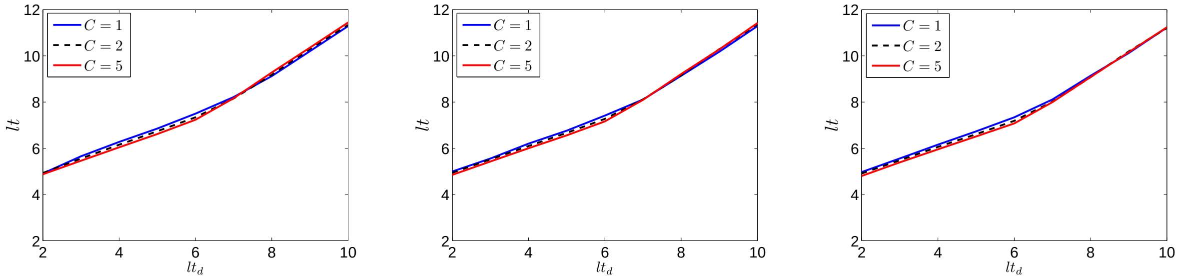


(a) $p = 0.9$ (b) $p = 0.95$ (c) $p = 0.99$

Figure 7.2: Production loss of per-cycle release with feedback control (identical machine case)

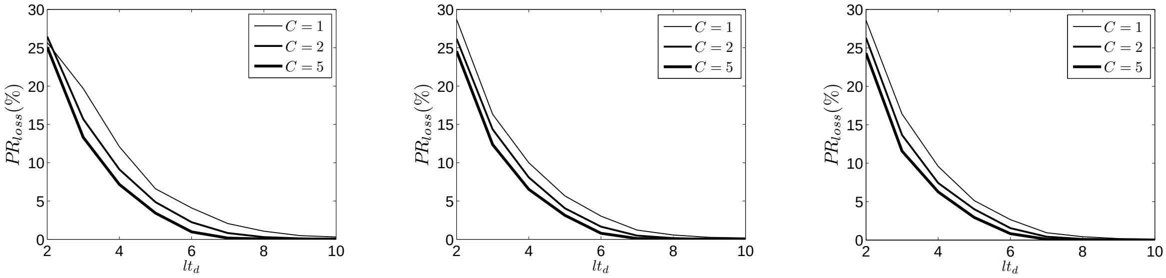
7.3.2 Non-identical machine case

We investigated control laws (7.3) and (7.4) using the three nominal lines given in (3.8). The actual machine efficiencies have been determined by selecting p_i randomly and equiprobably from



(a) p = 0.9 (b) p = 0.95 (c) p = 0.99

Figure 7.3: Relative *lt* of per-hour release with feedback control (identical machine case)



(a) p = 0.9 (b) p = 0.95 (c) p = 0.99

Figure 7.4: Production loss of per-hour release with feedback control (identical machine case)

the interval (7.2). The simulation procedure has been the same as in Subsection 7.3.1. The results obtained are similar to the identical machine case and are omitted due to space constraints.

Thus, *feedback control laws (7.3) and (7.4) can be used as a means to combat uncertainty in the producing machine efficiency for both identical and non-identical machine cases*.

8 Conclusions and Future Work

This paper provides a plausible explanation for excessively long lead time often encountered in cellular lines. Specifically, it shows that lead time as a function of the relative load on the system has a knee-type behavior, and operation at or close to the knee (which is desirable from the point of view of the production rate) has low robustness in systems with a large number of machines per cell. The problem is exacerbated for operating points beyond the knee, where the production rate remains practically constant, but the lead time grows without bounds. To avoid this undesirable behavior, the paper offers a means for calculating the optimal release rate that ensures the desired lead time, while maximizing the production rate, and offers a feedback control law to maintain the desired lead time.

Numerous problems, however, remain open. Some of them are as follows:

- Extension of the results obtained to system with Bernoulli machines, which are not necessarily asymptotically reliable (i.e., having their efficiency on $(0, 1)$).
- Extension of the results obtained to system with machines obeying the exponential reliability model.
- Extension of the results obtained to system with non-Markovian machines, e.g., having Weibull, gamma, log-normal, etc., reliability models.
- Investigation of lead time in cellular assembly systems.
- Investigation of lead time in re-entrant lines, where problems with excessive lead time are notoriously complex.
- Last but not least, application of the results obtained in practical systems.

Solutions of these problems will lead to a relatively complete theory of lead time for cellular production systems.

Appendix

Proof of Theorem 3.1: To prove the first statement of the theorem, we observe that, since the buffer is infinite, the production rate of the system equals the production rate of the release cell.

Therefore,

$$
\begin{aligned}
PR &= \sum_{i=0}^{C} i \binom{C}{i} p_0^i (1-p_0)^{C-i} \\
&= \sum_{i=1}^{C} i \frac{C!}{i!(C-i)!} p_0^i (1-p_0)^{C-i} \\
&= Cp_0 \sum_{i=1}^{C} \frac{(C-1)!}{(i-1)!(C-i)!} p_0^{i-1} (1-p_0)^{C-i} \\
&= Cp_0 \sum_{i=0}^{C-1} \frac{(C-1)!}{i!(C-1-i)!} p_0^{i} (1-p_0)^{C-1-i} \\
&= Cp_0.
\end{aligned} \tag{A.1}
$$

To prove the second statement, note that, since $p = 1 - \varepsilon k$,

$$
p^i = (1-\varepsilon k)^i = 1 - \varepsilon k i + O(\varepsilon^2). \tag{A.2}
$$

Thus,

$$
\begin{aligned}
p^i &= 1 - \varepsilon k i + O(\varepsilon^2) \\
&= 1 - \varepsilon k j + O(\varepsilon^2) - \varepsilon k (i-j) \\
&= p^j - \varepsilon k (i-j) + O(\varepsilon^2),\ 0 \leqslant j < i.
\end{aligned} \tag{A.3}
$$

From assumption (iii), it follows that the number of machines, which are up in cell c_i, $i = 0, 1$, is j with probability $\binom{C}{j} p_i^j (1-p_i)^{C-j}$, $j = 0, 1, \ldots, C$. Specifically, the probability that $C-1$ machines are up in cell c_i, $i = 0, 1$, is

$$
\binom{C}{C-1} p_i^{C-1}(1-p_i) = Cp_i^{C-1}(1-p_i). \tag{A.4}
$$

Based on (A.3), we have

$$
p_i^{C-1} = p_i^j - \varepsilon k_i (C-1-j) + O(\varepsilon^2),\ 0 \leqslant j < C-1, \tag{A.5}
$$

which implies

$$
Cp_i^{C-1} = p_i^{C-1} + \sum_{j=0}^{C-2} \left(p_i^j - \varepsilon k_i (C-1-j) + O(\varepsilon^2) \right) \tag{A.6}
$$

$$= \sum_{j=0}^{C-1} p_i^j - \varepsilon k_i \frac{C(C-1)}{2} + O(\varepsilon^2).$$

Thus,

$$\begin{aligned}
&\binom{C}{C-1} p_i^{C-1}(1-p_i) \\
&= \Big(\sum_{j=0}^{C-1} p_i^j - \varepsilon k_i \frac{C(C-1)}{2} + O(\varepsilon^2)\Big)(1-p_i) \\
&= (1-p_i)\sum_{j=0}^{C-1} p_i^j - (1-p_i)\Big(\varepsilon k_i \frac{C(C-1)}{2} + O(\varepsilon^2)\Big) \\
&= 1 - p_i^C - \varepsilon k_i\Big(\varepsilon k_i \frac{C(C-1)}{2} + O(\varepsilon^2)\Big) \\
&= 1 - p_i^C + O(\varepsilon^2).
\end{aligned} \tag{A.7}$$

Similarly,

$$\begin{aligned}
&\binom{C}{C-2} p_i^{C-2}(1-p_i)^2 = O(\varepsilon^2), \\
&\binom{C}{j} p_i^j (1-p_i)^{C-j} = o(\varepsilon^2),\ \forall j \leqslant C-3.
\end{aligned} \tag{A.8}$$

In other words, the probability that C machines are up in cell c_i, $i = 0, 1$, is p_i^C; $C-1$ machines are up with probability $1 - p_i^C + O(\varepsilon^2)$; and $C-2$ and $j \leqslant C-3$ machines are up with probability $O(\varepsilon^2)$ and $o(\varepsilon^2)$, respectively.

Let P_{ij} denote the transition probability of buffer occupancy from j to i. Then, using the above arguments,

$$\begin{aligned}
P_{C-1,j} &= 1 - p_0^C + O(\varepsilon^2),\ j = 0, 1, \ldots, C-1, \\
P_{C,j} &= p_0^C + O(\varepsilon^2),\ j = 0, 1, \ldots, C-1, \\
P_{ii} &= p_0^C p_1^C + (1-p_0^C)(1-p_1^C) + O(\varepsilon^2),\ i \geqslant C, \\
P_{i-1,i} &= (1-p_0^C)p_1^C + O(\varepsilon^2),\ i \geqslant C, \\
P_{i+1,i} &= p_0^C(1-p_1^C) + O(\varepsilon^2),\ i \geqslant C, \\
P_{ij} &= 0 \text{ or } O(\varepsilon^2) \text{ or } o(\varepsilon^2),\ \text{for other } i \text{ and } j.
\end{aligned} \tag{A.9}$$

Let $\hat{P}_{ij}$ denote the estimate of P_{ij} defined by

$$\begin{aligned}
\hat{P}_{C-1,j} &= 1 - p_0^C,\ j = 0, 1, \ldots, C-1, \\
\hat{P}_{C,j} &= p_0^C,\ j = 0, 1, \ldots, C-1, \\
\hat{P}_{ii} &= p_0^C p_1^C + (1 - p_0^C)(1 - p_1^C),\ i \geqslant C, \\
\hat{P}_{i-1,i} &= (1 - p_0^C) p_1^C,\ i \geqslant C, \\
\hat{P}_{i+1,i} &= p_0^C (1 - p_1^C),\ i \geqslant C, \\
\hat{P}_{ij} &= 0,\ \text{for other } i \text{ and } j.
\end{aligned} \tag{A.10}$$

Based on the perturbation theory [24],

$$\left|\hat{P}_{ij} - P_{ij}\right| = O(\varepsilon^2),\ i, j \in \{0, 1, \ldots\}. \tag{A.11}$$

Using (A.10) and the conservation law

$$\hat{P}_i = \sum_{j=0}^{\infty} \hat{P}_{ij} \hat{P}_j,\ \forall i, \tag{A.12}$$

we obtain the following balance equations:

$$\begin{aligned}
\hat{P}_i &= 0,\ i = 0, 1, \ldots, C-2, \\
\hat{P}_{C-1} &= (1 - p_0^C) \hat{P}_{C-1} + (1 - p_0^C) p_1^C \hat{P}_C,
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\hat{P}_C &= p_0^C \hat{P}_{C-1} + [p_0^C p_1^C + (1 - p_0^C)(1 - p_1^C)] \hat{P}_C \\
&\quad + (1 - p_0^C) p_1^C \hat{P}_{C+1}, \\
\hat{P}_{i+1} &= p_0^C (1 - p_1^C) \hat{P}_i + [p_0^C p_1^C + (1 - p_0^C)(1 - p_1^C)] \hat{P}_{i+1} \\
&\quad + (1 - p_0^C) p_1^C \hat{P}_{i+2},\ i \geqslant C.
\end{aligned}$$

The last three rows of the above equations can be re-written as follows:

$$\hat{P}_C = \frac{p_0^C}{p_1^C (1 - p_0^C)} \hat{P}_{C-1},\ \hat{P}_{i+1} = \frac{p_0^C (1 - p_1^C)}{p_1^C (1 - p_0^C)} \hat{P}_i,\ i \geqslant C. \tag{A.14}$$

To obtain the solution, let

$$\alpha = \frac{p_0^C(1 - p_1^C)}{p_1^C(1 - p_0^C)}. \tag{A.15}$$

Then,

$$\hat{P}_i = \frac{\alpha^{i-C+1}}{1 - p_1^C}\hat{P}_{C-1},\ i \geqslant C. \tag{A.16}$$

Taking into account

$$\sum_{i=0}^{\infty} \hat{P}_i = 1 \tag{A.17}$$

and $\hat{P}_i = 0, \forall i = 0, 1, \dots, C - 2$ (see the first row of (A.13)), we obtain

$$\hat{P}_{C-1} + \sum_{i=C}^{\infty} \frac{\alpha^{i-C+1}}{1 - p_1^C}\hat{P}_{C-1} = 1. \tag{A.18}$$

Since $p_0 < p_1$ (implying that $\alpha < 1$), we have

$$\hat{P}_{C-1} = \frac{1}{1 + \frac{\alpha}{1-p_1^C}\frac{1}{1-\alpha}} = \frac{1}{1 + \frac{p_0^C}{p_1^C - p_0^C}} = 1 - \frac{p_0^C}{p_1^C}. \tag{A.19}$$

Therefore,

$$
\begin{aligned}
\widehat{WIP} &= \sum_{i=C-1}^{\infty} i\hat{P}_i \\
&= (C-1)\Big(1-\frac{p_0^C}{p_1^C}\Big) + \sum_{i=C}^{\infty} i\frac{\alpha^{i-C+1}}{1-p_1^C}\Big(1-\frac{p_0^C}{p_1^C}\Big) \\
&= (C-1)\Big(1-\frac{p_0^C}{p_1^C}\Big) + \frac{\alpha^{-C+1}}{1-p_1^C}\Big(1-\frac{p_0^C}{p_1^C}\Big)\sum_{i=C}^{\infty} i\alpha^i \\
&= (C-1)\Big(1-\frac{p_0^C}{p_1^C}\Big) \\
&\quad + \frac{\alpha^{-C+1}}{1-p_1^C}\Big(1-\frac{p_0^C}{p_1^C}\Big)\Big[\frac{C\alpha^C}{1-\alpha} + \frac{\alpha^{C+1}}{(1-\alpha)^2}\Big] \\
&= (C-1)\Big(1-\frac{p_0^C}{p_1^C}\Big) \\
&\quad + \frac{1}{1-p_1^C}\Big(1-\frac{p_0^C}{p_1^C}\Big)\Big[\frac{C\alpha}{1-\alpha} + \Big(\frac{\alpha}{1-\alpha}\Big)^2\Big] \\
&= (C-1)\Big(1-\frac{p_0^C}{p_1^C}\Big) + \frac{1}{1-p_1^C}\frac{p_1^C-p_0^C}{p_1^C} \\
&\quad \times\Big[C\frac{p_0^C(1-p_1^C)}{p_1^C-p_0^C} + \Big(\frac{p_0^C(1-p_1^C)}{p_1^C-p_0^C}\Big)^2\Big] \\
&= (C-1)\Big(1-\frac{p_0^C}{p_1^C}\Big) + C\frac{p_0^C}{p_1^C} + \frac{p_0^{2C}(1-p_1^C)}{p_1^C(p_1^C-p_0^C)} \\
&= C-1+\frac{p_0^C}{p_1^C} + \frac{p_0^{2C}(1-p_1^C)}{p_1^C(p_1^C-p_0^C)} \\
&= C-1+\frac{p_0^C(1-p_0^C)}{p_1^C-p_0^C},
\end{aligned}
\tag{A.20}
$$

which coincides with (3.2) (having p_1 denoted as p). Using the Little's law [25], we obtain

$$
\widehat{LT} = \frac{\widehat{WIP}}{PR} = \frac{C-1}{Cp_0} + \frac{p_0^{C-1}(1-p_0^C)}{C(p_1^C-p_0^C)}, \tag{A.21}
$$

and, due to (A.11),

$$
\begin{aligned}
\Big|\widehat{WIP} - WIP\Big| &= O(\varepsilon^2), \\
\Big|\widehat{LT} - LT\Big| &= O(\varepsilon^2).
\end{aligned}
\tag{A.22}
$$

■

To prove Theorem 4.1, we need the following lemma:

Lemma A.1 *Function $\widehat{lt}(\rho)$ defined by (4.4) is convex.*

Proof: First, we re-write (4.4) as follows:

$$\widehat{lt}(\rho) = \frac{(C-1)p^{-1}}{C\rho} + \frac{(p\rho)^{C-1}}{C} + \frac{(p\rho)^{C-1}(p^{-C}-1)}{C(1-\rho^C)}. \tag{A.23}$$

If all three terms of (A.23) are convex, then $\widehat{lt}(\rho)$ is also convex (see [26]). Clearly, the first two terms are convex. So, what remains to prove is that

$$f(\rho) := \frac{(p\rho)^{C-1}(p^{-C}-1)}{C(1-\rho^C)} \tag{A.24}$$

is also convex. For $C = 1$,

$$f(\rho) = \frac{p^{-1}-1}{1-\rho}, \tag{A.25}$$

which is convex. For $C > 1$, let

$$g(\rho) := \rho^{C-1}, \tag{A.26}$$

$$h(x) := \frac{p^{C-1}x(p^{-C}-1)}{1-x^{\frac{C}{C-1}}} \tag{A.27}$$

and, therefore,

$$f(\rho) = h(g(\rho)). \tag{A.28}$$

In the following, we prove that $h(x)$ is convex, i.e., for all $0 < x_1, x_2 < 1$,

$$\frac{h(x_1)+h(x_2)}{2} \geqslant h\Big(\frac{x_1+x_2}{2}\Big). \tag{A.29}$$

From (A.27), we have

$$
\begin{aligned}
& h(x_1)+h(x_2)-2h\Big(\frac{x_1+x_2}{2}\Big) \\
&= (p^{-1}-p^{C-1})\Bigg[\frac{x_1}{1-x_1^{\frac{C}{C-1}}}+\frac{x_2}{1-x_2^{\frac{C}{C-1}}}-2\frac{\frac{x_1+x_2}{2}}{1-(\frac{x_1+x_2}{2})^{\frac{C}{C-1}}}\Bigg] \\
&= (p^{-1}-p^{C-1})\Bigg\{\frac{x_1\Big[x_1^{\frac{C}{C-1}}-(\frac{x_1+x_2}{2})^{\frac{C}{C-1}}\Big]}{\Big(1-x_1^{\frac{C}{C-1}}\Big)\Big[1-(\frac{x_1+x_2}{2})^{\frac{C}{C-1}}\Big]} \\
&\qquad + \frac{x_2\Big[(\frac{x_1+x_2}{2})^{\frac{C}{C-1}}-x_2^{\frac{C}{C-1}}\Big]}{\Big(1-x_2^{\frac{C}{C-1}}\Big)\Big[1-(\frac{x_1+x_2}{2})^{\frac{C}{C-1}}\Big]}\Bigg\}.
\end{aligned}
\tag{A.30}
$$

Let

$$
K=\frac{p^{-1}-p^{C-1}}{\Big(1-x_1^{\frac{C}{C-1}}\Big)\Big(1-x_2^{\frac{C}{C-1}}\Big)\Big[1-(\frac{x_1+x_2}{2})^{\frac{C}{C-1}}\Big]},
\tag{A.31}
$$

then

$$
\begin{aligned}
& h(x_1)+h(x_2)-2h\Big(\frac{x_1+x_2}{2}\Big) \\
&= K\Bigg\{x_1^{\frac{2C-1}{C-1}}+x_2^{\frac{2C-1}{C-1}}-2\Big(\frac{x_1+x_2}{2}\Big)^{\frac{2C-1}{C-1}}+x_1x_2\Big(\frac{x_1+x_2}{2}\Big) \\
&\quad \times\Big[\Big(\frac{x_1+x_2}{2}\Big)^{\frac{1}{C-1}}\Big(x_1^{\frac{1}{C-1}}+x_2^{\frac{1}{C-1}}\Big)-2(x_1x_2)^{\frac{1}{C-1}}\Big]\Bigg\}.
\end{aligned}
\tag{A.32}
$$

Since $\frac{2C-1}{C-1}>2$, $x^{\frac{2C-1}{C-1}}$ is convex. Thus, for any $x_1, x_2>0$,

$$
x_1^{\frac{2C-1}{C-1}}+x_2^{\frac{2C-1}{C-1}}-2\Big(\frac{x_1+x_2}{2}\Big)^{\frac{2C-1}{C-1}}\geqslant 0.
\tag{A.33}
$$

Also, for any $x_1, x_2>0$, we have

$$
\Big(\frac{x_1+x_2}{2}\Big)^{\frac{1}{C-1}}\geqslant (x_1x_2)^{\frac{1}{2(C-1)}}
\tag{A.34}
$$

and

$$
\frac{x_1^{\frac{1}{C-1}}+x_2^{\frac{1}{C-1}}}{2}\geqslant (x_1x_2)^{\frac{1}{2(C-1)}}.
\tag{A.35}
$$

Multiplying the above two inequalities gives

$$\Big(\frac{x_1 + x_2}{2}\Big)^{\frac{1}{C-1}}\Big(x_1^{\frac{1}{C-1}} + x_2^{\frac{1}{C-1}}\Big) - 2(x_1x_2)^{\frac{1}{C-1}} \geqslant 0. \tag{A.36}$$

Based on (A.33) and (A.36) and observing that K in (A.31) is positive, we conclude that (A.29) holds and, thus, $h(x)$ is convex. Since $h(x)$ is also monotonically increasing (see (A.27)) and $g(\rho)$ is convex (see (A.26)), based on the property of the composition of convex functions [26], $f(\rho)$ is also convex, which implies that $\widehat{lt}(\rho)$ is convex. ∎

Proof of Theorem 4.1: From (4.4), we have

$$lt_d = \frac{C-1}{C\hat{p}_0} + \frac{\hat{p}_0^{C-1}(1-\hat{p}_0^C)}{C(p^C - \hat{p}_0^C)}. \tag{A.37}$$

Multiplying both sides by $C\hat{p}_0(p^C - \hat{p}_0^C)$ gives (4.14). Since from Lemma A.1, $\widehat{lt}(\rho)$ is convex, $\widehat{lt}(p_0)$ is also convex for fixed $p \in (0, 1)$. Thus, (4.14) has at most two solutions on $(0, 1)$, which proves the first statement of the theorem. If lt_d is smaller than the minimum of $\widehat{lt}(p_0)$, (4.14) has no solutions on $(0, 1)$ and thus, lt_d is infeasible, which proves the second statement.

If (4.14) has a unique solution on $(0, 1)$ and this solution is in the interval (3.10) or (4.14) has two solutions on $(0, 1)$ and only one is in the interval (3.10), then this solution is the optimal release rate, $\hat{p}_0^*$, which proves the third statement.

If (4.14) has two solutions in the interval of (3.10), then the largest of them is the optimal release rate, $\hat{p}_0^*$, (due to (3.7)), which completes the proof of the theorem. ∎

Proof of Theorem 5.2: From (3.6), we have

$$\mathcal{LT}_d = \sum_{i=1}^{M}\Big[\frac{C-1}{C\hat{p}_0} + \frac{\hat{p}_0^{C-1}(1-\hat{p}_0^C)}{C(p_i^C - \hat{p}_0^C)}\Big]. \tag{A.38}$$

Multiplying both sides by $C\hat{p}_0 \prod_{i=1}^{M}(p_i^C - \hat{p}_0^C)$ gives (5.3).

Similar to Lemma A.1, it can be proved that $\widehat{\mathcal{LT}}(p_0)$ is convex. Based upon this fact, Theorem 5.2 is proved in the same manner as Theorem 4.1. ∎

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