Efficiency Analysis of Series Machines under Operation-Dependent-Failures

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May 31, 2017

Abstract

Series machines, i.e., machines connected in series with no buffering, are pervasive in production systems. In the analysis, design, and optimization of the series-machine system, the efficiency analysis is one of the most fundamental issues. There are not a lot of researches analyzing the efficiency of the series-machine system under the assumption of Operation-Dependent-Failures (ODF) mechanism and almost all of them assumed that machines break down in terms of ODF-I (i.e., the breakdown of a machine could make all other series machines forced down) rather than under ODF-II (i.e., the breakdown of a machine does not make any other series machines forced down). The reason that ODF-I is usually assumed in the literature is that the analysis of the series-machine system under ODF-II is much more complex than under ODF-I, although ODF-II might be more common in practice. To systematically analyze the efficiency of the series-machine system, in this paper, we propose an analytical method to investigate the efficiency under both ODF-I and ODF-II failure mechanisms. Different from under ODF-I, analytical expressions of the efficiency of series-machine systems under ODF-II are hard to obtain and thus, limit bounds of the efficiency are derived and an algorithm is developed to calculate its exact value. Results show that the series-machine system under ODF-II is more efficient than under ODF-I, which, intuitively making sense, is the reason that ODF-II are more common in the industry.

Keywords: Efficiency, Series machines, Operation-Dependent-Failures, Production lines.

1 Introduction

Machines connected in series with no buffering are pervasive in production systems, e.g., in general assembly systems of the automotive industry [1]. This configuration is called series-machine system, which is shown in Figure 1.1, where circles represent machines $m_1, m_2, \ldots, m_N$. In general, series machines are connected without buffering in the plant mainly due to two considerations: 1) buffers may be very expensive, e.g., they take up huge and costly space in the automotive industry; 2) although buffers could improve the throughput, they cause higher work-in-process and longer production lead time, which conflicts the leanness concept in the production systems. Thus, series-machine systems are inevitable and important in the production industry.

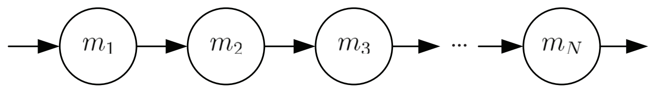

Figure 1.1: A series-machine system with N machines

In production systems, machines are usually unreliable, i.e., they break down randomly during the production process and the repair time is also random. In the literature, the failure mechanism of a machine could be either Time-Dependent-Failures (TDF) or Operation-Dependent-Failures (ODF). Their difference is that, under ODF, a machine can break down only when it is operational (i.e., up and busy); while under TDF, it could break down as well when it is idle (i.e., up but forced down). More details of ODF and TDF can be found in monographs [2]-[4] and survey [5]. Since there are no intermediate buffers between the series machines, for each of the above two failure mechanisms, the breakdown of a machine could either make all the other series machines forced down or not make any series machines forced down, which we call type-I and type-II mechanisms, respectively. Thus, we have four combinations of failure mechanisms for the series-machine system and call them TDF-I, TDF-II, and ODF-I, ODF-II. In this paper, we concentrate on ODF-I and ODF-II mechanisms. Differences of ODF-I and ODF-II failure mechanisms are illustrated by exemplifying two series machines m_1 and m_2 in Figure 1.2, where cycle times of machines are deterministic and identical. Note that these two machines are synchronous (i.e., jobs do not move to the next machine until all of them are completed on the current machine).

The efficiency is one of the most important measures of the series-machine system. It indicates

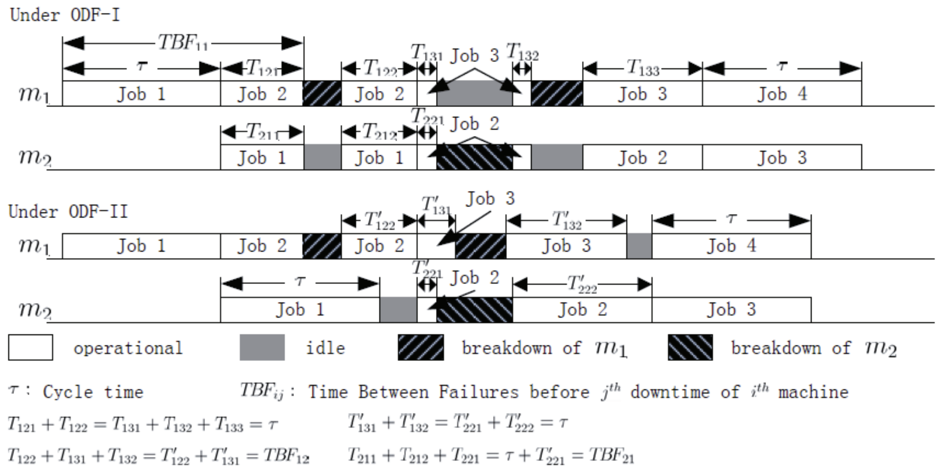

Figure 1.2: Illustration of ODF-I and ODF-II on two series machines

the extent to which machines' downtimes impact on the throughput of the series-machine system. Since it is closely related to the throughput, the efficiency plays a key role in analysis, design, and optimization of production lines such as performance analysis of lines with stages consisting of series machines [1], buffer allocation [6], stage aggregation [7], and performance analysis and optimization of production systems obtained by stage aggregation [8, 9], etc. Clearly, if the efficiency is not accurately estimated, errors of the throughput and other performance measures may be large and unacceptable, which would lead to totally wrong conclusions and/or wrong decisions in analysis, design, and optimization of production lines. Therefore, it is critical to analyze the efficiency of the series-machine systems.

In the literature, Buzacott initiated the efficiency analysis of the series-machine system in the seminal work [10] and [11]. However, almost all existing researches focus on the efficiency analysis of series-machine systems under type-I failure mechanisms (i.e., ODF-I and TDF-I). The reason is that, on the basis of some assumptions, the efficiency of the series-machine system under type-II failure mechanisms (i.e., ODF-II and TDF-II) is approximated in practice by the efficiency under corresponding type-I mechanisms. One of these assumptions usually used is that in the series-machine system, only one machine could break down at any time (see [11] and monographs [2,3]). Another assumption for the approximation analysis is that the machine cycle time is much shorter than the mean time between failures [12], [13]. In addition to the approximation analysis, another reason that existing literature on type-II failure mechanisms is very limited is the complexity of efficiency analysis under this type of failure mechanisms. Although it is easy to analyze the efficiency of the series-machine system under type-I failure mechanisms, it is very difficult under type-II mechanisms. In [14], a bound analysis approach and an approximation approach were developed to analyze the throughput of the series-machine system under ODF-II. In the bound analysis approach, a numerical method was proposed to calculate the lower and upper bounds of the expectation of the cycle completion time. Based on this method, the gap between the lower and upper bounds could be as small as possible. In [2], a similar concept for the efficiency analysis under ODF-II was also mentioned. In [7], an estimate of the efficiency under ODF-II was obtained by an approximate approach.

Although type-I failure mechanisms are used for performance analysis, series-machine systems usually operate under type-II failure mechanisms because these mechanisms make more sense and are intuitively, more efficient. However, the accuracy of approximating type-II failure mechanisms by type-I may be unacceptable for efficiency analysis, especially when there are a lot of series machines and efficiencies of the machines are not close to 1 (which will be demonstrated in Section 5). Thus, this paper intends to analyze the efficiency of the series-machine system under type-II failure mechanism. Due to space limitations, we focus on the analysis under ODF-II.

In this paper, we propose an approach to investigate the efficiency of the series-machine system under ODF mechanisms. It can also be used to analyze the series-machine systems under TDF mechanisms. As in the literature, for ODF-I, we obtain closed-form expressions of the efficiency. While for ODF-II, it is very hard to derive closed-form expressions. Thus, we derive limit bounds of the efficiency analytically and develop an algorithm to numerically calculate the efficiency under ODF-II.

The main contribution of this paper is as follows: it analyzes limit bounds of the efficiency of the series-machine system under ODF-II and develops an algorithm to calculate the efficiency; it preliminarily investigates some properties of the efficiency under ODF-II, which may provide some insights for deriving closed-form analytical expressions of the efficiency.

The remainder of the paper is organized as follows: In Section 2, the series-machine system investigated in this paper is modeled and an approach for efficiency analysis is proposed. Based on this approach, the efficiency of the series-machine system under ODF-I is analyzed. The efficiency under ODF-II is analyzed in Section 3, where limit bounds of the efficiency are derived. In Section 4, an algorithm for estimating the efficiency under ODF-II is developed and based on the algorithm, the limit bounds of the efficiency are numerically verified. Discussions and insights are presented in Section 5. The conclusions and topics for future work are addressed in Section 6. Proofs of theorems are provided in the Appendix.

2 Modeling of Series-Machine System and Approach to Efficiency Analysis

In Subsection 2.1, we will model the series-machine system investigated in this paper. In Subsection 2.2, an approach to analyze the efficiency of the series-machine system is proposed and based on this approach, the efficiency under ODF-I is analyzed.

2.1 Model

We make the following assumptions for the series-machine system investigated in this paper:

(i) The system consists of N machines, $m_1, m_2, \ldots, m_N$, which are connected in series without buffering.

(ii) The cycle time of machine m_i, $i = 1, 2, \ldots, N$, is denoted as τ_i.

(iii) All machines are unreliable and are characterized by the reliability model, i.e., continuous random variables that define machine's Time Between Failures (*TBF*) and Time To Repair (*TTR*). Mean values of *TBF* and *TTR* are Mean Time Between Failures (*MTBF*) and Mean Time To Repair (*MTTR*), respectively. All random variables are mutually independent. The efficiency of machine m_i, $i = 1, 2, \ldots, N$, is $e_i = \frac{MTBF_i}{MTBF_i + MTTR_i}$.

(iv) All series machines have the same failure mechanism, which is either Operation-Dependent-Failures-I (ODF-I) or Operation-Dependent-Failures-II (ODF-II).

(v) All series machines process jobs in pace with each other, i.e., at the beginning of a processing cycle, all machines start to process jobs; when all of them complete the processing, jobs are synchronously moved to the next adjacent machine.

2.2 Analysis approach

Although the efficiency of the series-machine system under ODF-I has been analyzed in the literature, e.g., [2] and [3], existing methods are not applicable to analyze the efficiency under ODF-II. In the following, we propose an approach for the efficiency analysis and verify its effectiveness under ODF-I.

Denote the maximum value of the cycle times as τ_{max}, i.e.,

$$\tau_{max} = \max_{1 \leqslant i \leqslant N} \tau_i. \tag{2.1}$$

Let Y denote the cycle completion time of the series-machine system. Since the series machines process jobs in pace with each other (see assumption (v) in Subsection 2.1), Y is the maximum completion time of the series machines. Clearly, it is a random variable because of random failures and repairs of the machines. Then, based on the definitions of throughput and capacity in [7], the throughput and capacity of the series-machine system can, respectively, be expressed as

$$TP = \frac{1}{E_Y[Y]}, \ c = \frac{1}{\tau_{max}}. \tag{2.2}$$

Thus, based on the definition of efficiency in [7], the efficiency of the series-machine system is

$$E = \frac{TP}{c} = \frac{\tau_{max}}{E_Y[Y]}. \tag{2.3}$$

To derive the efficiency of the series-machine system, we need to calculate the denominator in (2.3), i.e., the average cycle completion time $E_Y[Y]$. In the following, we verify the effectiveness of the above approach.

Under ODF-I, the breakdown of a machine will make all the other series machines forced down. In other words, there is no overlapping between the operational time of a machine and downtimes of the other machines. Since a machine could break down only when it is operational, downtimes of all machines are non-overlapping, which can be observed in Figure 1.2. Thus, the cycle completion time Y of the series-machine system can be expressed as the sum of two parts: the necessary processing time τ_{max} and the total downtime in a processing cycle of the series-machine system. For machine m_i, $i = 1, 2, \ldots, N$, the average downtime in a processing cycle is $\frac{MTTR_i}{MTBF_i}\tau_i$, where $MTBF_i$ and $MTTR_i$ are Mean Time Between Failures and Mean Time To Repair of m_i, respectively. Therefore,

$$\begin{aligned} E_Y[Y] =& \tau_{max} + \sum_{i=1}^{N} \frac{MTTR_i}{MTBF_i}\tau_i \\ =& \tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i \end{aligned} \tag{2.4}$$

and thus, based on (2.3), the efficiency of the series-machine system defined by model (i)-(v) under ODF-I is

$$E_{odf1} = \frac{\tau_{max}}{\tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i}. \tag{2.5}$$

The efficiency in (2.5) is in accordance with the result in [2] and [3], which indicates the effectiveness of the approach for efficiency analysis of the series-machine system.

3 Efficiency Analysis under Operation-Dependent-Failures-II

Using the proposed approach in Subsection 2.2, the efficiency of the series-machine system could be derived by analyzing the average cycle completion time $E_Y[Y]$. However, it is extremely hard to calculate $E_Y[Y]$ analytically under ODF-II because downtimes of machines may overlap with each other. In the following, we will present the expression of $E_Y[Y]$. Since it is difficult to calculate its exact value, we derive limit bounds for it. Based on the bounds of $E_Y[Y]$, we obtain lower and upper bounds of the efficiency of the series-machine system under ODF-II.

Inspired by the analysis method in [14], we now provide the expression of $E_Y[Y]$. Let W_i, $i = 1, 2, \ldots, N$, denote the number of failures of machine m_i in a processing cycle. (Clearly, the values of W_i's are non-negative integers.) Corresponding repair times are denoted by X_{ij},

$i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, W_i$. Thus, the completion time (i.e., the processing time and repair times) of m_i to process a job is $\tau_i + \sum_{j=1}^{W_i} X_{ij}$. Since the breakdown of a machine cannot make any other series machines forced down (see Figure 1.2), the cycle completion time of the series-machine system can be expressed as

$$Y = \max_{1 \leqslant i \leqslant N} \Big(\tau_i + \sum_{j=1}^{W_i} X_{ij}\Big). \tag{3.1}$$

In expression (3.1), the distribution of W_i, $i = 1, 2, \ldots, N$, depends on τ_i and on the distribution of *TBF* of machine m_i; *TBF* and *TTR* (i.e., X_{ij}'s) follow general continuous distributions.

Since it is hard to calculate the exact value of $E_Y[Y]$, in the following, we derive limit bounds for it. For this purpose, we first investigate the limits of $E_Y[Y]$ based on the assumption of exponential distributions of *TBF* and deterministic repair times, and then prove for general distributions of *TBF* and *TTR*, $E_Y[Y]$ is bounded by the above limits.

To derive the limits of $E_Y[Y]$, for the sake of simplicity, we assume that *TBF* of m_i, $i = 1, 2, \ldots, N$, are independent and identically distributed with an exponential distribution with parameter λ_i and repair times are deterministic. That is to say, $W_i \sim \text{Poisson}(\lambda_i \tau_i)$ and $X_{ij} \equiv \frac{1}{\mu_i}$, where failure rate $\lambda_i = \frac{1}{MTBF_i}$, repair rate $\mu_i = \frac{1}{MTTR_i}$, and $MTBF_i$ and $MTTR_i$ are Mean Time Between Failures and Mean Time To Repair of machine m_i, respectively. Thus, expression (3.1) is rewritten as

$$Y = \max_{1 \leqslant i \leqslant N} \Big(\tau_i + \frac{W_i}{\mu_i}\Big), \tag{3.2}$$

where $W_i \sim \text{Poisson}(\lambda_i \tau_i)$, $i = 1, 2, \ldots, N$, and W_i's are independent.

Having λ_i's and μ_i's go to zero or infinity while e_i fixed, we could derive the limits of $E_Y[Y]$. To do that, we choose $2N$ constants $(\lambda_1^*, \mu_1^*, \lambda_2^*, \mu_2^*, \ldots, \lambda_N^*, \mu_N^*)$ such that $\frac{\mu_i^*}{\lambda_i^* + \mu_i^*} = e_i$ and let $\lambda_i = \alpha \lambda_i^*$ and $\mu_i = \alpha \mu_i^*$, where $\alpha \in (0, +\infty)$. Now expression (3.2) becomes

$$Y_\alpha = \max_{1 \leqslant i \leqslant N} \Big(\tau_i + \frac{W_i}{\alpha \mu_i^*}\Big), \tag{3.3}$$

where $W_i \sim \text{Poisson}(\alpha \lambda_i^* \tau_i)$, $i = 1, 2, \ldots, N$. As a result, we obtain:

Proposition 3.1 *When α goes to zero (i.e., all failure and repair rates go to zero), the mean* *value of the cycle completion time Y_α in (3.3) goes to $\tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i$. Mathematically,*

$$\lim_{\alpha \to 0^+} E_{Y_\alpha}[Y_\alpha] = \tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i. \tag{3.4}$$

Proof: See the Appendix.

Proposition 3.2 *When α goes to infinity (i.e., all failure and repair rates go to infinity), the mean value of the cycle completion time Y_α in (3.3) goes to $\max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i}$ with probability one. Mathematically,*

$$\lim_{\alpha \to +\infty} E_{Y_\alpha}[Y_\alpha] = \max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i}, \ w.p.1. \tag{3.5}$$

Proof: See the Appendix.

Lemma 3.1 *The right-hand sides of equations (3.4) and (3.5) are, respectively, upper and lower bounds of the mean value of the cycle completion time Y in (3.1), i.e.,*

$$\max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i} \leqslant E_Y[Y] \leqslant \tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i. \tag{3.6}$$

Proof: See the Appendix.

Remark 3.1 *Although equations (3.4) and (3.5) are derived under the assumption of exponential distributions of time between failures and constant repair times, it is clear that the proof of Lemma 3.1 is independent of distributions of TBF and TTR. Therefore, Lemma 3.1 holds for general reliability models.*

Theorem 3.1 *The efficiency of the series-machine system defined by model (i)-(v) under ODF-II is bounded by*

$$\frac{\tau_{max}}{\tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i} \leqslant E_{odf2} \leqslant \frac{\tau_{max}}{\max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i}}. \tag{3.7}$$

Based on (2.3), it follows from Lemma 3.1 that (3.7) holds. Note that the lower bound is the efficiency of the series-machine system under ODF-I.

Remark 3.2 *From (3.7) one can observe that if for some* i', $e_{i'} \leqslant 1$ *and* $\tau_{i'} = \tau_{max}$, *while* $e_i = 1$, $\forall i = 1, 2, \ldots, N$, $i \neq i'$, *then the lower and upper bounds of* E_{odf2} *are both equal to* $e_{i'}$, *and thus, the efficiency* $E_{odf2} = e_{i'}$. *Clearly, a special case is that* $e_{i'} = 1$. *In other words, if* $e_i = 1$, $\forall i = 1, 2, \ldots, N$, *the lower and upper bounds are* 1, *and the efficiency* $E_{odf2} = 1$.

Remark 3.3 *From (3.7), we can see that* E_{odf1} *is a lower bound of* E_{odf2}. *This can be intuitively explained under the assumption of the exponential reliability model of machines: when failure rates* λ_i*'s and repair rates* μ_i*'s are close to* 0, *downtimes of different series machines hardly overlap with each other, which implies that the average cycle completion time under ODF-II is close to that under ODF-I. As for the upper bound of* E_{odf2}, *we can see that if* $\tau_i = \tau$, $i = 1, 2, \ldots, N$, *then* $E_{odf2} \leqslant \min_{1 \leqslant i \leqslant N} e_i$, *which intuitively makes sense.*

4 Numerical Verification of Limit Bounds of the Efficiency

Although Theorem 3.1 provides the lower and upper bounds, no closed-form expressions of E_{odf2} are derived so far (note that the error of the estimate in [7] may be large). To precisely estimate E_{odf2} in terms of (2.3), based on the mechanism of ODF-II, we propose a numerical method to calculate the expected cycle completion time, $E_Y[Y]$. This method is shown in Algorithm 1.

The output of the above algorithm, Y, is an estimate of $E_Y[Y]$. In this algorithm, N, as before, is the number of series machines, T the total number of processing cycles simulated, k_i the number of failures of machine m_i, and R_i^f represents m_i's residual uptime to its next failure.

Algorithm 1 can be used to verify the generalization of (3.7) for all reliability models. To do that, we numerically verify (3.6) by estimating the expected cycle completion time (i.e., $E_Y[Y]$) of the series-machine system. For this purpose, 5,000 series-machine systems are constructed with parameters selected randomly and equiprobably from the following sets:

$$\begin{aligned} & N \in [2, 10], \ \tau \in \{0.9, 0.95, 1, 1.05, 1.1\}, \\ & e \in \{0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 0.99\}, \\ & \alpha \in \{0.1, 1, 10, 100\}, \ MTTR^* \in \{0.5, 1, 1.5, 2, 2.5\}, \end{aligned} \tag{4.1}$$

Algorithm 1 Calculation of Expected Cycle Completion Time under ODF-II

$k_i \leftarrow 1, \forall i \in \{1, 2, \ldots, N\}$
$Y \leftarrow 0, t \leftarrow T$
for $i = 1 : N$ **do**
    Generate TBF_{ik_i} following the specified distribution of Time Between Failures
    Generate TTR_{ik_i} following the specified distribution of Time To Repair
    $R_i^f \leftarrow TBF_{ik_i}$
end for
while $t > 0$ **do**
    $t \leftarrow t - 1$
    for $i = 1 : N$ **do**
        $S_i^{r,new} \leftarrow 0$
        while $R_i^f < \tau_i$ **do**
            $S_i^{r,new} \leftarrow S_i^{r,new} + TTR_{ik_i}$
            $k_i \leftarrow k_i + 1$
            Generate TBF_{ik_i} following the specified distribution of Time Between Failures
            Generate TTR_{ik_i} following the specified distribution of Time To Repair
            $R_i^f \leftarrow R_i^f + TBF_{ik_i}$
        end while
        $S_i = Y + \tau_i + S_i^{r,new}, R_i^f \leftarrow R_i^f - \tau_i$
    end for
    $Y \leftarrow \max_{1 \leqslant i \leqslant N} S_i$
end while
$Y \leftarrow \frac{Y}{T}$

where α and $MTTR^*$ are used to calculate $MTTR$ in terms of $MTTR = \frac{MTTR^*}{\alpha}$.

As for the reliability models, we consider exponential distribution and Weibull, gamma, and log-normal distributions with various coefficient of variation (CV). For the sake of simplicity, distributions and CV's of all times between failures and repair times of machines in a series-machine system are assumed to be identical. The reliability model and CV are selected randomly and equiprobably from the following set:

$$\begin{aligned} &\text{Reliability model} \in \{\text{Exp, Weibull, gamma, log-normal}\}, \\ &CV \in \{0.1, 0.25, 0.5, 0.75, 1\}. \end{aligned} \tag{4.2}$$

The analysis is confined to $CV \leqslant 1$ since, most of manufacturing equipment has CV's of times between failures and repair times less than 1 (see [15] and [16]).

The expected cycle completion time of the series-machine system is estimated based on Algorithm 1 and the estimation result with 99% confidence level is denoted as $[\bar{Y}^{LB}_{odf2}, \bar{Y}^{UB}_{odf2}]$. To verify (3.6), we denote

$$LB_{odf2} = \max_{1\leqslant i\leqslant N} \frac{\tau_i}{e_i},\ UB_{odf2} = \tau_{max} + \sum_{i=1}^{N} \frac{1-e_i}{e_i}\tau_i, \tag{4.3}$$

and define

$$\epsilon = \left|\min(\bar{Y}^{LB}_{odf2} - LB_{odf2}, 0) + \min(UB_{odf2} - \bar{Y}^{UB}_{odf2}, 0)\right|. \tag{4.4}$$

Clearly, if $\epsilon = 0$, then the interval $[\bar{Y}^{LB}_{odf2}, \bar{Y}^{UB}_{odf2}]$ locates in $[LB_{odf2}, UB_{odf2}]$, which implies that $E_Y[Y]$ is in $[LB_{odf2}, UB_{odf2}]$ with a probability greater than 99%; if ϵ is very small, say, less than 4×10^{-5}, then either $\bar{Y}^{LB}_{odf2}$ is a little less than LB_{odf2} or $\bar{Y}^{UB}_{odf2}$ a little greater than UB_{odf2}, which implies that $E_Y[Y]$ is in $[LB_{odf2}, UB_{odf2}]$ with a probability close to 99%. Taking into consideration that the estimate based on simulations is noisy, we regard $E_Y[Y]$ locates in $[LB_{odf2}, UB_{odf2}]$ (in other words, (3.6) holds) if

$$\epsilon < 4 \times 10^{-5}. \tag{4.5}$$

The expected cycle completion time of these constructed $5,000$ series-machine systems has been estimated using the following procedure: 20 repetitions of Algorithm 1 were carried out and for each repetition, the simulation run for $200,000,000$ processing cycles (i.e., T in Algorithm 1 was $200,000,000$). As a result, we have:

Numerical Fact 4.1 *For all* $5,000$ *series-machine systems constructed in terms of (4.1) and (4.2),* $\epsilon = 0$ *for* 4970 *of the systems; for the other* 30 *systems,* $\epsilon < 4 \times 10^{-5}$. *Thus, (3.6) holds for all* $5,000$ *systems analyzed, which implies (3.7) holds.*

5 Discussions and Insights

In this section, we compare the efficiency of the series-machine system under ODF-I and ODF-II and get some insights from the efficiency analysis.

From (2.5), we can see that under ODF-I, the efficiency of the series-machine system does not depend on failure and repair rates. In other words, for a given N, if cycle times and efficiencies of machines are fixed, the efficiency of the series-machine system under ODF-I is a constant.

However, this conclusion does not hold for ODF-II (see (3.4) and (3.5)). Although the exact value of the real efficiency under ODF-II is hard to obtain, Theorem 3.1 indicates that

$$E_{odf1} \leqslant E_{odf2}. \tag{5.1}$$

Inequality (5.1) indicates that, E_{odf1} is a lower bound of E_{odf2}. Since it is hard to analytically calculate E_{odf2}, in practical applications, E_{odf1} is usually used to approximate it. The accuracy of this approximation is very high for systems with small N, large relative $MTTR$ (i.e., large $\frac{MTTR}{\tau}$), and e close to 1, which could be observed in Table 5.1 (see the case that $N = 2$, $e = 0.99$, $MTTR = 5$), where E_{odf2} is calculated based on Algorithm 1. However, for systems with large N and small e and relative $MTTR$, the approximation error could be as large as 49.93% (see the case that $N = 8$, $e = 0.7$, $MTTR = 0.25$). Based on the results in Table 5.1, one can expect that the approximation accuracy is much worse for larger N and smaller e and relative $MTTR$. Thus, we should carefully use the efficiency approximation in performance analysis and continuous improvement of practical systems.

Table 5.1: Efficiency of the series-machine system with identical machines ($\tau = 1$)

N	e	$MTTR$	E_{odf1}	E_{odf2}
2	0.7	0.25	0.5385	0.5998
		5	0.5385	0.5437
	0.99	0.25	0.9802	0.9804
		5	0.9802	0.9803
8	0.7	0.25	0.2258	0.4510
		5	0.2258	0.2510
	0.99	0.25	0.9252	0.9298
		5	0.9252	0.9256

From (3.7), we can see that when $e_i = 1$, $\forall i = 1, 2, \ldots, N$, the efficiency of the series-machine system under both ODF-I and ODF-II are 1. In other words, if all machines are (nearly) reliable, operations of the system under both failure mechanisms are (almost) identical, which can also be observed in Figure 1.2 and Table 5.1.

6 Conclusions and Future Work

The efficiency of the series-machine system is important in production systems engineering. However, almost all researches focus on type-I failure mechanisms (i.e., ODF-I and TDF-I), under which the breakdown of a machine makes all other series machines forced down. Nevertheless, type-II failure mechanisms (i.e., ODF-II and TDF-II), under which the breakdown of a machine does not make any other series machines forced down, are more common in the industry. In this paper, we proposed an approach to investigate the efficiency of the series-machine system under ODF-I and ODF-II mechanisms. Although it is very hard to derive the closed-form expression of the efficiency under ODF-II, we analyzed its lower and upper bounds by limit analysis and developed an algorithm to calculate the efficiency. The results show that, ODF-II is more efficient than ODF-I.

The future work will concentrate on the efficiency analysis under TDF-II mechanism and fitting closed-form analytical expressions of the efficiency under type-II failure mechanisms. Based on these expressions, not only we can easily calculate the efficiency of the series-machine system, but also analytically investigate the impact of system parameters on the efficiency without using the time-consuming algorithms.

Acknowledgement

The authors appreciate the help from Dr. H. Zhang in China University of Mining and Technology, Beijing, for proving Proposition 3.2.

A Appendix: Proofs of Theorems

A.1 Proof of Proposition 3.1

Proof: Due to space limitation, we prove it for $N = 2$ and $\tau_1 = \tau_2 = \tau$. The general case is proved similarly.

Based on expression (3.2) and the total probability formula, we have

$$\begin{aligned}
&E_Y[Y]-\tau \\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P\{W_1=m,W_2=n\}\max\Big(\frac{m}{\mu_1},\frac{n}{\mu_2}\Big) \\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P\{W_1=m\}P\{W_2=n\}\Big(\frac{m}{\mu_1}+\frac{n}{\mu_2}\Big)-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P\{W_1=m\}P\{W_2=n\}\min\Big(\frac{m}{\mu_1},\frac{n}{\mu_2}\Big) \\
=&\frac{E_{W_1}[W_1]}{\mu_1}+\frac{E_{W_2}[W_2]}{\mu_2}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P\{W_1=m\}P\{W_2=n\}\min\Big(\frac{m}{\mu_1},\frac{n}{\mu_2}\Big) \\
=&\sum_{i=1}^{2}\frac{1-e_i}{e_i}\tau-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\lambda_1\tau)^m}{m!}\frac{(\lambda_2\tau)^n}{n!}e^{-(\lambda_1+\lambda_2)\tau}\cdot\min\Big(\frac{m}{\mu_1},\frac{n}{\mu_2}\Big).
\end{aligned} \tag{A.1}$$

Replacing λ_i and μ_i by $\alpha\lambda_i^*$ and $\alpha\mu_i^*$, respectively, we have

$$\begin{aligned}
&E_{Y_\alpha}[Y_\alpha]-\tau \\
=&\sum_{i=1}^{2}\frac{1-e_i}{e_i}\tau-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^{m+n-1}\frac{(\lambda_1^*\tau)^m}{m!}\frac{(\lambda_2^*\tau)^n}{n!}\cdot e^{-\alpha(\lambda_1^*+\lambda_2^*)\tau}\cdot\min\Big(\frac{m}{\mu_1^*},\frac{n}{\mu_2^*}\Big).
\end{aligned} \tag{A.2}$$

Clearly,

$$\lim_{\alpha\to0^+}(E_{Y_\alpha}[Y_\alpha]-\tau)=\sum_{i=1}^{2}\frac{1-e_i}{e_i}\tau, \tag{A.3}$$

i.e.,

$$\lim_{\alpha\to0^+}E_{Y_\alpha}[Y_\alpha]=\tau+\sum_{i=1}^{2}\frac{1-e_i}{e_i}\tau. \tag{A.4}$$

■

A.2 Proof of Proposition 3.2

Proof: To prove the proposition, we construct a sub-series of Y_α and prove it converges to $\max\limits_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i}$ with probability 1.

Let $S_{i,\alpha}=\tau_i+\frac{W_i}{\alpha\mu_i^*}$, where $W_i\sim\text{Poisson}(\alpha\lambda_i^*\tau_i)$. We construct N series $\{S_{i,k}\}_{k=1}^{\infty}$, where $S_{i,k}$ is $S_{i,\alpha}$ with $k=\lceil\alpha\rceil$ and $b=k-\alpha$, $i=1,2,\ldots,N$, $k=1,2,\ldots$, and $b\in[0,1)$. Clearly, there is a one-to-one mapping between α and (k,b). Based on $\{S_{i,k}\}_{k=1}^{\infty}$, we construct $\{Y_k\}_{k=1}^{\infty}$, where

$Y_k = \max_{1 \leqslant i \leqslant N} S_{i,k}$. Thus, $\{Y_k\}_{k=1}^{\infty}$ is a sub-series of Y_α and for any fixed b, $\lim_{\alpha \to +\infty} Y_\alpha = \lim_{k \to \infty} Y_k$. Hence, to prove the proposition, we only need to prove that, $\forall b \in [0, 1)$,

$$P\Big\{ \lim_{k \to \infty} Y_k = \max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i} \Big\} = 1. \tag{A.5}$$

To prove (A.5), we need an auxiliary inequality of $S_{i,k}$. Clearly,

$$E[S_{i,k}] = \tau_i + \frac{\lambda_i^*}{\mu_i^*} \tau_i = \frac{\tau_i}{e_i} \tag{A.6}$$

and

$$Var[S_{i,k}] = \frac{\lambda_i^* \tau_i}{(k-b)(\mu_i^*)^2} = \frac{1-e_i}{e_i} \frac{\tau_i}{(k-b)\mu_i^*}. \tag{A.7}$$

Based on Chebyshev's inequality [17] and taking into account the above two equations, we have

$$P\Big\{ \Big| S_{i,k} - \frac{\tau_i}{e_i} \Big| > \epsilon \Big\} \leqslant \frac{Var[S_{i,k}]}{\epsilon^2} = \frac{1-e_i}{\epsilon^2 e_i} \frac{\tau_i}{(k-b)\mu_i^*}, \forall \epsilon > 0. \tag{A.8}$$

Then, we prove equation (A.5). Without loss of generality, assume $\frac{\tau_1}{e_1} = \max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i}$. Thus, we have

$$\begin{aligned}
&P\Big\{ \Big| Y_k - \max_{1 \leqslant i \leqslant N} \frac{\tau_i}{e_i} \Big| > \epsilon \Big\} \\
=&P\Big\{ \Big| \max_{1 \leqslant i \leqslant N} S_{i,k} - \frac{\tau_1}{e_1} \Big| > \epsilon \Big\} \\
=&P\Big\{ \max_{1 \leqslant i \leqslant N} S_{i,k} > \frac{\tau_1}{e_1} + \epsilon \Big\} + P\Big\{ \max_{1 \leqslant i \leqslant N} S_{i,k} < \frac{\tau_1}{e_1} - \epsilon \Big\} \\
=&P\Big\{ \bigcup_{i=1}^{N} \Big\{ S_{i,k} > \frac{\tau_1}{e_1} + \epsilon \Big\} \Big\} + P\Big\{ \bigcap_{i=1}^{N} \Big\{ S_{i,k} < \frac{\tau_1}{e_1} - \epsilon \Big\} \Big\} \\
\leqslant& \sum_{i=1}^{N} P\Big\{ S_{i,k} > \frac{\tau_1}{e_1} + \epsilon \Big\} + P\Big\{ S_{1,k} < \frac{\tau_1}{e_1} - \epsilon \Big\} \\
\leqslant&P\Big\{ \Big| S_{1,k} - \frac{\tau_1}{e_1} \Big| > \epsilon \Big\} + \sum_{i=2}^{N} P\Big\{ S_{i,k} > \frac{\tau_i}{e_i} + \epsilon \Big\} \\
\leqslant& \sum_{i=1}^{N} P\Big\{ \Big| S_{i,k} - \frac{\tau_i}{e_i} \Big| > \epsilon \Big\},
\end{aligned} \tag{A.9}$$

i.e.,

$$
\begin{aligned}
&P\left\{\left|Y_k - \max_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i}\right| < \epsilon\right\}\\
\geqslant& 1 - \sum_{i=1}^{N} P\left\{\left|S_{i,k} - \frac{\tau_i}{e_i}\right| > \epsilon\right\}\\
\geqslant& 1 - \sum_{i=1}^{N}\frac{1-e_i}{\epsilon^2 e_i}\frac{\tau_i}{(k-b)\mu_i^*}\\
=& 1 - \frac{1}{k-b}\sum_{i=1}^{N}\frac{(1-e_i)\tau_i}{\epsilon^2 e_i\mu_i^*}.
\end{aligned}
\tag{A.10}
$$

As a result, $\forall b \in [0,1)$, we have

$$
\begin{aligned}
&\lim_{K\to\infty}\frac{\sum_{k=1}^{K} P\left\{\left|Y_k - \max_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i}\right| < \epsilon\right\}}{K}\\
\geqslant& \lim_{K\to\infty}\frac{\sum_{k=1}^{K}\left(1 - \frac{1}{k-b}\sum_{i=1}^{N}\frac{(1-e_i)\tau_i}{\epsilon^2 e_i\mu_i^*}\right)}{K}\\
=& 1 - \lim_{K\to\infty}\frac{1}{K}\sum_{i=1}^{N}\frac{(1-e_i)\tau_i}{\epsilon^2 e_i\mu_i^*}\sum_{k=1}^{K}\frac{1}{k-b}\\
=& 1,
\end{aligned}
\tag{A.11}
$$

which implies

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{\left|Y_k - \max_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i}\right| < \epsilon\right\}\\
=& \lim_{K\to\infty}\sum_{k=1}^{K} P\left\{\left|Y_k - \max_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i}\right| < \epsilon\right\}\\
=& \infty.
\end{aligned}
\tag{A.12}
$$

Based on Borel-Cantelli Lemma [17], equation (A.5) holds, which implies

$$
P\left\{\lim_{\alpha\to+\infty} Y_\alpha = \max_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i}\right\} = 1. \tag{A.13}
$$

Finally, we prove equation (3.5). Based on the monotone convergence theorem [18], we obtain

$$
\lim_{\alpha\to+\infty} E_{Y_\alpha}[Y_\alpha] = E[\lim_{\alpha\to+\infty} Y_\alpha] = \max_{1\leqslant i\leqslant N}\frac{\tau_i}{e_i},\ w.p.1. \tag{A.14}
$$

∎

A.3 Proof of Lemma 3.1

Proof: First, we prove the right-hand side inequality. Let $S_i = \tau_i + \sum_{j=1}^{W_i} X_{ij}$, $i = 1, 2, \dots, N$. Since $\sum_{j=1}^{W_i} X_{ij}$ is the total repair time of m_i in its cycle time τ_i, its mean value is

$$E\Big[\sum_{j=1}^{W_i} X_{ij}\Big] = \frac{MTTR_i}{MTBF_i}\tau_i = \frac{1-e_i}{e_i}\tau_i, \tag{A.15}$$

i.e., $\forall i = 1, 2, \dots, N$,

$$E_{S_i}[S_i - \tau_i] = \frac{1-e_i}{e_i}\tau_i,\ E_{S_i}[S_i] = \frac{\tau_i}{e_i}. \tag{A.16}$$

As a result, we have

$$\begin{aligned}
&E_Y[Y] - \tau_{max}\\
=&E_{S_i}\Big[\max_{1\leqslant i\leqslant N} S_i\Big] - \tau_{max}\\
=&E_{S_i}\Big[\max_{1\leqslant i\leqslant N}(S_i - \tau_{max})\Big]\\
\leqslant&E_{S_i}\Big[\max_{1\leqslant i\leqslant N}(S_i - \tau_i)\Big]\\
\leqslant&E_{S_i}\Big[\sum_{i=1}^{N}(S_i - \tau_i)\Big]\\
=&\sum_{i=1}^{N}\frac{1-e_i}{e_i}\tau_i.
\end{aligned} \tag{A.17}$$

Then, we prove the left-hand side inequality. Since

$$\max_{1\leqslant i\leqslant N} S_i \geqslant S_{i'},\ \forall i' \in \{1, 2, \dots, N\}, \tag{A.18}$$

we have

$$E_{S_i}\Big[\max_{1\leqslant i\leqslant N} S_i\Big] \geqslant E_{S_{i'}}[S_{i'}] = \frac{\tau_{i'}}{e_{i'}},\ \forall i', \tag{A.19}$$

which implies

$$E_Y[Y] = E_{S_i}\left[\max_{1\leqslant i\leqslant N} S_i\right] \geqslant \max_{1\leqslant i\leqslant N} \frac{\tau_i}{e_i}. \tag{A.20}$$

■

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