

# Lower Bound Theory of Nonzero Entries in Solutions of $\ell_2$ - $\ell_p$ Minimization

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**Abstract.** Recently, variable selection and sparse reconstruction are solved by finding an optimal solution of a minimization model where the objective function is the sum of a data-fitting term in  $\ell_2$  norm and a regularization term in  $\ell_p$  norm ( $0 < p < 1$ ). Since it is a non-convex model, most algorithms for solving the problem can only provide an approximate local optimal solution, where nonzero entries in the solution cannot be identified theoretically. In this paper, we establish lower bounds for the absolute value of nonzero entries in every local optimal solution of the model, which can be used to eliminate zero entries precisely in any numerical solution. Therefore, we have developed a lower bound theorem to classify zero and nonzero entries in its every local solution. These lower bounds clearly show the relationship between the sparsity of the solution and the choice of the regularization parameter and norm, so that our theorem can be used for selecting desired model parameters and norms. Furthermore, we also develop error bounds for verifying accuracy of numerical solutions of the  $\ell_2$ - $\ell_p$  minimization model. To demonstrate applications of our theory, we propose a hybrid orthogonal matching pursuit-smoothing gradient (OMP-SG) method for solving the nonconvex, non-Lipschitz continuous  $\ell_2$ - $\ell_p$  minimization problem. Computational results show the effectiveness of the lower bounds for identifying nonzero entries in numerical solutions and the OMP-SG method for finding a high quality numerical solution.

**Keywords:** Variable selection, sparse solution, linear least-squares problem,  $\ell_p$  regularization, smoothing approximation, first order condition, second order condition.

**AMS Subject Classifications:** 90C26, 90C46, 90C90

## 1 Introduction

We consider the following minimization problem

$$\min_{x \in R^n} \|Ax - b\|_2^2 + \lambda \|x\|_p^p, \quad (1.1)$$

where  $A \in R^{m \times n}$ ,  $b \in R^m$ ,  $\lambda \in (0, \infty)$ ,  $p \in (0, 1)$ . Recently, minimization problem (1.1) attracted great attention in variable selection and sparse reconstruction [3, 5, 7, 8, 9, 17, 28].

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The objective function of (1.1),

$$f(x) := \|Ax - b\|_2^2 + \lambda \|x\|_p^p$$

consists of a data fitting term  $\|Ax - b\|_2^2$  and a regularization term  $\lambda \|x\|_p^p$ . Problem (1.1) is intermediate between the  $\ell_2$ - $\ell_0$  minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 + \lambda \|x\|_0 \quad (1.2)$$

and the  $\ell_2$ - $\ell_1$  minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 + \lambda \|x\|_1, \quad (1.3)$$

in the sense

$$\|x\|_0 = \lim_{p \rightarrow 0} \|x\|_p^p = \lim_{p \rightarrow 0} \sum_{i=1}^n |x_i|^p = \#\{i \mid x_i \neq 0\}$$

and

$$\|x\|_1 = \lim_{p \rightarrow 1} \|x\|_p^p = \lim_{p \rightarrow 1} \sum_{i=1}^n |x_i|^p = \sum_{i=1}^n |x_i|.$$

Naturally, one expects that using the  $\ell_p$  norm in the regularization term can find sparser solution than using the  $\ell_1$  norm, which was evidenced in extensive computational studies [3, 5, 7, 8, 9, 17, 28]. However, some major theoretical issues remain open. Is there any theoretical justification for solving minimization problem (1.1) with  $p < 1$ ? What are the solution characteristics of (1.1)? Is there theory to dictate the choice of the regularization parameter  $\lambda$  and norm  $p$ ? Our first main contribution of this paper is to answer these questions. We have established lower bounds for the absolute value of nonzero entries in every local optimal solution of (1.1) only when  $p < 1$ . Therefore, we have developed a lower bound theorem to classify zero and nonzero entries in every local solution of (1.1). These lower bounds clearly show the relationship between the sparsity of the solution and the choice of the regularization parameter and norm, so that the theorem can be used to guide the selection of desired model parameters and norms in (1.1). It can be also used to eliminate zero entries precisely in any computational algorithm and to identify nonzero entries in the numerical optimal solution.

More specifically, using the first order necessary condition for a local minimizer, we present a lower bound

$$L = \left( \frac{\lambda p}{2\beta} \right)^{\frac{1}{1-p}} \quad (1.4)$$

for the absolute value of nonzero entries in every local optimal solution  $x^*$  of (1.1), that is, for any  $x^*$ ,

$$L \leq |x_i^*|, \quad \text{for all } x_i^* \neq 0, i \in \mathcal{N},$$

which is equivalent to the following statement

$$x_i^* \in (-L, L) \quad \Rightarrow \quad x_i^* = 0, \quad i \in \mathcal{N}.$$

Here,  $\beta$  is an upper bound for the error tolerance, that is,

$$\|A^T(Ax^* - b)\| < \beta.$$

Then, using the second order necessary condition for a local minimizer, we present a component-wise lower bound

$$L_i = \left( \frac{\lambda p(1-p)}{2\|a_i\|^2} \right)^{\frac{1}{2-p}} \quad (1.5)$$

for each nonzero entry  $x_i^*$  of any local optimal solution  $x^*$ , that is, for any  $x^*$ ,

$$L_i \leq |x_i^*|, \quad \text{for all } x_i^* \neq 0, i \in \mathcal{N}.$$

Here,  $a_i$  is the  $i$ th column of the matrix  $A$ .

The lower bounds in (1.4) and (1.5) are not only useful for identification of zero entries in all local optimal solutions from approximation ones, but also for selection of the regularization parameter  $\lambda$  and norm  $\|\cdot\|_p$ . In particular, for a given norm  $\|\cdot\|_p$ , the lower bounds can help us to choose the regularization parameter  $\lambda$  for controlling the sparsity level of the solution. On the other hand, for a given  $\lambda$ , the lower bounds can also help us to understand the  $\ell_2$ - $\ell_p$  problem with different values  $p \in (0, 1)$ .

We need to mention that Nikolova [22] proved a similar bound based on the second order condition in a different context. Her result is important: it shows that using non-convex potential functions is good for piecewise constant image restoration. However, the result has not been used in practical algorithms, because one needs to solve an optimization problem to construct the bound. On the other hand, our first and second order bounds have explicit close forms and are easily checkable. Moreover, we have found that the bound based on the first order condition seems more effective in practice.

Our second main contribution is on some numerical issues for solving (1.1). Let  $\mathcal{N} = \{1, 2, \dots, n\}$  and

$$X_0 = \{x \mid x_i = 0, \text{ for some } i \in \mathcal{N}\}.$$

The  $\ell_1$  norm  $\|\cdot\|_1$  is convex and Lipschitz continuous with the Lipschitz constant 1 in  $R^n$ , that is,  $|\|x\| - \|y\|| \leq \|x - y\|$  for any  $x, y \in R^n$ . Moreover, it is differentiable in  $R^n \setminus X_0$  and the Clarke subgradient [13] is

$$(\partial\|x\|_1) = \left\{ g \in R^n \mid g_i = \begin{cases} 1 & x_i > 0 \\ -1 & x_i < 0 \\ \theta & x_i = 0, \quad \theta \in [-1, 1], \end{cases} \quad i = 1, \dots, n \right\}.$$

However, the  $\ell_p$  norm  $\|\cdot\|_p$  for  $0 < p < 1$  is continuous in  $R^n$  but neither convex nor Lipschitz continuous at  $x \in X_0$ . The Clarke subgradient cannot be defined at  $x \in X_0$ . Thus, solving the nonconvex, non-Lipschitz continuous minimization problem (1.1) is very difficult.

Most optimization algorithms are only efficient for smooth and convex problems, and they can only find local optimal solutions. Nevertheless, some algorithms for nonsmooth and nonconvex optimization problems have been developed recently [4, 12, 20, 29]. However, the Lipschitz continuity remains a necessary condition to define the Clarke subgradient in these algorithms. To overcome the non-Lipschitz continuity, some approximation methods have been considered for solving (1.1). For example, at the  $k$ th iteration, replacing  $\|x\|_p^p$  by the following terms [3, 5, 7, 23]

$$\sum_{i=1}^n \frac{x_i^2}{((x_i^{k-1})^2 + \varepsilon_i)^{p/2-1}}, \quad \sum_{i=1}^n (|x_i| + \varepsilon_i)^p \quad \text{or} \quad \sum_{i=1}^n \frac{|x_i|}{(|x_i^{k-1}| + \varepsilon_i)^{1-p}}.$$

Here  $\varepsilon \in R^n$  is a small positive vector. The question is: are there error bounds for verifying accuracy of numerical solutions of these approximation methods? We have resolved this question by developing several error bounds.

More precisely, we consider smoothing methods for nonconvex, nonsmoothing optimization problems, for example, the smoothing gradient method [29]. We choose a smoothing function  $s_\varepsilon(t)$  of  $|t|$ , such that  $s_\varepsilon^p$  is continuously differentiable for any fixed scalar  $\varepsilon > 0$  and satisfies

$$0 \leq (s_\varepsilon(t))^p - |t|^p \leq \left(\frac{\varepsilon}{2}\right)^p.$$

See section 3. Let the smoothing objective function of  $f$  be

$$f_\varepsilon(x) := \|Ax - b\|_2^2 + \sum_{i=1}^n s_\varepsilon(x_i)^p.$$

We can show that the solution  $x_\varepsilon^*$  of the smoothing minimization problem

$$\min_{x \in R^n} f_\varepsilon(x) \tag{1.6}$$

converges to a solution of (1.1) as  $\varepsilon \rightarrow 0$ . For some small  $\varepsilon < \frac{L}{2}$ , let

$$(\bar{x}_\varepsilon^*)_i = \begin{cases} 0 & \text{if } |(x_\varepsilon^*)_i| \leq \varepsilon \\ (x_\varepsilon^*)_i & \text{otherwise.} \end{cases}$$

We show that there is  $x^*$  in the solution set of (1.1) such that

$$|(\bar{x}_\varepsilon^*)_i| = 0 \quad \text{if and only if} \quad x_i^* = 0, \quad i \in \mathcal{N}$$

and

$$\|\bar{x}_\varepsilon^* - x^*\| \leq \kappa \|\nabla f_\varepsilon(\bar{x}_\varepsilon^*)\|, \tag{1.7}$$

where  $\kappa$  is a computable constant.

To demonstrate the significance of the absolute lower bounds (1.4), (1.5) and error bounds (1.7), we propose a hybrid orthogonal matching pursuit-smoothing gradient (OMP-SG) method for the nonconvex, non-Lipschitz  $\ell_2$ - $\ell_p$  minimization problem (1.1). We first use the orthogonal matching pursuit method to select candidates of nonzero entries in the solution. Next we use the smoothing gradient method in [29] to find an approximate solution of (1.1). Both before and after the SG method, we use the lower bound theory to eliminate zero entries in the solution.

Our preliminary numerical results show that using OMP-SG with elimination of small entries in the numerical solution by the lower bounds for  $\ell_2$ - $\ell_p$  minimization problem (1.1) can provide more sparse solutions with smaller predictor error compared with several well-known approaches for variable selection.

This paper is organized as follows. In section 2, we present absolute lower bounds (1.4) and (1.5) for nonzero entries in any local solution of  $\ell_2$ - $\ell_p$  minimization problem (1.1). In section 3, we present the computable error bound (1.7) for numerical solutions. In section 4, we give the hybrid OMP-SG method for solving the  $\ell_2$ - $\ell_p$  minimization problem (1.1). Numerical results are given to demonstrate the effectiveness of the lower bounds, the error bounds and the OMP-SG method.

**Notations** For  $x \in R^n$  and  $p \in (0, 1)$ ,  $\|x\|_p$  denotes the  $\ell_p$  norm and  $\|x\|_p^p = \sum_{i=1}^n |x_i|^p$ . Throughout the paper,  $\|\cdot\|$  denotes the  $\ell_2$  norm. For any  $x, y \in R^n$ ,  $x \cdot y$  represents the vector  $(x_1y_1, \dots, x_ny_n)^T$ . Let  $\mathcal{X}_p^*$  and  $\mathcal{X}_1^*$  denote the set of local solutions of (1.1) and (1.3) respectively. For a vector  $x \in R^n$ ,  $\Lambda = \text{support}\{x\} = \{i \mid |x_i| \neq 0, \text{ for some } i \in \mathcal{N}\}$  denotes the support set of  $x$ .

## 2 Lower bounds for nonzero entries in solutions

In this section we present two lower bounds for nonzero entries in any local solution of  $\ell_2$ - $\ell_p$  minimization problem (1.1).

Since  $f(x) \geq \lambda\|x\|_p^p$ , the objective function  $f(x)$  is bounded below and  $f(x) \rightarrow \infty$  if  $\|x\| \rightarrow \infty$ . Moreover, the set  $\mathcal{X}_p^*$  is nonempty and bounded. A bounded and closed ball containing  $\mathcal{X}_p^*$  is given in Theorem 2.2.

Let  $\beta$  be a positive constant such that for any  $x^* \in \mathcal{X}_p^*$

$$\|A^T(Ax^* - b)\| < \beta.$$

In practice, one can simply choose  $\beta = \|A\|\|b\|$ , since for any  $x$  satisfying  $f(x) \leq f(0)$ , we have

$$\|A^T(Ax - b)\|^2 \leq \|A^T\|^2(\|Ax - b\|^2 + \lambda\|x\|_p^p) \leq \|A^T\|^2\|b\|^2 = (\|A\|\|b\|)^2. \quad (2.1)$$

**Theorem 2.1** (*The first order bound*) Let  $L = \left(\frac{\lambda p}{2\beta}\right)^{\frac{1}{1-p}}$ . Then for any  $x^* \in \mathcal{X}_p^*$ , we have

$$x_i^* \in (-L, L) \quad \Rightarrow \quad x_i^* = 0, \quad i \in \mathcal{N}.$$

**Proof:** For  $x^* \in \mathcal{X}_p^*$ , with  $\|x^*\|_0 = k$ , without loss of generality, we assume

$$x^* = (x_1^*, \dots, x_k^*, 0, \dots, 0)^T.$$

Let  $z^* = (x_1^*, \dots, x_k^*)^T$  and  $B \in R^{m \times k}$  be the submatrix of  $A$ , whose columns are the first  $k$  columns of  $A$ . Define a function  $g : R^k \rightarrow R$  by

$$g(z) = \|Bz - b\|^2 + \lambda\|z\|_p^p.$$

We have

$$f(x^*) = \|Ax^* - b\|^2 + \lambda\|x^*\|_p^p = \|Bz^* - b\|^2 + \lambda\|z^*\|_p^p = g(z^*).$$

Since  $|z^*| > 0$ ,  $g$  is continuously differentiable at  $z^*$ . Moreover from

$$\begin{aligned} g(z^*) = f(x^*) &\leq \min\{f(x) \mid x_i = 0, i = k+1, \dots, n\} \\ &= \min\{g(z) \mid z \in R^k\} \end{aligned} \quad (2.2)$$

in a neighborhood of  $x^*$ , we find that  $z^*$  must be a local minimizer of the function  $g$ . Hence the first order necessary condition for

$$\min_{z \in R^k} g(z)$$

holds at  $z^*$ . This gives

$$2B^T(Bz^* - b) + \lambda p(|z^*|^{p-1} \cdot \text{sign}(z^*)) = 0.$$

Therefore, we obtain

$$\lambda p \| |z^*|^{p-1} \| = 2 \| B^T(Bz^* - b) \| = 2 \| B^T(Ax^* - b) \| \leq 2 \| A^T(Ax^* - b) \| \leq 2\beta.$$

This implies

$$2\beta \geq \lambda p \| |z^*|^{p-1} \| \geq \lambda p \left( \min_{1 \leq i \leq k} |z_i^*| \right)^{p-1}.$$

Note that  $0 < p < 1$ . We find

$$\min_{1 \leq i \leq k} |z_i^*| \geq \left( \frac{\lambda p}{2\beta} \right)^{\frac{1}{1-p}} = L.$$

Since  $x^* \in \mathcal{X}_p^*$  is arbitrarily chosen, we can claim that for any  $x^* \in \mathcal{X}_p^*$ , its nonzero components are no less than  $L$ . In other words, if  $x_i^* \in (-L, L)$  then  $x_i^* = 0$  for  $i \in \mathcal{N}$ .  $\blacksquare$

We now develop another lower bound as well as other bounds on the sparsity of local minimizers using the second order necessary optimality conditions.

**Theorem 2.2** (The second order bound) *Let  $L_i = \left( \frac{\lambda p(1-p)}{2\|a_i\|^2} \right)^{\frac{1}{2-p}}$ ,  $i \in \mathcal{N}$ . Then for any  $x^* \in \mathcal{X}_p^*$ , the following statements hold.*

(1)

$$x_i^* \in (-L_i, L_i) \quad \Rightarrow \quad x_i^* = 0, \quad i \in \mathcal{N}.$$

(2) *The columns of the sub-matrix  $B := A_\Lambda \in \mathbb{R}^{m \times |\Lambda|}$  of  $A$  are linearly independent, where  $\Lambda = \text{support}\{x^*\}$ , and  $|\Lambda|$  is the cardinality of the set  $\Lambda$ .*

(3)

$$\|B^T A(x^* - b)\| \leq \frac{\lambda p}{2} \cdot \sqrt{|\Lambda|} \left( \min_{1 \leq i \leq |\Lambda|} L_i \right)^{p-1}.$$

*In particular, if  $\|a_i\| = 1$  for all  $i \in \mathcal{N}$  (that is,  $A$  is column-wise normalized), then*

$$\|B^T A(x^* - b)\| \leq \sqrt{|\Lambda|} \left( \frac{\lambda p}{2} \right)^{\frac{1}{2-p}} \left( \frac{1}{1-p} \right)^{\frac{1-p}{2-p}}.$$

(4) *Let  $\|a_i\| = 1$  for all  $i \in \mathcal{N}$  and  $x^*$  be any local minimizer of (1.1) satisfying  $f(x^*) \leq f(0)$ . Then, the number of nonzero entries in  $x^*$  is bounded by*

$$|\Lambda| \leq \|b\|^2 \cdot \left( \frac{1}{\lambda} \right)^{\frac{2}{2-p}} \left( \frac{2}{p(1-p)} \right)^{\frac{p}{2-p}}.$$

(5)

$$\mathcal{X}_p^* \subseteq \left\{ x \mid \|x\| \leq \|(B^T B)^{-1} B^T b\| + \frac{\lambda p}{2} \|(B^T B)^{-1}\| \left( \min_{1 \leq i \leq |\Lambda|} L_i \right)^{p-1} \right\}.$$

If  $\|a_i\| = 1$  for all  $i \in \mathcal{N}$ , then

$$\mathcal{X}_p^* \subseteq \left\{ x \mid \|x\| \leq \|(B^T B)^{-1} B^T b\| + \|(B^T B)^{-1}\| \left( \frac{\lambda p}{2} \right)^{\frac{1}{2-p}} \left( \frac{1}{1-p} \right)^{\frac{1-p}{2-p}} \right\}.$$

**Proof:** (1) Recall the function  $g$  in the proof of Theorem 2.1. From (2.2), the second order necessary condition for

$$\min_{z \in R^k} g(z)$$

holds at  $z^*$ . This gives that the matrix

$$2B^T B + \lambda p(p-1) \text{diag}(|z^*|^{p-2})$$

is positive semi-definite. Therefore, we obtain

$$2e_i^T B^T B e_i + \lambda p(p-1) |z_i^*|^{p-2} \geq 0, \quad i = 1, \dots, k$$

where  $e_i$  is the  $i$ th column of the identity matrix of  $R^{k \times k}$ .

Note that  $\|a_i\|^2 = e_i^T B^T B e_i$ . We find that

$$|z_i^*|^{p-2} \leq \frac{2\|a_i\|^2}{\lambda p(1-p)}, \quad i = 1, \dots, k$$

which implies that

$$|z_i^*| \geq \left( \frac{\lambda p(1-p)}{2\|a_i\|^2} \right)^{\frac{1}{2-p}} = L_i, \quad i = 1, \dots, k.$$

Hence for any  $x^* \in \mathcal{X}_p^*$ , if  $x_i^* \in (-L_i, L_i)$  then  $x_i^* = 0$ ,  $i \in \mathcal{N}$ .

(2) Since the matrix  $2B^T B + \lambda p(p-1) \text{diag}(|z^*|^{p-2})$  is positive semi-definite, and  $\lambda p(p-1) \text{diag}(|z^*|^{p-2})$  is negative definite, the matrix  $B^T B$  must be positive definite. Hence the columns of  $B$  must be linearly independent.

(3) Since  $z^*$  is a local minimizer of  $g$ , the first order necessary condition must hold at  $z^*$ . Hence, we find, with  $Bz^* = Ax^*$ ,

$$\|B^T(Ax^* - b)\| = \|B^T(Bz^* - b)\| = \frac{\lambda p}{2} \| |z^*|^{p-1} \| \leq \frac{\lambda p}{2} \cdot \sqrt{|\Lambda|} \left( \min_{1 \leq i \leq |\Lambda|} L_i \right)^{p-1}.$$

If  $\|a_i\| = 1$  for all  $i \in \mathcal{N}$ , then  $L_i = \left( \frac{\lambda p(1-p)}{2} \right)^{\frac{1}{2-p}}$  for all  $i \in \Lambda$ , which implies the second bound in (3).

(4) At  $x = 0$ , the objective value  $f(0) = \|b\|^2$ . Thus, for any local minimizer  $x^*$  satisfying  $f(x^*) \leq f(0)$ , we have  $f(x^*) \leq \|b\|^2$  which implies

$$\lambda \|x^*\|_p^p \leq \|b\|^2.$$

But from (1) any nonzero entry of  $x^*$  is bounded from below by  $\left(\frac{\lambda p(1-p)}{2}\right)^{\frac{1}{2-p}}$ . Thus, they together imply the desired inequality in (4).

(5) Recall the first order necessary condition for (2.2),

$$2B^T B z^* = 2B^T b - \lambda p |z^*|^{p-1} \text{sign}(z^*).$$

From (1) and (2) of this theorem, we know that  $|z_i^*| \geq L_i$  and  $B^T B$  is nonsingular. Hence, we obtain result (5) for the general case. Moreover, for the case where  $\|a_i\| = 1$ ,  $i \in \mathcal{N}$ , we have

$$\begin{aligned} \|x^*\| = \|z^*\| &\leq \|(B^T B)^{-1} B^T b\| + \|(B^T B)^{-1}\| \frac{1}{2} \lambda p \|z^*\|^{p-1} \\ &\leq \|(B^T B)^{-1} B^T b\| + \|(B^T B)^{-1}\| \left(\frac{\lambda p}{2}\right)^{\frac{1}{2-p}} \left(\frac{1}{1-p}\right)^{\frac{1-p}{2-p}}. \end{aligned}$$

■

**Remark 2.1.** Note that the lower bound (1) of Theorem 2.2 depends only on  $\lambda$  and  $p$ . Result (2) of Theorem 2.2 implies that columns of  $A$  corresponding to nonzero entries of  $x^*$  must form a basis as long as  $0 < p < 1$ , while bound (3) shows that  $x^*$  approaches the least squares solution of  $\|Ax - b\|$  (restricted to the support of  $x^*$ ) as  $\lambda \rightarrow 0$ . Result (4) of Theorem 2.2 indicates that for  $\lambda$  sufficiently large but finite, the number of nonzero entries in any local minimizer of (1.1) reduces to 0 for  $0 < p < 1$ . Finally, (5) of Theorem 2.2 presents a closed ball which contains all local minimizers of (1.1), and an upper bound for all nonzero entries in any local minimizer.

The lower bound theory can be extended to the following problem

$$\min_{x \in R^n} \|Ax - b\|_2^2 + \lambda \sum_{i=1}^r \phi(|d_i^T x_i|^p), \quad (2.3)$$

where  $D \in R^{r \times n}$  is the first or second order difference matrix with rows  $d_i$ , and  $\phi$  is a non-Lipschitz potential function; see Table 4.5. In fact, as we mentioned earlier, Nikolova [22] proved that there is  $\theta > 0$  such that every local minimizer  $x^*$  of (2.3) satisfies

$$\text{either } |d_i^T x^*| = 0 \quad \text{or} \quad |d_i^T x^*| \geq \theta$$

by using the second order necessary condition for (2.3). However, the result has not been used in practical algorithms, because one needs to solve an optimization problem to construct  $\theta$ . Nikolova [22] also stated that it is difficult to get an explicit solution from the optimization problem for constructing  $\theta$ .

Lower bounds (1.4) and (1.5) clearly show the relationship between the sparsity of the solution and the choice of the regularization parameter  $\lambda$  and norm  $\|\cdot\|_p$ . Hence our lower bound theory can be used for selecting model parameters  $\lambda$  and  $p$ . In Figure 1, we show some properties of the function  $L(\lambda, p) = (\lambda p(1-p))^{\frac{1}{2-p}}$ . The sub-figure (a) plots

$$p^*(\lambda) = \arg \max_{0 \leq p \leq 1} (\lambda p(1-p))^{\frac{1}{2-p}}$$

for  $\lambda \in (0, 5]$ . The sub-figure (b) plots the function  $L(\lambda, p)$  for  $\lambda = (0, 5]$  and  $p \in [0, 1]$ .



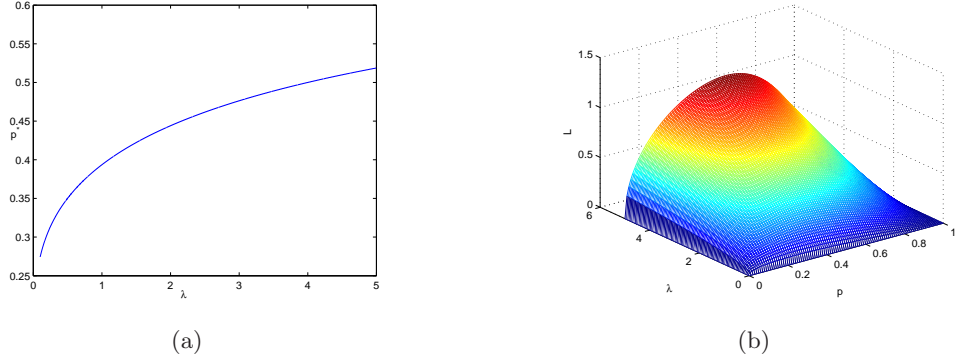


Figure 1: Function  $(\lambda p(1-p))^{\frac{1}{2-p}}$

From Figure 1, we can see clearly that for any given  $\lambda > 0$ ,  $(\lambda p(1-p))^{\frac{1}{2-p}}$  is a nonnegative and concave function of  $p$  on  $[0, 1]$ . It takes the minimum value at  $p = 0$  and  $p = 1$ , and the maximum value at  $p \in (0.26, 0.55)$  for any  $\lambda \in (0, 5]$ . It perhaps suggests that in general using  $\|\cdot\|_p$  with  $p \in (0.26, 0.55)$  may provide more sparse solutions than using  $\|\cdot\|_1$ .

### 3 Error Bounds derived from lower bound theory

Smoothing approximations are widely used in optimization and scientific computing. In the following we consider a smoothing function of  $f$  and give a smooth version of Theorem 2.1 and Theorem 2.2.

For  $\mu \in (0, \infty)$ , let

$$s_\mu(t) = \begin{cases} |t| & |t| > \mu \\ \frac{t^2}{2\mu} + \frac{\mu}{2} & |t| \leq \mu. \end{cases}$$

Then  $s_\mu(t)$  is continuously differentiable and

$$((s_\mu(t))^p)' = \begin{cases} p|t|^{p-1}\text{sign}(t) & |t| > \mu \\ p\left(\frac{t^2}{2\mu} + \frac{\mu}{2}\right)^{p-1} \frac{t}{\mu} & |t| \leq \mu. \end{cases}$$

However,  $s_\mu(t)$  is not twice differentiable at  $t = \mu$ . For  $t \in (-\mu, \mu)$ , the second derivative of  $(s_\mu(t))^p$  satisfies

$$\begin{aligned} (s_\mu(t))^p{}'' &= p(p-1)\left(\frac{t^2}{\mu} + \frac{\mu}{2}\right)^{p-2} \left(\frac{t}{\mu}\right)^2 + p\left(\frac{t^2}{\mu} + \frac{\mu}{2}\right)^{p-1} \frac{1}{\mu} \\ &\geq p(p-1)\left(\frac{t^2}{\mu} + \frac{\mu}{2}\right)^{p-2} \left(\frac{t^2}{\mu} + \frac{\mu}{2}\right) \frac{1}{\mu} + p\left(\frac{t^2}{\mu} + \frac{\mu}{2}\right)^{p-1} \frac{1}{\mu} \\ &= p^2 \left(\frac{t^2}{\mu} + \frac{\mu}{2}\right)^{p-1} \frac{1}{\mu} > 0. \end{aligned}$$

Hence  $s_\mu(t)$  is strictly convex in  $(-\mu, \mu)$ . Moreover, from  $s_\mu(t) = |t| \left( \frac{t^2 + \mu^2}{2\mu|t|} \right) \geq |t|$  and  $0 = \operatorname{argmax}(s_\mu(t) - |t|)$  for  $t \in (-\mu, \mu)$ , we have that for any  $t \in R$

$$0 \leq (s_\mu(t))^p - |t|^p \leq \left( \frac{\mu}{2} \right)^p. \quad (3.1)$$

Let

$$\psi_\mu(x) = (s_\mu(x_1), \dots, s_\mu(x_n))^T$$

and

$$\Psi_\mu(x) = \left( ((s_\mu(x_1))^p)', \dots, ((s_\mu(x_n))^p)' \right)^T.$$

We define a smoothing approximation of the objective function  $f(x)$

$$f_\mu(x) = \|Ax - b\|^2 + \lambda \|\psi_\mu(x)\|_p^p,$$

and consider the smooth minimization problem (1.6). The smoothing objective function  $f_\mu$  is continuously differentiable in  $R^n$ , and strictly convex on the set  $\{x \mid \|x\|_\infty \leq \mu\}$ .

Let  $\mathcal{X}_{p,\mu}^*$  denote the set of local solutions of (1.6). By the definition of  $\psi_\mu$ , for any  $x$  we have

$$\lambda n \left( \frac{\mu}{2} \right)^p \geq f_\mu(x) - f(x) \geq 0.$$

Since  $\|x\| \rightarrow \infty$  implies  $f(x) \rightarrow \infty$ , we deduce  $f_\mu(x) \rightarrow \infty$  if  $\|x\| \rightarrow \infty$ . Moreover, for any  $x \in R^n$ ,  $\lim_{\mu \downarrow 0} f_\mu(x) = f(x)$ . Without loss of generality, we consider local solutions  $x^*$  of (1.6) satisfying  $f_\mu(x^*) \leq f(0)$ . In such case, from (2.1), we have

$$\|A^T(Ax_\mu^* - b)\| \leq \|A\| \|b\| =: \beta, \quad \text{for } x_\mu^* \in \mathcal{X}_{p,\mu}^*.$$

The following theorem presents the smooth version of the first and second lower bounds:

**Theorem 3.1** *Let  $L = \left( \frac{\lambda p}{2\beta} \right)^{\frac{1}{1-p}}$  and  $L_i = \left( \frac{\lambda p(1-p)}{2\|a_i\|^2} \right)^{\frac{1}{2-p}}$ ,  $i \in \mathcal{N}$ . For any  $\mu > 0$  and any  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$ , we have*

(1) *(The first order bound)*

$$(x_\mu^*)_i \in (-L, L) \quad \Rightarrow \quad |(x_\mu^*)_i| \leq \mu, \quad i \in \mathcal{N}.$$

(2) *(The second order bound)*

$$(x_\mu^*)_i \in (-L_i, L_i) \quad \Rightarrow \quad |(x_\mu^*)_i| \leq \mu, \quad i \in \mathcal{N}.$$

**Proof:** (1) Since  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$ , the first order necessary condition for (1.6) gives

$$\nabla f_\mu(x_\mu^*) = 2(A^T Ax_\mu^* - A^T b) + \lambda \Psi_\mu(x_\mu^*) = 0, \quad (3.2)$$

which implies

$$\|\lambda \Psi_\mu(x_\mu^*)\| = 2\|A^T Ax_\mu^* - A^T b\| \leq 2\beta. \quad (3.3)$$

Suppose  $(x_\mu^*)_i \in (-L, L)$  but  $|(x_\mu^*)_i| > \mu$  then

$$\lambda \|\Psi_\mu(x_\mu^*)\| \geq \lambda |\Psi_\mu(x_\mu^*)_i| = \lambda p |(x_\mu^*)_i|^{p-1}. \quad (3.4)$$

From (3.3) and (3.4) we can get

$$|(x_\mu^*)_i|^{p-1} \leq \frac{2\beta}{\lambda p}.$$

Note that  $0 < p < 1$ , we find

$$|(x_\mu^*)_i| \geq \left(\frac{\lambda p}{2\beta}\right)^{\frac{1}{1-p}} = L.$$

This is a contradiction to  $(x_\mu^*)_i \in (-L, L)$ . Since  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$  is arbitrarily chosen, we can claim that for any  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$ , if  $(x_\mu^*)_i \in (-L, L)$  then  $|(x_\mu^*)_i| \leq \mu$  for  $i \in \mathcal{N}$ .

(2) Since  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$ , the second order necessary condition for (1.6) implies that the matrix

$$\nabla^2 f_\mu(x_\mu^*) = 2A^T A + \lambda \Psi'_\mu(x)$$

is positive semi-definite. Suppose  $(x_\mu^*)_i \in (-L_i, L_i)$  but  $|(x_\mu^*)_i| > \mu$  then from

$$e_i^T (2A^T A + \lambda \Psi'_\mu(x)) e_i = 2\|a_i\|^2 + \lambda p(p-1) |(x_\mu^*)_i|^{p-2} \geq 0,$$

we can get

$$|(x_\mu^*)_i| \geq \left(\frac{\lambda p(1-p)}{2\|a_i\|^2}\right)^{\frac{1}{2-p}} = L_i.$$

This is a contradiction to  $(x_\mu^*)_i \in (-L, L)$ . Since  $\mu > 0$  and  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$  are arbitrarily chosen, we can claim that for any  $\mu > 0$  and  $x_\mu^* \in \mathcal{X}_{p,\mu}^*$ , if  $(x_\mu^*)_i \in (-L_i, L_i)$  then  $|(x_\mu^*)_i| \leq \mu$  for  $i \in \mathcal{N}$ .  $\blacksquare$

The function  $f$  is not Lipschitz continuous. We use (2.2) to define the first order necessary condition and the second order necessary condition for (1.1).

**Definition 3.1** For  $x \in R^n$ , let  $X = \text{diag}(x)$ .

(1)  $x$  is said to satisfy the first order necessary condition of (1.1) if

$$2XA^T(Ax - b) + \lambda p(|x|^p \cdot \text{sign}(x)) = 0. \quad (3.5)$$

(2)  $x$  is said to satisfy the second order necessary condition of (1.1) if

$$2XA^T AX + \lambda p(p-1)\text{diag}(|x|^p) \quad (3.6)$$

is positive semidefinite.

Obviously, the zero vector in  $R^n$  satisfies the first and second necessary condition of (1.1).

**Theorem 3.2** (1) Let  $\{x_\mu\}$  be a sequence of vectors satisfying the first order necessary condition (3.2) of (1.6). Then any accumulation point of  $\{x_\mu\}$  as  $\mu \rightarrow 0$  satisfies the first order necessary condition (3.5) of (1.1).

(2) Let  $\{x_\mu\}$  be a sequence of vectors satisfying the second order necessary condition of (1.6), that is,  $\nabla^2 f_\mu(x_\mu^*)$  is positive semi-definite. Then any accumulation point of  $\{x_\mu\}$  as  $\mu \rightarrow 0$  satisfies the second order necessary condition of (1.1), that is, the matrix in (3.6) at  $x^*$  is positive semi-definite.

(3) Let  $\{x_\mu\}$  be a sequence of vectors being global minimizers of (1.6). Then any accumulation point of  $\{x_\mu\}$  as  $\mu \rightarrow 0$  is a global minimizer of (1.1).

**Proof:** Let  $\bar{x}$  be an accumulation point of  $\{x_\mu\}$ . Without loss of generality, we assume that  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, 0, \dots, 0)^T$ . Let  $B \in R^{m \times k}$  be the matrix whose columns are the first  $k$  columns of  $A$ . Let  $z \in R^k$ . Let

$$L_2 = \max\{L, \min_{1 \leq i \leq n} L_i\}, \quad \ell = \min_{1 \leq i \leq k} |\bar{x}_i|, \quad \text{and} \quad \varepsilon = \frac{1}{2} \min\{L_2, \ell\}.$$

Since  $\bar{x}$  is an accumulation point of  $\{x_\mu\}$ , there is a  $\bar{\mu} > 0$ , and a subsequence of  $\{x_\mu\}$  such that for all  $\mu \in (0, \bar{\mu})$ , the subsequence lies in the ball

$$\mathcal{B} := \{x \mid \|x - \bar{x}\|_\infty \leq \varepsilon\}.$$

Without loss of generality, we assume that the sequence  $\{x_\mu\}$  in  $\mathcal{B}$  and  $x_\mu \rightarrow \bar{x}$ , as  $\mu \rightarrow 0$ . Hence, we have

$$|x_\mu|_i \geq \frac{1}{2}\ell, \quad i = 1, \dots, k, \quad \text{and} \quad |(x_\mu)_i| \leq \frac{1}{2}L_2, \quad i = k+1, \dots, n.$$

Let  $\mu \leq \frac{1}{2}\ell$ . From the definition of  $s_\mu$ , we have

$$s_\mu((x_\mu)_i) = |x_\mu|_i \geq \frac{1}{2}\ell, \quad i = 1, \dots, k.$$

Suppose that  $|(x_\mu)_i| > \mu$ , for  $i = k+1, \dots, n$ . By the same argument in the proof of Theorem 3.1, we can show  $|(x_\mu)_i| \geq L_2$ , which is a contradiction. Hence, we have

$$|(x_\mu)_i| \leq \mu, \quad i = k+1, \dots, n.$$

Therefore, we find

$$\lim_{\mu \downarrow 0} s_\mu((x_\mu)_i) = |\bar{x}|_i, \quad i = 1, \dots, k, \quad \text{and} \quad \lim_{\mu \downarrow 0} (x_\mu)_i = 0, \quad i = k+1, \dots, n.$$

Let  $X_\mu = \text{diag}(x_\mu)$ .

(1) From the first order necessary condition (3.2) for (1.6), we have

$$X_\mu \nabla f_\mu(x_\mu) = 2X_\mu(A^T A x_\mu - A^T b) + \lambda X_\mu \Psi_\mu(x_\mu) = 0.$$

By the definition of  $\Psi_\mu$  and the above argument, we have

$$(X_\mu \Psi_\mu(x_\mu))_i = p|x_\mu|_i^p, \quad i = 1, \dots, k,$$

and

$$(X_\mu \Psi_\mu(x_\mu))_i = p\left(\frac{(x_\mu^2)_i}{2\mu} + \frac{\mu}{2}\right)^{p-1} \frac{(x_\mu^2)_i}{\mu} \leq p\left(\frac{(x_\mu^2)_i}{\mu}\right)^p \leq p|x_\mu|_i^p \leq p\mu^p, \quad i = k+1, \dots, n.$$

Therefore, we have

$$\lim_{\mu \downarrow 0} (X_\mu \Psi_\mu(x_\mu))_i = p|\bar{x}_i|^p, \quad i = 1, \dots, k, \quad \text{and} \quad \lim_{\mu \downarrow 0} (X_\mu \Psi_\mu(x_\mu))_i = 0 \quad i = k+1, \dots, n.$$

Hence  $\bar{x}$  satisfies the first order necessary condition for (1.1).

(2) From the second order necessary condition (3.2) for (1.6), we have

$$X_\mu \nabla^2 f(x_\mu) X_\mu = 2X_\mu A^T A X_\mu + \lambda X_\mu \Psi'_\mu(x_\mu) X_\mu$$

is positive semi-definite. Using the above argument, we have

$$(X_\mu \Psi'_\mu(x_\mu) X_\mu)_{ii} = p(p-1)|x_{\mu i}|^p, \quad i = 1, \dots, k,$$

and

$$|(X_\mu \Psi'_\mu(x_\mu) X_\mu)_{ii}| \leq \left(\frac{(x_\mu^2)_i}{2\mu} + \frac{\mu}{2}\right)^{p-2} \frac{(x_\mu^4)_i}{\mu^2} + \left(\frac{(x_\mu^2)_i}{2\mu} + \frac{\mu}{2}\right)^{p-1} \frac{(x_\mu^2)_i}{\mu} \leq \mu^p, \quad i = k+1, \dots, n.$$

Therefore, we have

$$\lim_{\mu \downarrow 0} (X_\mu \Psi'_\mu(x_\mu) X_\mu)_{ii} = p(p-1)|\bar{x}_i|^p, \quad i = 1, \dots, k,$$

and

$$\lim_{\mu \downarrow 0} (X_\mu \Psi'_\mu(x_\mu) X_\mu)_{ii} = 0, \quad i = k+1, \dots, n.$$

Hence  $\bar{x}$  satisfies the second order necessary condition for (1.1).

(3) Let  $x^*$  be a global minimizer of (1.1). Then from the following three inequalities

$$f(x_\mu) \leq f_\mu(x_\mu) \leq f_\mu(x^*) \leq f(x^*) + \lambda n \left(\frac{\mu}{2}\right)^p,$$

we deduce that  $\bar{x}$  is a global minimizer of (1.1). ■

Now, we focus on the smooth minimization problem (1.6). We show that the above (approximate) lower bound theory leads to checkable error bounds for approximate solutions of original problem (1.1) from exact solutions of smooth problem (1.6). Problem (1.6) is a continuously differentiable, unconstrained optimization problem. We can use some optimization toolboxes to solve (1.6). In this section, we present a computable error bound for KKT solutions (satisfying the first order necessary condition) of the smoothing minimization problem (1.6) to approximate a KKT solution of the non-Lipschitz optimization problem (1.1).

Let  $\mathcal{X}_{p,\varepsilon}$  be the set of KKT solutions of (1.6) and  $\mathcal{X}_p$  be the set of KKT solutions of (1.1).

**Theorem 3.3** *There is  $\bar{\varepsilon} > 0$ , such that for any  $\varepsilon \in (0, \bar{\varepsilon}]$  and any  $x_\varepsilon^* \in \mathcal{X}_{p,\varepsilon}$ , there is  $x^* \in \mathcal{X}_p$  such that*

$$\Gamma_\mu := \{i \mid |(x_\varepsilon^*)_i| \leq \varepsilon, \quad i \in \mathcal{N}\} = \{i \mid |x_i^*| = 0, \quad i \in \mathcal{N}\} =: \Gamma. \quad (3.7)$$

Define

$$(\bar{x}_\varepsilon^*)_i = \begin{cases} 0 & i \in \Gamma \\ (x_\varepsilon^*)_i & i \in \mathcal{N} \setminus \Gamma. \end{cases} \quad (3.8)$$

Let  $B$  be the submatrix of  $A$  whose columns are indicated by  $\mathcal{N} \setminus \Gamma$ . Suppose  $\lambda_{\min}(B^T B) > \frac{\lambda p(1-p)}{2} L^{p-2}$ , then

$$\|\bar{x}_\varepsilon^* - x^*\| \leq \|G^{-1}\| \|\nabla f_\varepsilon(\bar{x}_\varepsilon^*)\|, \quad (3.9)$$

where  $G = 2B^T B + \lambda p(p-1)L^{p-2}I$ , and  $\lambda_{\min}(B^T B)$  denotes the smallest eigenvalue of the matrix  $B^T B$ .

**Proof:** From (1) of Theorem 3.2, we have  $\lim_{\mu \downarrow 0} \text{dist}(x_\mu^*, \mathcal{X}_p) = 0$ . Hence, there is  $\bar{\varepsilon} \in (0, \frac{L}{2})$  such that for any  $\varepsilon \in (0, \bar{\varepsilon}]$  and any  $x_\varepsilon \in \mathcal{X}_{p,\varepsilon}$ , there is  $x^* \in \mathcal{X}_p$  such that

$$\text{dist}(x_\varepsilon^*, \mathcal{X}_p) = \|x_\varepsilon^* - x^*\| < \frac{L}{2}.$$

Then

$$|x_i^*| - |(x_\varepsilon^*)_i| \leq |x_i^* - (x_\varepsilon^*)_i| = |(x^* - x_\varepsilon^*)_i| \leq \|x^* - x_\varepsilon^*\| < \frac{L}{2}.$$

For all  $i \in \Gamma_\mu$ , we have

$$|x_i^*| < |(x_\varepsilon^*)_i| + \frac{L}{2} < L.$$

From Theorem 2.1 we know  $|x_i^*| = 0$ , then  $i \in \Gamma$ . That is  $\Gamma_\mu \subset \Gamma$ . On the other hand, if  $i \in \Gamma$  then  $x_i^* = 0$ . We have

$$|(x_\varepsilon^*)_i| = |(x^* - x_\varepsilon^*)_i| \leq \|x^* - x_\varepsilon^*\| < \frac{L}{2} < L.$$

From Theorem 3.1 we know  $|(x_\varepsilon^*)_i| \leq \varepsilon$ , then  $i \in \Gamma_\mu$ . Hence  $\Gamma \subset \Gamma_\mu$ . We obtain (3.7).

Without loss of generality, we assume that  $\mathcal{N} \setminus \Gamma = \{1, 2, \dots, k\}$ . Define the function  $g: R^k \rightarrow R$  by

$$g(z) = \|Bz - b\|_2^2 + \lambda \|z\|_p^p.$$

Similar to the argument of (2.2) in Theorem 2.1, we have

$$\nabla g(z^*) = 2B(Bz^* - b) + \lambda p |z^*|^{p-1} \cdot \text{sign}(z^*) = 0$$

at  $z^* = (x_1^*, \dots, x_k^*)^T$ . Furthermore, let  $\bar{z}_\varepsilon^* = ((x_\varepsilon^*)_1, \dots, (x_\varepsilon^*)_k)^T$ , then

$$\begin{aligned} \nabla g(\bar{z}_\varepsilon^*) &= \nabla g(\bar{z}_\varepsilon^*) - \nabla g(z^*) \\ &= 2B^T B(\bar{z}_\varepsilon^* - z^*) + \lambda p |\bar{z}_\varepsilon^*|^{p-1} \cdot \text{sign}(\bar{z}_\varepsilon^*) - \lambda p |z^*|^{p-1} \cdot \text{sign}(z^*). \end{aligned}$$

Note that  $\text{sign}(\bar{z}_\varepsilon^*) = \text{sign}(z^*)$ . By using the mean value theorem, there exists  $\tilde{z}_\varepsilon^*$  such that

$$\begin{aligned} \nabla g(\bar{z}_\varepsilon^*) &= 2B^T B(\bar{z}_\varepsilon^* - z^*) + \lambda p \text{sign}(\bar{z}_\varepsilon^*) \cdot (|\bar{z}_\varepsilon^*|^{p-1} - |z^*|^{p-1}) \\ &= (2B^T B + \lambda p(p-1) D)(\bar{z}_\varepsilon^* - z^*), \end{aligned} \quad (3.10)$$

where  $D \in R^{k \times k}$  is a diagonal matrix whose diagonal elements are  $|\tilde{z}_\varepsilon^*|_i^{p-2}$ , which is between  $(\bar{z}_\varepsilon^*)_i$  and  $z_i^*$ ,  $i = 1, 2, \dots, k$ .

Since the matrix  $2B^T B + \lambda p(p-1)D$  is symmetric,  $0 < p < 1$  and  $|(z_\varepsilon^*)_i| \geq L$  for all  $i \in \mathcal{N} \setminus \Gamma$ , for any  $z \in R^k$  with  $\|z\| = 1$ , we have

$$\begin{aligned} z^T(2B^T B + \lambda p(p-1)D)z &= z^T(2B^T B)z + \lambda p(p-1)z^T D z \\ &\geq 2z^T(B^T B)z + \lambda p(p-1)L^{p-2}\|z\| \\ &\geq 2\lambda_{\min}(B^T B) + \lambda p(p-1)L^{p-2} \\ &> 0, \end{aligned}$$

where the last inequality uses the assumption of this theorem. Hence the matrix  $2B^T B + \lambda p(p-1)D$  is invertible. We conclude from (3.8) and (3.10) that

$$\begin{aligned} \|\bar{x}_\varepsilon^* - x^*\| = \|\bar{z}_\varepsilon^* - z^*\| &\leq \|(2B^T B + \lambda p(p-1)D)^{-1}\|\|\nabla g(\bar{z}_\varepsilon^*)\| \\ &\leq \|(2B^T B + \lambda p(p-1)L^{p-2}I)^{-1}\|\|\nabla g(\bar{z}_\varepsilon^*)\| \\ &= \|G^{-1}\|\|\nabla g(\bar{z}_\varepsilon^*)\| \\ &\leq \|G^{-1}\|\|\nabla f_\varepsilon(\bar{x}_\varepsilon^*)\|, \end{aligned}$$

where the last inequality uses  $\|\nabla g(\bar{z}_\varepsilon^*)\| \leq \|\nabla f_\varepsilon(\bar{x}_\varepsilon^*)\|$ , which can be shown as follows,

$$\begin{aligned} \|\nabla g(\bar{z}_\varepsilon^*)\| &= \|2B^T(B\bar{z}_\varepsilon^* - b) + \lambda p|\bar{z}_\varepsilon^*|^{p-1} \cdot \text{sign}(\bar{z}_\varepsilon^*)\| \\ &= \|2B^T(A\bar{x}_\varepsilon^* - b) + \lambda p|\bar{z}_\varepsilon^*|^{p-1} \cdot \text{sign}(\bar{z}_\varepsilon^*)\| \\ &= \|2B^T(A\bar{x}_\varepsilon^* - b) + \lambda\Psi_\varepsilon(\bar{z}_\varepsilon^*)\| \\ &\leq \|2A^T(A\bar{x}_\varepsilon^* - b) + \lambda\Psi_\varepsilon(\bar{x}_\varepsilon^*)\| \\ &= \|\nabla f_\varepsilon(\bar{x}_\varepsilon^*)\|, \end{aligned}$$

where the inequality uses  $(\bar{x}_\varepsilon^*)_i = 0$  for  $i \in \Gamma$  and  $((s_\varepsilon(0))^p)' = 0$ . ■

**Corollary 3.1** *Let  $\lambda_1$  be the smallest nonzero eigenvalue of  $A^T A$ . Suppose  $2\lambda_1 > \lambda p(1-p)L^{p-2}$ . Then for sufficiently small  $\varepsilon > 0$ , we have*

$$\text{dist}(\bar{x}_\varepsilon^*, \mathcal{X}_p) \leq (2\lambda_1 + \lambda p(p-1)L^{p-2})^{-1}\|\nabla f_\varepsilon(\bar{x}_\varepsilon^*)\|.$$

## 4 Hybrid OMP-SG algorithm using lower bound theory

The lower bound theory can be applied to improve existing algorithms and develop new algorithms. To demonstrate the application, we use a hybrid Orthogonal Matching Pursuit-smoothing gradient (OMP-SG) method to solve the  $\ell_2$ - $\ell_p$  minimization problem (1.1). Our limited computational experiment in this section does not intend to develop a new algorithm for sparse reconstruction, but to show how our theory could improve any existing algorithm to achieve a higher quality performance.

The OMP algorithm is well-known in the literature of signal processing. It is also called the basis pursuit method [10, 14, 21] in other fields. The following algorithm is a standard version of the OMP algorithm [3], but has a different stop criterion.

**Algorithm 1 Orthogonal Matching Pursuit(OMP)**

**Parameters:** Given the  $m \times n$  matrix  $A$ , the vector  $b \in R^m$  and the error threshold  $\beta_0$ .

**Initialization:** Initialize  $k = 0$ , and set

- the initial solution  $x^0 = 0$ .
- the initial residual  $r^0 = b - Ax^0 = b$ .
- the initial solution support  $\Lambda_0 = \emptyset$ .

**Main Iteration:** Increment  $k$  by 1 and perform the following steps:

- Find the index  $j_k$  that solves the optimization problem

$$j_k = \arg \max \frac{\|(Ax^{k-1} - b)^T a_j\|_2^2}{\|a_j\|} \text{ for } j \in \mathcal{N} \setminus \Lambda_{k-1}.$$

- Let  $\Lambda_k = \Lambda_{k-1} \cup \{j_k\}$ .
- Compute  $x^k$ , the minimizer of  $\|Ax - b\|_2^2$  subject to  $\text{support}\{x\} = \Lambda^k$ .
- Calculate the new residual  $r^k = Ax^k - b$ .
- If  $\|A^T r^k\| < \beta_0$ , stop, and let  $\Lambda = \Lambda_k$ .

**Output:** The support set  $\Lambda$  and matrix  $B = A_\Lambda \in R^{m \times |\Lambda|}$ .

The smoothing gradient method (SG) [29] is a simple method for Lipschitz continuous but nonsmooth nonconvex minimization problems. Define  $g_k = \nabla f_{\mu_k}(x^k)$ , the SG method is described as follows.

**Algorithm 2 Smoothing Gradient(SG)**

**Step 1.** Choose constants  $\sigma, \rho \in (0, 1)$ , and an initial point  $x^0$ . Set  $k = 0$ .

**Step 2.** Compute the step size  $\nu_k$  by the Armijo line search, where  $\nu_k = \max\{\rho^0, \rho^1, \dots\}$  and  $\rho^i$  satisfies

$$f_{\mu_k}(x^k - \rho^i g_k) \leq f_{\mu_k}(x^k) - \sigma \rho^i g_k^T g_k.$$

Set  $x^{k+1} = x^k - \nu_k g_k$ .

**Step 3.** If  $\|\nabla f_{\mu_k}(x^{k+1})\| \geq n\mu_k$ , then set  $\mu_{k+1} = \mu_k$ ; otherwise, choose  $\mu_{k+1} = \sigma\mu_k$ .

Now we present the hybrid OMP-SG algorithm for solving  $\ell_2$ - $\ell_p$  minimization problem (1.1) with the lower bound  $L$  defined in (1.4).

**Algorithm 3 Hybrid OMP-SG**

**Step 1.** Using the OMP algorithm to get the submatrix  $B$  and the support set  $\Lambda$ .

**Step 2.** Using the SG algorithm to find

$$y^* = \arg \min g(y) := \|By - b\|_2^2 + \lambda \|y\|_p^p.$$

**Step 3.** Choose  $\beta > \beta_0$ . Output a numerical solution  $x^*$ , where

$$x_j^* = \begin{cases} y_j^* & y_j^* \geq L \text{ and } j \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$



**Remark 4.1** In the hybrid OMP-SG method, we first choose candidates of columns of  $A$  which correspond to nonzero entries in a solution of (1.1). Next, we use the global convergence smoothing gradient method to find a minimizer of the reduced problem  $\min g(y)$ . Both before and after the SG method, we use the lower bound theory to eliminate zero entries in the solution precisely. It is worth noting that the lower bound theory is algorithms independent. For instance, we can replace the SG by the smoothing conjugate gradient (SCG) method [12] in Step 2 of the hybrid OMP-SG method to accelerate the algorithm, and have a hybrid OMP-SCG method.

Now we report numerical results to compare the performance of the hybrid OMP-SG method and OMP-SCG method for solving (1.1) with several other approaches to find sparse solutions. Our preliminary computational results indicate that the variable elimination according to our theory makes a significant difference. The computational test was conducted on a Philips PC (2.36 GHz, 1.96GB of RAM) with using Matlab 7.4.

We consider the following four approaches.

- LASSO: Solve the  $\ell_2$ - $\ell_1$  problem (1.3) by the least squares algorithm (Lars) proposed in [16].
- ConApp: Solve the  $\ell_2$ - $\ell_p$  problem (1.1) with  $p = \frac{1}{2}$  by using the following  $\ell_2$ - $\ell_1$  convex approximation [5]

$$\min \|Ax - b\|_2^2 + \lambda \sum_{i=1}^n \frac{|x_i|}{\sqrt{|x_i^{k-1}| + \varepsilon}} \quad (4.1)$$

at the  $k$ th iteration, where  $\varepsilon > 0$  is a parameter. We use the Lars to solve (4.1).

- OMP-SG: Solve the  $\ell_2$ - $\ell_p$  problem (1.1) by the hybrid OMP-SG.
- OMP-SCG: Solve the  $\ell_2$ - $\ell_p$  problem (1.1) by the hybrid OMP-SCG.

#### 4.1 Variable selection

This example is artificially generated and was firstly used in Tibshirani [25] to test the effectiveness of Lasso. The true solution is  $x^* = (3, 1.5, 0, 0, 2, 0, 0, 0)^T$ . We simulated 100 data sets consisting of  $n$  observations from the model

$$Ax = b + \sigma\varepsilon,$$

where  $\varepsilon$  is a noise vector generated by the standard normal distribution. We select three cases to discuss the performance of the three approaches LASSO, ConApp and OMP-SCG. The first case is  $m = 40$ ,  $\sigma = 3$ , the second case is  $m = 40$ ,  $\sigma = 1$  and the last case is  $m = 60$ ,  $\sigma = 1$ . We choose 20 data as the test set. The mean squared errors ( $MSE$ ) over the test set are summarized in Table 4.1. The average number of correctly identified zero coefficient ( $ANZ$ ) and the average number of the coefficients erroneously set to zero ( $NANZ$ ) over test set are also presented in Table 4.1. In our numerical experiment, we select  $p = 0.5$  in the  $\ell_2$ - $\ell_p$  problem (1.1).

From Table 4.1, we observe that OMP-SCG performs the best, followed by LASSO and ConApp.

Table 4.1: Results for variable selection

m	$\sigma$	Approach	MSE	ANZ	NANZ
40	3	LASSO	0.4730	4.77	0.23
		ConApp	0.4688	4.83	0.17
		OMP-SCG	0.4755	4.88	0.12
40	1	LASSO	0.1595	4.77	0.23
		ConApp	0.1541	4.86	0.14
		OMP-SCG	0.1511	4.91	0.09
60	1	LASSO	0.3582	4.92	0.08
		ConApp	0.3503	4.93	0.07
		OMP-SCG	0.3464	4.95	0.05

## 4.2 Signal reconstruction

The signal reconstruction has been studied extensively in the past decades [6, 15]. According to Donoho [15], the signal reconstruction can be solved by the  $\ell_2$ - $\ell_1$  model (1.3). In this subsection we apply the  $\ell_2$ - $\ell_p$  model to solve signal reconstruction problems.

Consider a real-valued, finite-length signal  $x \in R^n$ . Suppose  $x$  is  $T$ -sparse, that is, only  $T$  of the signal coefficients are nonzero and the others are zero. The sensing matrix  $A \in R^{m \times n}$  is the Gaussian random matrix. Our aim is to obtain good reconstructions of  $x$  by using the sensing matrix  $A$  with less number of  $n$ .

Let sampling be uniform in  $[0, 512]$ . We applied OMP-SCG, LASSO and ConApp to reconstruct the signal. The error between the reconstructed signal and the original one is computed by 2-norm.

In Table 4.2, we present numerical results of three sets of signal examples. The CPU time is given in second. From Table 4.2, we observe that the three approaches can reconstruct the original signal with  $m = 512, T = 60, n = 184$ , while OMP-SCG has the highest accuracy. Moreover, the LASSO can not reconstruct the signal with  $m = 512, T = 60, n = 182$ , but OMP-SCG and ConApp can reconstruct the original signal, while OMP-SCG has small error. Furthermore, if the original signal has  $m = 512, T = 130, n = 225$ , LASSO and ConApp algorithm can not reconstruct this signal, but OMP-SCG can reconstruct this signal perfectly with error=0.41. OMP-SG gives similar results as OMP-SCG, but uses more time. See Figure 2 for more details.

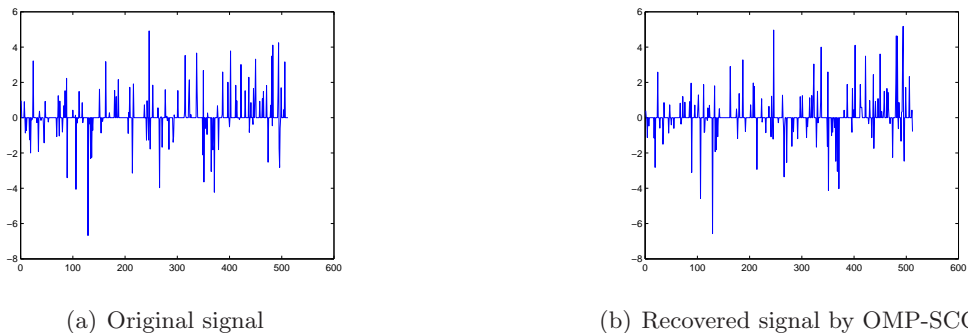
Figure 2: The signal with  $m = 512, T = 130, n = 225$

Table 4.2: Results for signal reconstruction

Problem	LASSO	ConApp	OMP-SCG			
	(Error,Time)	(Error,Time)	$L$	$\lambda$	Error	Time
$m = 512$ $T = 60$ $n = 184$	$(5.33 \times 10^{-4},$ 0.653)	$(1.29 \times 10^{-5},$ 6.82)	0.8	0.002	$1.12 \times 10^{-16}$	1.02
$m = 512$ $T = 60$ $n = 182$	(38.64, 0.43)	$(2.41 \times 10^{-5},$ 7.84)	0.7	0.001	$1.03 \times 10^{-16}$	1.34
$m = 512$ $T = 130$ $n = 225$	(122.25, 0.69)	(119.43, 19.99)	0.00001	0.00006	0.41	4.03

### 4.3 Prostate cancer

The data set in this subsection is from examination of the correlation between the level of prostate specific antigen and a number of clinical measures who were about to receive a radical prostatectomy. We download the data from UCI Standard database [1]. The data set consists of the medical records of 97 patients who were about to receive a radical prostatectomy. The predictors are eight clinical measures: lcavol, lweight, age, lbph, svi, lcp, gleason and pgg45. More detailed explanation can be found in the UCI Standard database. This is a variable selection problem. One of our main aims is to identify which predictors are most significant in predicting the response.

The prostate cancer data were divided into two parts: a training set with 67 observations and a test set with 30 observations. The prediction error is the mean squared errors over the test set. The numerical results of Ridge regression [19] and Best Subset [2] were derived from [18]. In this example, we also select  $p = 0.5$  in the  $\ell_2$ - $\ell_p$  model (1.1).

From Table 4.3 we find that OMP-SG and OMP-SCG succeeded in finding three main factors and have smaller prediction accuracy than ConApp and LASSO. This implies that OMP-SG and OMP-SCG can find more sparse solution with smaller prediction error than LASSO.

Table 4.3: Results for prostate cancer

Parameter	LASSO	Ridge	Best Subset	ConApp	OMP-SG	OMP-SCG
$x_1$ (lcavol)	0.545	0.389	0.740	0.6187	0.6436	0.6436
$x_2$ (lweight)	0.237	0.238	0.367	0.2362	0.2804	0.2804
$x_3$ (lage)	0	-0.029	0	0	0	0
$x_4$ (lbph)	0.098	0.159	0	0.1003	0	0
$x_5$ (svi)	0.165	0.217	0	0.1858	0.1856	0.1857
$x_6$ (lcp)	0	0.026	0	0	0	0
$x_7$ (gleason)	0	0.042	0	0	0	0
$x_8$ (pgg45)	0.059	0.123	0	0	0	0
Number of nonzero	5	8	2	4	3	3
Prediction error	0.478	0.5395	0.5723	0.468	0.4418	0.4419

Now we apply Theorem 3.1 to compute the error bound of  $\bar{x}_\varepsilon^*$  to  $x^* \in \mathcal{X}_p^*$ , for a given  $\varepsilon > 0$ . We set  $\varepsilon < 0.01$  and  $p = 0.5$ . The numerical results are listed in Table 4.4.

Table 4.4: Error bounds for  $\|\bar{x}_\varepsilon^* - x^*\|$

$\varepsilon$	L	$\lambda$	error bound
0.001	0.015	0.1304	$1.5793 \times 10^{-5}$
0.0001	0.0119	0.1164	$5.7310 \times 10^{-6}$
0.00001	0.0119	0.1164	$5.5721 \times 10^{-6}$

It is worth noting that the absolute lower bounds, the error bounds and the hybrid OMP-SCG method can be extended to

$$\min_{x \in R^n} \|Ax - b\|_2^2 + \sum_{i=1}^n \varphi(|x_i|^p), \quad (4.2)$$

where the function  $\varphi : R_+ \rightarrow R$  is a potential function, e.g. [23], which includes (1.1) as a special case. Table 4.5 lists some well-used potential functions (left) and their extensions (right).

Table 4.5: Potential functions (PFs) where  $\alpha \in (0, 1)$  is a parameter

	Convex	Non Lipschitz
$f_1$	$\varphi( t ) =  t $	$\varphi( t ^p) =  t ^p$
	Non convex	Non Lipschitz
$f_2$	$\varphi( t ) =  t ^p$	$\varphi( t ^p) = ( t ^p)^\alpha$
$f_3$	$\varphi( t ) = \frac{\alpha t }{1 + \alpha t }$	$\varphi( t ^p) = \frac{\alpha t ^p}{1 + \alpha t ^p}$
$f_4$	$\varphi( t ) = \log(\alpha t  + 1)$	$\varphi( t ^p) = \log(\alpha t ^p + 1)$

The numerical results with different potential functions and  $\alpha = 0.1699$  are listed in Table 4.6. We observe that choosing  $p \leq 0.5$  seems good for this example, since using  $p \leq 0.5$  can find three main factors with smaller prediction error than  $p > 0.5$ .

Table 4.6: Comparisons of different  $p$  with different PFs

$p$	$(L, \text{Number of nonzero}, \text{Prediction error})$			
	$f_1$	$f_2$	$f_3$	$f_4$
0.9	(0.0001, 4, 0.4754)	(0.011, 4, 0.473)	(2.500, 4, 0.475)	(2.040, 4, 0.474)
0.8	(0.0015, 4, 0.4740)	(0.013, 4, 0.468)	(1.990, 4, 0.474)	(1.851, 4, 0.474)
0.7	(0.0050, 4, 0.4741)	(0.012, 4, 0.465)	(1.755, 4, 0.474)	(1.550, 4, 0.474)
0.6	(0.0084, 4, 0.4661)	(0.015, 3, 0.446)	(1.545, 4, 0.475)	(1.344, 4, 0.475)
0.5	(0.0119, 3, 0.4419)	(0.016, 3, 0.445)	(1.420, 3, 0.477)	(1.200, 3, 0.483)
0.4	(0.0148, 3, 0.4456)	(0.014, 3, 0.445)	(1.480, 3, 0.477)	(1.114, 3, 0.484)
0.3	(0.0176, 3, 0.4429)	(0.012, 3, 0.443)	(1.590, 3, 0.484)	(1.190, 3, 0.483)
0.2	(0.0196, 3, 0.4359)	(0.018, 3, 0.443)	(1.955, 3, 0.483)	(1.240, 3, 0.482)

## 5 Final remark

Using the first and second order necessary condition for a local minimizer, we establish lower bounds for nonzero entries in any local optimal solution of a minimization model where the

objective function is the sum of a data-fitting term in  $\ell_2$  norm and a regularization term in  $\ell_p$  norm ( $0 < p < 1$ ). This establishes a theoretical justification by “zeroing” those entries in an approximate solution whose values are small enough, and explanation why the model generates more sparse solutions when the norm parameter  $p < 1$ .

Moreover, the lower bounds clearly show the relationship between the sparsity of the solution and the choice of the regularization parameter and norm. These provide a systematic mechanism for selecting the model parameters, such as regularization weight  $\lambda$  and norm  $p$ . Based on these relations, we propose a hybrid orthogonal matching pursuit-smoothing gradient (OMP-SG) method for the nonconvex, non-Lipschitz continuous  $\ell_2$ - $\ell_p$  minimization problem. Numerical results show that using the OMP-SG method to solve the  $\ell_2$ - $\ell_p$  minimization problem (1.1) can provide more sparse solutions with smaller predictor error compared with several well-known approaches for variable selection.

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