Optimization Methods and Software
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/goms20

Sparse portfolio rebalancing model based on inverse optimization

Meihua Wang [a], Fengmin Xu [b] & Guan Wang [b]
[a] School of Economics and Finance, Xi'an Jiaotong University, Xi'an, 710061, People's Republic of China
[b] Department of Mathematics, Xi'an Jiaotong University, Xi'an, 710049, People's Republic of China
Published online: 09 Oct 2012.

Optimization Methods & Software
iFirst, 2012, 1–13

Sparse portfolio rebalancing model based on inverse optimization

Meihua Wang[a], Fengmin Xu[b]* and Guan Wang[b]

[a]*School of Economics and Finance, Xi'an Jiaotong University, Xi'an 710061, People's Republic of China;*
[b]*Department of Mathematics, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China*

(*Received 16 December 2011; final version received 31 May 2012*)

This paper considers a sparse portfolio rebalancing problem in which rebalancing portfolios with minimum number of assets are sought. This problem is motivated by the need to understand whether the initial portfolio is worthwhile to adjust or not, inducing sparsity on the selected rebalancing portfolio to reduce transaction costs (TCs), out-of-sample performance and small changes in portfolio. We propose a sparse portfolio rebalancing model by adding an l_1 penalty item into the objective function of a general portfolio rebalancing model. In this way, the model is sparse with low TCs and can decide whether and which assets to adjust based on inverse optimization. Numerical tests on four typical data sets show that the optimal adjustment given by the proposed sparse portfolio rebalancing model has the advantage of sparsity and better out-of-sample performance than the general portfolio rebalancing model.

Keywords: portfolio rebalancing; sparse; inverse optimization; second-order cone program

1. Introduction

Much research in portfolio selection theory and practice has been made in recent six decades, and most research concentrates on initial investments. However, with the elapse of time, the initial portfolio may become not optimal. If an investor hopes to hold the investment on the portfolio for a long time period, it is necessary to adjust the portfolio based on either the maximizing expected return of the portfolio or minimizing the risk of the portfolio. This is called portfolio rebalancing (revision or adjusting) problem [12].

From a perspective on supplying additional investment, there exist two kinds of main strategies in portfolio rebalancing. The first rebalance strategy needs the investor to supply an additional fixed amount of money. Fang *et al*. give a mean-absolute deviation model for portfolio rebalancing, in which linear transaction costs (TCs) and minimal purchase unit are integrated into the rebalancing model, and an equality constraint is introduced to balance the investment of buying, selling, TCs and additional investment amount [9]. Wang *et al*. [20] assume that the return rates of assets satisfy t-distribution and VaR is selected as the risk measure of portfolios. Another kind of strategy is called the self-finance strategy [4,13,14]. It is assumed that the investor will not supply any additional

*Corresponding author. Email: fengminxu@mail.xjtu.edu.cn

ISSN 1055-6788 print/ISSN 1029-4937 online

http://dx.doi.org/10.1080/10556788.2012.700309
http://www.tandfonline.com

investment amount, and a self-finance constraint is included in the portfolio rebalancing model. The investor aims either to maximize the expected return of the resulting portfolio after paying TCs under a given tolerated level of risk, or to minimize the total rebalance TCs subject to a specified requirement on the expected return of the portfolio, risk and self-finance constraints. Zhang *et al.* [23] regard the returns of risky assets as fuzzy variables and then solve the portfolio rebalancing problem with TCs on the basis of the credibility theory. They propose a sequential minimal optimization algorithm for calculating the optimal portfolio adjusting strategy in [24]. Multiple criteria such as risk, return, short-selling, skewness and kurtosis are considered and then five portfolio rebalancing models are constructed in [21].

Recently, statistical regularization approaches have attracted extensive attention and have been successfully applied to the mean–variance portfolio selection and the portfolio adjustment problem with only a budget constraint which can promote out-of-sample properties and decrease TCs [5–8]. Brodie *et al.* add an l_1 penalty to the traditional mean–variance model. This penalty regularizes the portfolio selection problem, encourages sparse portfolios, and allows to account for TCs [5]. DeMiguel *et al.* [6,7] discuss several different regularization techniques for the portfolio construction problem, including the imposition of constraints on l_p ($p = 1$ or 2) norms of the portfolio weight vector. The main contribution of Fan *et al.* is the provision of deep mathematical insights into the utility approximations with the gross-exposure constraint [8]. Moreover, the regularization parameter has an important impact on the results of the performance, there are many statistical criteria to use [1,17,19].

However, some important criteria are not considered in the existing general rebalancing model, such as sparsity, when and whether the initial portfolio is worthwhile to adjust or not. Since the optimal rebalance portfolio determined by the history data may not be optimal after the rebalance, these criteria should be considered so as to be prevented from over-adjustment. And it is important that small changes in portfolio may be a good choice because of the existing of TCs and other transaction limitations. In this paper, we consider a sparse portfolio rebalancing model by adding an l_1 penalty into the objective function of general portfolio rebalancing model. In this way, the model is sparse with low TCs and can decide whether and which assets to adjust based on inverse optimization.

Summarily, the main contribution of our work is to construct a sparse portfolio rebalancing model and then propose a method to decide whether the initial portfolio is needed to adjust based on inverse optimization [11,22]. If needed, the optimal adjustment in the portfolio obtained by the sparse portfolio rebalancing model has the advantage of sparsity, stability and good performance in the out-of-sample time period and this will be tested in numerical tests.

The rest of the paper is organized as follows. The sparse portfolio rebalancing model with an l_1 penalty and the corresponding solving method are proposed in Section 2. Numerical tests are stated in Section 3. Conclusions and future research are given in Section 4.

Throughout the paper prime ($'$) denotes transposition of vectors without special declaration. The notation s_i or $(s)_i$ is used to denote the ith component of the vector s.

2. Sparse portfolio rebalancing

2.1 *General portfolio rebalancing model*

In this subsection, we will describe the general portfolio rebalancing model and the relative optimization method based on the work of Lobo *et al.* [14] without TCs.

Assumed that the investor holds a given portfolio invested in n risky assets, which is considered to be adjusted by a number of transactions. Let $w = (w_1, w_2, \dots, w_n)'$ be the current holdings in each asset and $\sum_{i=1}^{n} w_i = 1$, where w_i denotes the investment weight invested on the ith asset. Let

$r = (r_1, r_2, \ldots, r_n)'$ be the return rates of the assets and assumed that $r \sim \mathcal{N}(\mu, V)$, $V \succ 0$ (V is a symmetric positive matrix), i.e.

$$\mathbf{E}r = \mu, \quad \mathbf{E}(r-\mu)(r-\mu)' = V,$$

where $\mathbf{E}$ denotes expectation.

Let $x = (x_1, x_2, \ldots, x_n)'$ be the weight transacted in all assets, with $x_i > 0$ for buying and $x_i < 0$ for selling. After transacting, the final portfolio is $w + x$. The expected return after rebalance is $\mu'(w+x)$, and the variance is $(w+x)'V(w+x)$.

The investor's goal is to maximize the expected return of the portfolio, subject to a set of constraints. Then the general portfolio rebalancing model without TCs and any additional investment can be formulated as

$$\begin{aligned} \max_{x} \quad & \mu'(w+x) \\ \text{s.t.} \quad & e'x = 0 \\ & (w+x)'V(w+x) \le \sigma_{\max}^2 \\ & Ax \le b, \end{aligned} \tag{1}$$

where $\sigma_{\max}^2$ ($\sigma_{\max} > 0$) denotes the risk level acceptable to the investor. e denotes the vector with all entries as one. The constraints $Ax \le b$ involve the lower and upper limitations of investments on different assets $l \le w + x \le u$ and so on.

To get the optimal solution of problem (1), second-order cone programs (SOCPs) is applied to solve this problem, which has been the focus of research both in terms of algorithms and application. So we replace the constraint

$$(w+x)'V(w+x) \le \sigma_{\max}^2$$

by

$$\| L(w+x) \| \le \sigma_{\max}$$

with $\| \cdot \|$ is the Euclidean or l_2 norm, L is the Cholesky decomposition matrix or the (symmetric) matrix square root of V (i.e. $V = L'L$), problem (1) can be converted into an SOCP problem. Then the general portfolio rebalancing problem can be efficiently solved by a software package such as SeDuMi [18], which is an efficient tool for handling SOCP problems.

2.2 *Sparse and stable portfolio rebalancing model*

In this subsection, we will propose a sparse portfolio rebalancing model by adding an l_1 penalty constraint to the rebalance portfolio. In order to avoid frequently transacting, we use some criteria to decide whether the present portfolio is needed to be adjusted or not.

The sparse portfolio rebalancing model can be given by adding an l_1 penalty with a penalty factor $\lambda > 0$ into the objective function of (1). Three reasons are as follows. Firstly, adding an l_1 penalty on the general portfolio rebalancing model leads to a sparse and stable problem with low TCs. Secondly, the corresponding solution achieved by the sparse portfolio rebalancing model will have good out-of-sample performance in return which is one of the most concerns of the investor. Finally, the l_1 penalty item can be considered as a TC function. Then the sparse portfolio rebalancing problem can be written as

$$\begin{aligned} \min_{x} \quad & -\mu'(w+x) + \lambda \parallel x \parallel_1 \\ \text{s.t.} \quad & e'x = 0 \\ & \parallel L(w+x) \parallel \leq \sigma_{\max} \\ & Ax \leq b. \end{aligned} \tag{2}$$

To prevent from frequently purchasing or selling the assets, it is necessary to determine that when the portfolio is needed to rebalance. This is rarely considered in the previous work and we will deal with such problems from a new perspective based on inverse optimization [11]. The rebalance will be carried out and the adjustment weight of assets are the optimal solution of (2) when $\hat{x} = 0$ is not a feasible solution for problem (2). Otherwise, the idea of inverse optimization is introduced to deal with this problem. The rebalance will be carried out only when $\hat{x} = 0$ is no more an optimal solution for the problem (2).

In typical markets, the expected return rate vector μ varies frequently, but the covariance matrix V is relatively stable (see [15]). Let $\bar{\mu}$ is the current estimate of vector μ, $\phi(\hat{x} = 0)$ denote the set of all vectors μ that make $\hat{x} = 0$ the optimal solution of problem (2).

$$\phi(\hat{x} = 0) = \{\mu \mid \hat{x} = 0 \text{ is optimal to problem (2)}\}. \tag{3}$$

If $\bar{\mu}$ is not in the above set, the rebalance is needed, otherwise not.

Next we consider the following optimization problem:

$$\begin{aligned} \min_{\mu} \quad & \sigma(\mu) = \parallel \mu - \bar{\mu} \parallel^2_{V^{-1}} \\ \text{s.t.} \quad & \mu \in \phi(\hat{x} = 0), \end{aligned} \tag{4}$$

where $\parallel \mu - \bar{\mu} \parallel^2_{V^{-1}} = (\mu - \bar{\mu})' V^{-1} (\mu - \bar{\mu})$ denotes the weighted norm with a matrix V^{-1} (the inverse matrix of covariance matrix V).

The optimal solution of problem (4) is denoted by μ^*. It can be seen that if the current estimation $\bar{\mu}$ for μ in problem (2) satisfies $\bar{\mu} \in \phi(\hat{x} = 0)$, then $\sigma(\mu^*) = 0$ and $x = 0$ is optimal to the problem (2), that is, we hold the portfolio unchanged. However, if $\bar{\mu} \notin \phi(\hat{x} = 0)$, then $\sigma(\mu^*) > 0$ and the portfolio will be adjusted only when $\sigma(\mu^*) \geq C_\theta$ so as to manage the trade-off between TCs and higher returns, where the value of C_θ is determined by investor's confidence level θ. The value of C_θ can be selected as a fixed number or by any other criteria. According to the idea of Iyengar and Kang [11], $C_\theta = \chi^2_p(\theta)/p$ where $\chi^2_p(\theta)/p$ is the θ-critical value of a χ^2-distribution with p degrees of freedom. We can understand this roughly that $\hat{x} = 0$ is nearly optimal to the problem (2) when $\bar{\mu}$ is not so far away from the set $\phi(\hat{x} = 0)$ and hence rebalance is not needed, which can be avoid over-adjustment to cause large TCs.

2.3 *Reformulation*

For convenience of calculation, we will reformulate the problem (2) and simplify the set $\phi(\hat{x} = 0)$ for entirely solving the portfolio rebalance problem.

Since the l_1 penalty in the objective of the problem is non-smooth, then a group of variables $y_i (i = 1, 2, \ldots, n)$ are introduced to convert the problem (2) into a continuous and differentiable optimization problem equivalently, that is,

$$
\begin{aligned}
\min_{x,y} \quad & -\mu'(w+x)+\lambda e'y \\
\text{s.t.} \quad & \| L(w+x) \| \leq \sigma_{\max} \\
& -y \leq x \leq y \\
& e'x = 0 \\
& Ax \leq b.
\end{aligned} \tag{5}
$$

Clearly, problem (5) is an SOCP problem which is a type of conic programming problem and can be solved efficiently both in theory [16] and in practice [18]. If $(\hat{x};\hat{y})=0$ is not a feasible solution of problem (5), then the rebalance will be carried out and the adjustment is the optimal solution of problem (5); otherwise, optimal value $\sigma(\mu^*)$ for the inverse optimization problem (4) corresponding set $\phi((\hat{x};\hat{y})=0)=\{\mu \mid (\hat{x};\hat{y})=0$ is optimal to problem (5)$\}$.

ASSUMPTION 1 *Assumed that there exists at least one point* $(x^0;y^0)$, *satisfying* $e'x^0=0$, $\| L(w+x^0) \| < \sigma_{\max}, Ax^0 < b, x^0-y^0<0,$ *and* $x^0+y^0>0$.

THEOREM 1 *For an optimal solution* $(\hat{x};\hat{y})$ *of problem* (5), *the inverse optimization problem*

$$
\begin{aligned}
\min_{\mu} \quad & \sigma(\mu) = \| \mu - \bar{\mu} \|^2_{V^{-1}} \\
s.t. \quad & \mu \in \phi((\hat{x};\hat{y})) = \{\mu \mid (\hat{x};\hat{y}) \textit{ is optimal to problem } (5)\}
\end{aligned} \tag{6}
$$

can be written as a conic program which can be efficiently solved.

Proof We first simplify the set $\phi((\hat{x};\hat{y}))$ for an optimal solution $(\hat{x};\hat{y})$ to problem (5). From Assumption 1, it follows that the point $(\hat{x};\hat{y})$ is optimal for problem (5) iff $(\hat{x};\hat{y})$ satisfies Karush–Kuhn–Tucker conditions [11]: $\exists p^1,p^3,p^4 \in \mathbf{R^n}, p^2 \in \mathbf{R^m}, p^0,p^5 \in \mathbf{R}$, such that $p^0,p^2,p^3,p^4 \geq 0$, $p^0 \geq \|p^1\|$,

$$
\begin{bmatrix} -\mu \\ \lambda e \end{bmatrix} - \begin{bmatrix} L' \\ 0_{n\times n} \end{bmatrix} p^1 - \begin{bmatrix} -A' \\ 0_{n\times m} \end{bmatrix} p^2 - \begin{bmatrix} -I_n \\ I_n \end{bmatrix} p^3 - \begin{bmatrix} I_n \\ I_n \end{bmatrix} p^4 - \begin{bmatrix} e \\ 0_{n\times 1} \end{bmatrix} p^5 = 0, \tag{7}
$$

$$
p^0\sigma_{\max} + [p^1]'L(w+\hat{x}) = 0, \tag{8}
$$

$$
[p^2]'(A\hat{x}-b) = 0, \tag{9}
$$

$$
[p^3]'(-\hat{x}+\hat{y}) = 0, \tag{10}
$$

$$
[p^4]'(\hat{x}+\hat{y}) = 0. \tag{11}
$$

Then the relative problem (6) can be converter into a conic program:

$$
\begin{aligned}
\min_{\mu,p^0,p^1,p^2,p^3,p^4,p^5} \quad & \sigma(\mu) = (\mu-\bar{\mu})'V^{-1}(\mu-\bar{\mu}) \\
\text{s.t.} \quad & -\mu - L'p^1 + A'p^2 + p^3 - p^4 - p^5 = 0 \\
& \lambda e - p^3 - p^4 = 0
\end{aligned}
$$

$$
\begin{aligned}
& p^0\sigma_{\max} + [p^1]'L(w+\hat{x}) = 0 \\
& [p^2]'(A\hat{x} - b) = 0 \\
& [p^3]'(-\hat{x} + \hat{y}) = 0 \\
& [p^4]'(\hat{x} + \hat{y}) = 0 \\
& p^0 \geq \|p^1\| \\
& p^0, p^2, p^3, p^4 \geq 0.
\end{aligned} \tag{12}
$$

This problem can be efficiently solved [3]. ■

LEMMA 1 [2,11] *Fix $z \in \mathcal{K}$. Let $\mathcal{U} \subset \mathcal{K}^*$ ($\mathcal{K}^*$ is the dual cone of a cone $\mathcal{K}$) denote the set of vectors $q \in \mathcal{K}^*$ such that $q'z = 0$. Then for a second-order cone $\mathcal{K} = \mathcal{K}_{\text{so}} = \{(q_0; q) \in \mathbf{R}^{n+1} : q_0 \geq \sqrt{q'q}\} \subset \mathbf{R}^{n+1}$:*

$$
\mathcal{U} = \begin{cases} \mathcal{K}_{\text{so}}, & z = 0 \\ \left\{(q_0; q) : q = -\dfrac{q_0}{z_0}z, q_0 \geq 0\right\}, & z \in \mathbf{bd}(\mathcal{K}_{\text{so}})\backslash\{0\} \\ 0, & z \in \mathbf{int}(\mathcal{K}_{\text{so}}) \end{cases} \tag{13}
$$

For an optimal solution $(\hat{x}; \hat{y})$ of problem (5), we get from Lemma 1 that: if $\|L(w+\hat{x})\| = \sigma_{\max}$, then

$$p^1 = -\frac{p^0}{\sigma_{\max}}L(w+\hat{x});$$

otherwise $\|L(w+\hat{x})\| < \sigma_{\max}$, then

$$p^0 = 0; p^1 = 0.$$

Let $(\hat{x}; \hat{y}) = 0$, then problem (6) for determining whether to adjust or not can be reformulated as

$$
\begin{aligned}
\min_{\mu, p^0, p^2, p^3, p^4, p^5} \quad & \sigma(\mu) = (\mu - \bar{\mu})'V^{-1}(\mu - \bar{\mu}) \\
\text{s.t.} \quad & -\mu + \frac{p^0}{\sigma_{\max}}Vw + A'p^2 + p^3 - p^4 + p^5 = 0 \\
& \lambda e_n - p^3 - p^4 = 0 \\
& [p^2]'b = 0 \\
& p^0, p^3, p^4 \geq 0
\end{aligned} \tag{14}
$$

for $\|Lw\| = \sigma_{\max}$, or

$$
\begin{aligned}
\min_{\mu, p^2, p^3, p^4, p^5} \quad & \sigma(\mu) = (\mu - \bar{\mu})'V^{-1}(\mu - \bar{\mu}) \\
\text{s.t.} \quad & -\mu + A'p^2 + p^3 - p^4 + p^5 = 0 \\
& \lambda e_n - p^3 - p^4 = 0 \\
& [p^2]'b = 0 \\
& p^3, p^4 \geq 0
\end{aligned} \tag{15}
$$

for $\|Lw\| < \sigma_{\max}$. These two problems are quadratic programs which can be efficiently solved. Assumed that μ^* is the corresponding item of μ in the optimal solution for problem (14) or (15), if

$\sigma(\mu^*) < C_\theta$, we continue to hold the portfolio w; otherwise $\sigma(\mu^*) \geq C_\theta$, we solve the problem (5) to get an optimal adjustment x^* and finally we hold the portfolio $w + x^*$.

3. Numerical tests

In this section, we will test the proposed sparse portfolio rebalancing model with practical data.

3.1 *Test data sets and steps*

The model will be tested on the data described in [10]. The authors give four data sets taken from the German Stock Exchange Market (XETRA). Each data set includes daily return rates of 100 securities which compose the XETRA DAX 100 index at the date of 1 April 2005. Each data set consists of 6 months of in-sample daily observations and 6 months of out-of-sample ones. In order to consider all possible market trends they have constructed four data sets corresponding to different performance in-sample and out-of-sample time periods. The first data set is characterized by a market trend going up in the in-sample period as well as in the out-of-sample period (up–up data set), the second data set by a market increasing in the in-sample period and decreasing in the out-of-sample one (up–down data set), the third data set by a market going down in the in-sample period and going up in the out-of-sample period (down–up data set) and the last set by a market going down in both the in-sample and the out-of-sample periods (down–down data set). More details about the four data sets can be found in [10].

To show the superiority of the proposed sparse portfolio rebalancing method, general portfolio rebalancing model are also applied to solve the problem. Initial investment is assumed to be one unit of amount (such as dollar or euro) so as to show the results obviously. For each data sets, test experiments will be conducted in the following three stages:

Stage 1: The 6 months of in-sample daily observations are applied to construct the initial portfolio by the traditional mean-variance portfolio selection model (maximization the expected return of the portfolio, subject to different levels of risk tolerance $\sigma^2_{\max}$). Here, the matrix V is estimated by the entire data of the time period and expected return vector μ is estimated by the data of 6 months before the estimation time. The initial capital is assumed as one dollar. Then we hold it over 3 months.

Stage 2: At the end of the third month in out-of-sample period, the portfolio the investor held may not be optimal along with the new information or changes of return rate. General and the proposed sparse portfolio rebalancing models are used to get optimal adjustment portfolios. For sparse models, we solve the corresponding problem for different values of the penalty factor λ and confidence level θ. The portfolios after rebalance will be held until the end of out-of-sample time period.

Stage 3: According to the daily data in the data set, the wealth of the investor at the end of the out-of-sample time period is shown in figures for different rebalance methods.

All the problems are coded in Matlab and the SOCP problems can be efficiently solved by a matlab toolbox SeDuMi [18]. The computational tests are run on a personal computer with Pentium Pro 2.6 GHZ and a 2 GB memory.

3.2 *Results and analysis*

In the tests, the following setting is used: short-sales are permitted, no limitation on the lower and upper bound for each asset (There is no doubt that our proposed method can be applied to solve the portfolio rebalancing problem with any other convex constraints on portfolio.). For each

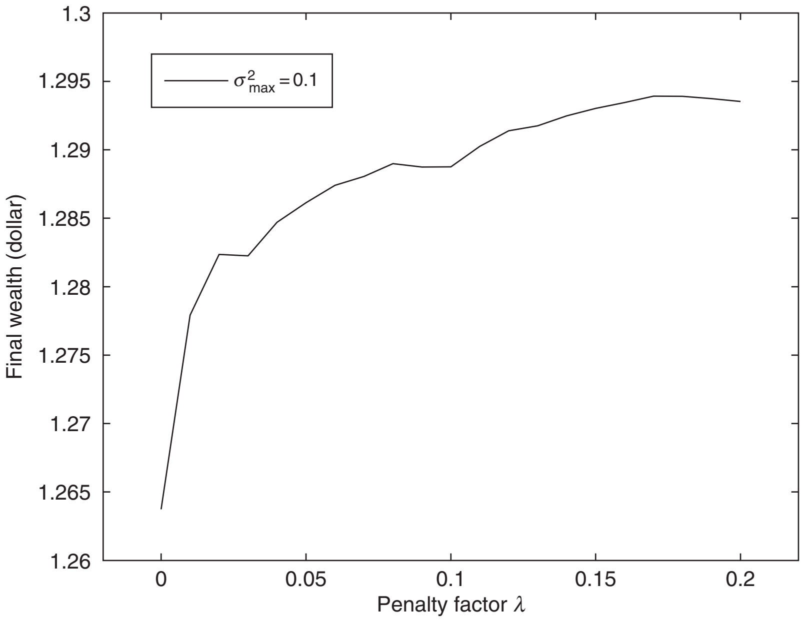

Figure 1. Final wealth determined by general and sparse models for up–up data set ($\sigma^2_{\max} = 0.1$).

data set, we show the results for the confidence level $\theta = 0.1$, $\theta = 0.5$ and $\theta = 0.9$ for a general model and a sparse model with different values of penalty factor λ. To show the results more clearly, the point with $\lambda = 0$ denote the value computed by a general portfolio rebalancing model. Figures 1–6 plot the curve of the final wealth the investor holds at the end of the out-of-sample time period. Among these, the curves are plots for different value of $\sigma^2_{\max}$ and C_θ.

Figure 1 shows the results of the final wealth for up–up data sets with the risk tolerance $\sigma^2_{\max} = 0.1$. For $\sigma^2_{\max} = 0.1$, the investment will be invested in a portfolio according to the description of *Stage 1* at the end of the in-sample period. Then the investor will hold the portfolio for the next 3 months. Since the unchanged portfolio $x = 0$ is not a feasible solution of the portfolio rebalancing model at the third month of the out-of-sample time, the rebalance must be conducted. We plot the curves of final wealth along with various values of λ. As can be seen, final wealth computed by our proposed sparse portfolio rebalancing model for various values of λ ($\lambda > 0$) is larger than that by the general portfolio rebalancing model ($\lambda = 0$). In Figure 2, when risk level $\sigma^2_{\max} = 0.2$, the unchanged portfolio $x = 0$ is a feasible solution of the portfolio rebalancing model at the third month of the out-of-sample time. Then the idea of inverse optimization is applied to solve the portfolio rebalancing problem. Figure 2 shows different cases for the selection methods of C_θ, respectively, the fixed valued $C_\theta = 0.1, 0.2, 0.3$ and $C_\theta = \chi^2_p(\theta)/p$ [11] for different confidence levels $\theta = 0.1, 0.5, 0.9$. In addition, we also give the results for $C_\theta = 0$ or $\theta = 0$ (In fact, this is the case of the sparse portfolio rebalanicng model without considering whether to adjust). The conclusion can be obtained from two subfigures in Figure 2: the proposed sparse portfolio model has a superiority than do the general portfolio rebalancing model in the value of final wealth (That is to say, final wealth for $\lambda > 0$ is all larger than that for $\lambda = 0$). Furthermore, whether to adjust is decided by the the value of C_θ. More larger the value of C_θ is, more less the possibility of rebalance is. For the second subfigure in Figure 2, rebalance is not needed when $C_\theta = 0.2$ and 0.3 for $\lambda = 0.05$; otherwise, rebalance is needed when $C_\theta = 0.1$ for $\lambda = 0.05$. This can be seen as a support of the above conclusion. Finally, comparing Figures 1 and 2, the higher the risk tolerance is, the higher the return of the portfolio is. When the investor wants a higher return, he must have a larger risk tolerance.

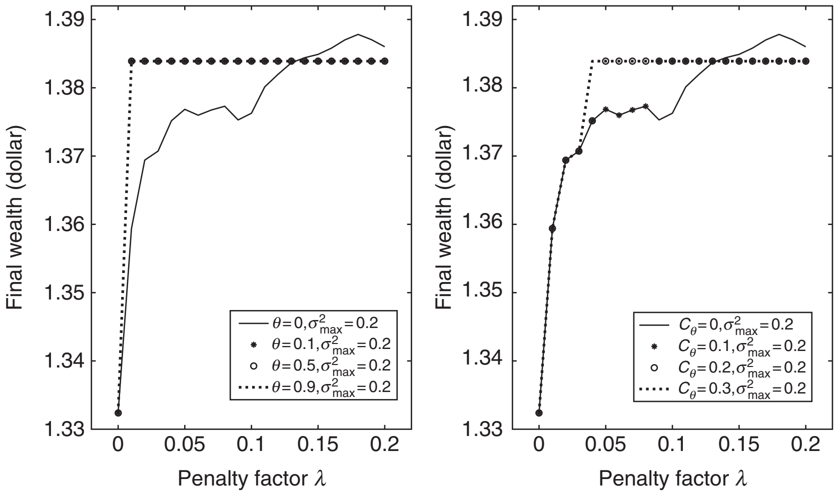

Figure 2. Final wealth determined by general and sparse models for up–up data set($\sigma^2_{\max} = 0.2$).

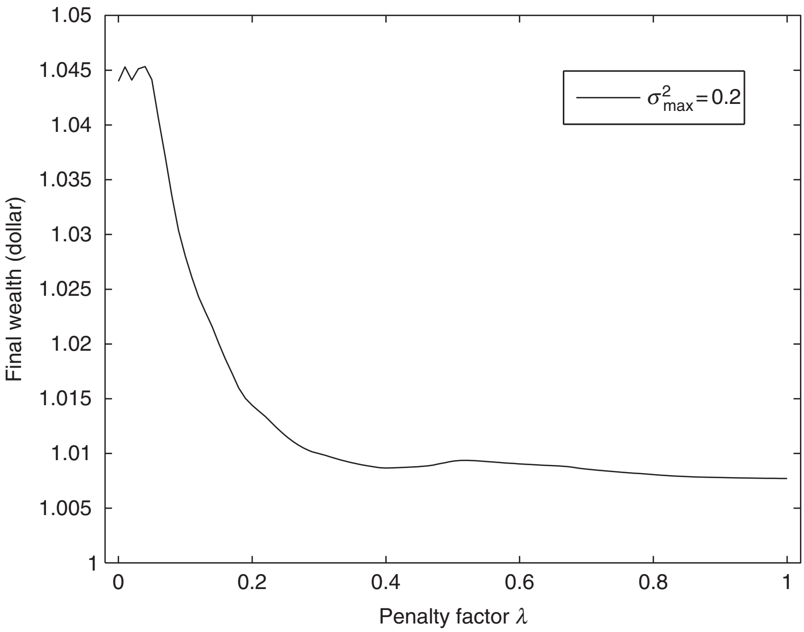

Figure 3. Final wealth determined by general and sparse models for up–down data set ($\sigma^2_{\max} = 0.2$).

As can be seen, the final wealth for initial investment(one dollar) determined by the proposed sparse model for different values of $\lambda > 0$ are always larger than the wealth determined by the general model for the up–up data set. The sparse model will solve the portfolio rebalancing problem better than the general model for this up–up data set. However, different data sets may lead to different results. So the results for the other three data sets are also shown in Figures 3–6. It can be seen that the final wealth determined by the sparse model is not always better than the value determined by the general model for different values of $\lambda > 0$. Especially for the down–down data set, the value computed by the sparse model without TC is slightly less than the value

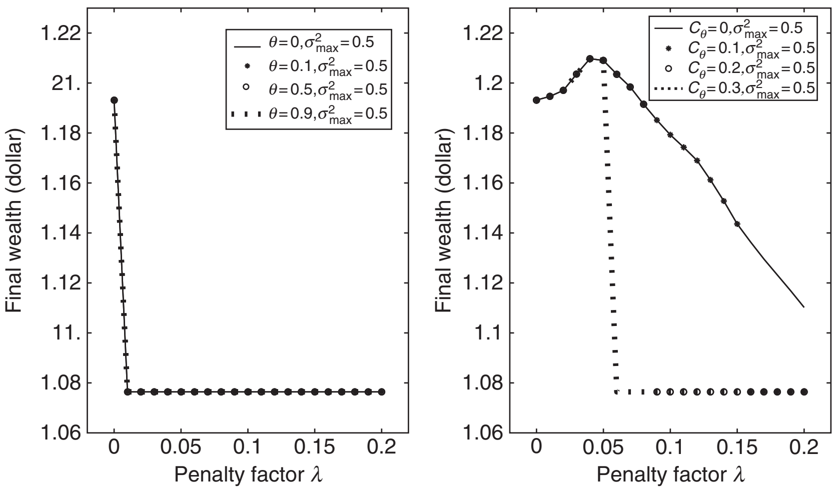

Figure 4. Final wealth determined by general and sparse models for up–down data set ($\sigma^2_{\max} = 0.5$).

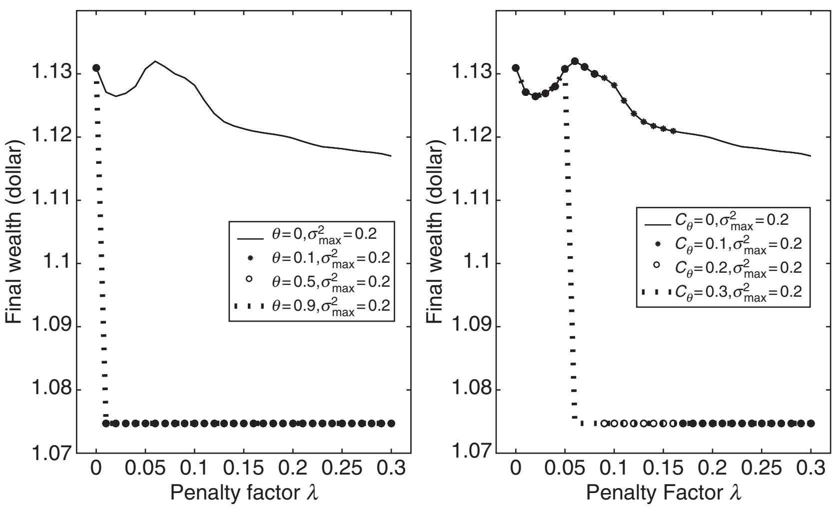

Figure 5. Final wealth determined by general and sparse models for down–up data set ($\sigma^2_{\max} = 0.2$).

computed by general model, but the former is larger than the latter after paying the linear TCs (the ratio of transaction cots is 0.005) in most cases. This is because the superiority of the sparse model is to control the number and amount of the rebalance portfolio so as to decrease the TCs. This can be also seen from Figures 7 and 8. However, we can find some points for the the sparse model better than the general model. We can get better results by adjusting the value of λ. It is highlighted that the sparse portfolio rebalancing model with TCs will have a much larger superiority than the general portfolio rebalancing model.

The value of penalty factor λ and C_θ is important for the investor to obtain large returns. In fact, this can be tuned according to the character of data sets or other considerations . This is not our main focus and the existing sparsity research [1,17,19] can help us determine the proper value of λ so as to obtain a high return. Our main contribution in this paper is to propose a new method to solve the portfolio rebalancing problem. For the four representative types of data sets, all or

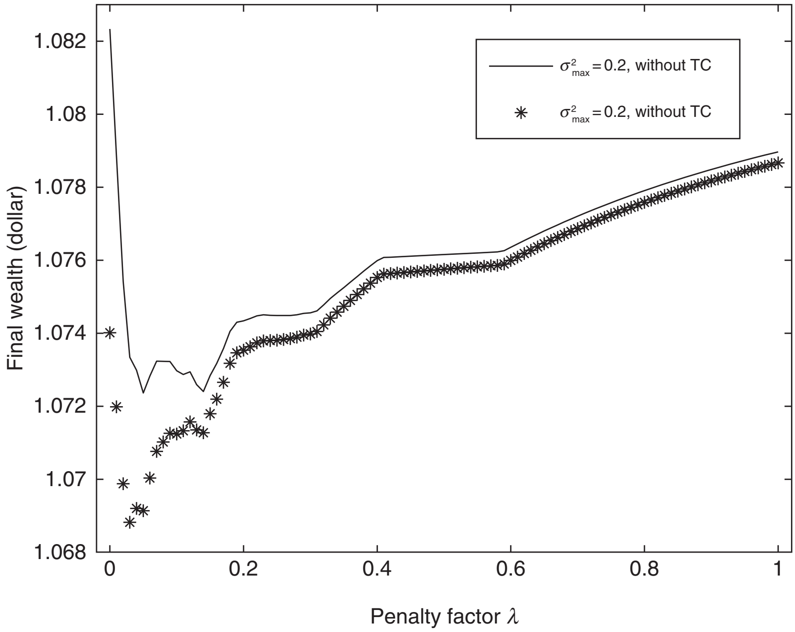

Figure 6. Final wealth determined by general and sparse models for down–down data set ($\sigma^2_{\max} = 0.2$).

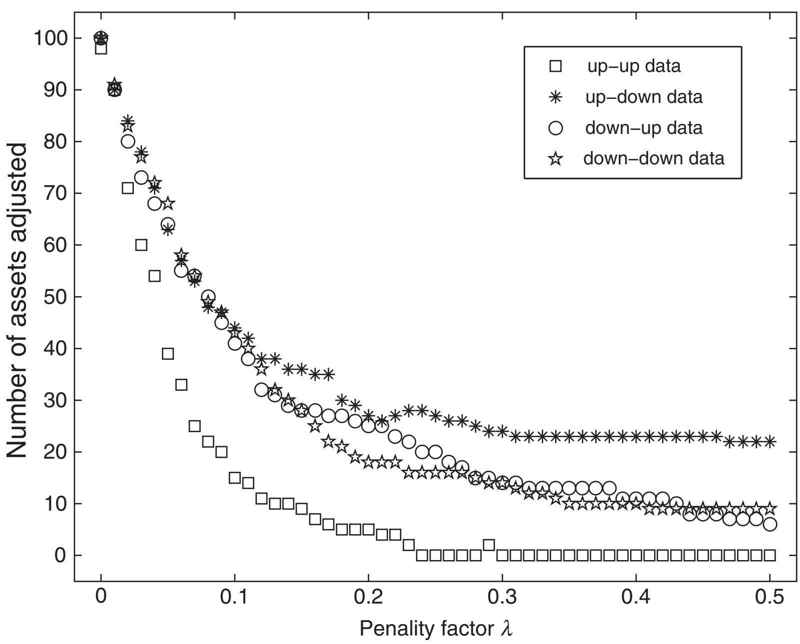

Figure 7. Number of assets adjusted determined by general and sparse models.

part of the results solved by the sparse portfolio rebalancing model are better than the one by the general model.

Additionally, we also show the changes of the number of assets adjusted and the total transaction weight $e'|x| = \sum_{i=1}^{n} |x_i|$ along with the changes of the values of λ in Figures 7 and 8. Similarly,

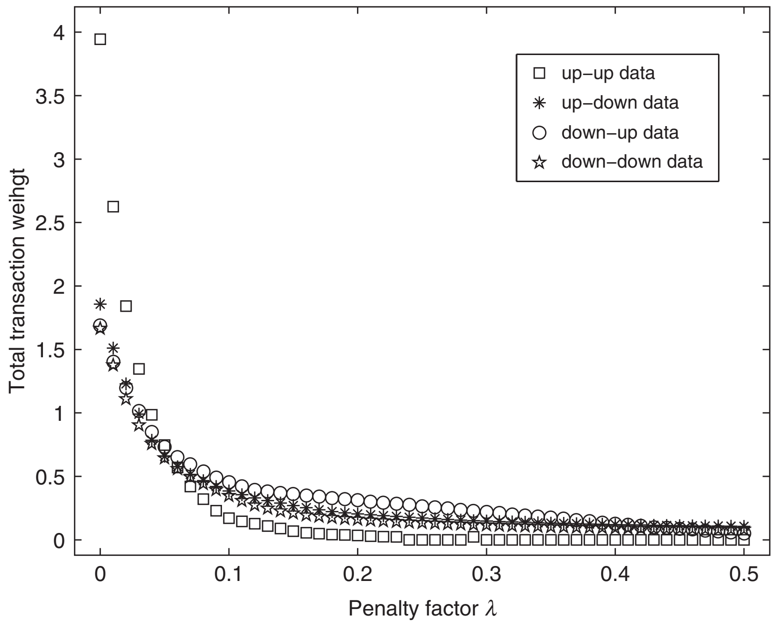

Figure 8. Total transaction weight determined by general and sparse models.

the points $\lambda = 0$ denotes the results determined by the general portfolio rebalancing model. Both of the two values have decreasing trends with an increase of the value of λ. More lager the value of penalty factor λ is, more sparser the optimal rebalance portfolio is. Introducing an l_1 penalty does have an effect on promoting the sparse solution. In addition to good performance on final wealth shown in Figures 1–6, if TCs are considered, the proposed the sparse portfolio rebalancing model will cause less costs as shown in Figure 8.

4. Conclusions and future research

We have proposed a sparse portfolio rebalancing model based on adding an l_1 penalty to rebalance the portfolio and the inverse optimization method. It becomes possible for the investor to decide whether the initial portfolio is worthwhile to adjust or not. Additionally, it can help us decide which assets are adjusted and prompt the performance of the portfolio after rebalance by smaller changes in number and weight in the rebalancing portfolio.

Numerical tests on four representative data sets described in [10] show that the proposed model is effective. It is better than the general portfolio rebalancing model in the aspects of out-of-sample performance, number of transaction assets and total transaction weight. Additionally, the l_1 penalty can be regarded as a function of TCs. This has good guidance in the practical investment area. Especially, the value of penalty factor λ is a key parameter to influence the performance of the portfolio rebalancing. The research results in the sparsity modelling area can be applied to determine the value of λ based on the data sets or other considerations.

In this paper, the risk of a portfolio is measured as variance. It is an interesting study to solve the sparse portfolio rebalancing problem with other risk measurements like MAD, CVaR and so on. Furthermore, other l_p (such as $p = 0$ or $p = \frac{1}{2}$) penalty can be considered to regularize the portfolio rebalancing problem. Finally more complicated functions of TCs or other limitations can be included into the model. Then the design of the algorithms seems more interesting and challenging. Some of these are under our current investigation.

Acknowledgements

This research is supported by the National Natural Science Foundations, PR China (Grant No. 10971162, 11101325 and 71171158). The authors thank Gianfranco Guastaroba and Renata Mansini for their grateful help.

References

[1] H. Akaike, *Information theory and an extension of the maximum likelihood principle*, in *Second International Symposium on Information Theory*, B.N. Petrov and F. Caki, eds., Akademiai Kiado, Budapest, 1973, pp. 267–281.
[2] F. Alizadeh and D. Goldfarb, *Second-order cone programming*, Math. Program. B 95 (2003), pp. 3–51.
[3] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MPS-SIAM Series in Optimization, SIAM, Philadelphia, PA, 2001.
[4] M.J. Best and J. Hlouskova, *Portfolio selection and transaction costs*, Comput. Optim. Appl. 24 (2003), pp. 95–116.
[5] J. Brodie, I. Daubechies, C.D. Mol, C. Giannone, and I. Loris, *Sparse and stable markowitz portfolios*, European Central Bank Working Paper Series, 936, 2008.
[6] V. DeMiguel, L. Garlappi, F.J. Nogales, and R. Uppal, *A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms*, Manage. Sci. 55(5) (2009), pp. 798–812.
[7] V. DeMiguel, L. Garlappi, and R. Uppal, *Optimal versus naive diversification: How inefficient is the* $1/N$ *portfolio strategy?* Rev. Financ. Stud. 22(5) (2009), pp. 1915–1953.
[8] J. Fan, J. Zhang, and K. Yu, *Asset allocation and risk assessment with gross exposure constraints for vast portfolios*, Ann. Statist. 25 (2008), pp. 1425–1432.
[9] Y. Fang, K.K. Lai, and S Wang, *Portfolio rebalancing with transaction costs and a minimal purchase unit*, Dyn. Contin. Discrete Impuls. Syst. Ser. B: Appl. Algorithms 12 (2005), pp. 499–515.
[10] G. Guastaroba, R. Mansini, and M.G. Speranza, *Models and simulations for portfolio rebalancing*, Comput. Econom. 33 (2009), pp. 237–262.
[11] G. Iyengar and W. Kang, *Inverse conic programming and applications*, Oper. Res. Lett. 33 (2005), pp. 319–330.
[12] H. Konno and A. Wijayanaake, *Mean-absolute deviation portfolio optimization model under transaction costs*, J. Oper. Res. Soc. Japan 42(4) (1999), pp. 422–435.
[13] M.S. Lobo, M. Fazel, and S. Boyd, *Portfolio optimization with linear and fixed transaction costs*, Tech. Rep., Department of Electrical Engineering, Stanford University, Stanford, CA 94305, September 2002.
[14] M.S. Lobo, M. Fazel, and S. Boyd, *Portfolio optimization with linear and fixed transaction costs*, Ann. Oper. Res. 152(1) (2007), pp. 341–365.
[15] R.O. Michaud, *Efficient Asset Management: A Practical Guide to Stock Portfolio Management and Asset Allocation*, Financial Management Association Survey and Synthesis Series, Harward Business School Press, Boston, 1998.
[16] Y. Nesterov and A. Nemirovskii, *Interior-point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, PA, 1993.
[17] G. Schwarz, *Estimating the dimension of a model*, Ann. Statist. 6 (1978), pp. 461–464.
[18] J.F. Sturm, *Using SeDuMi 1.02, a matlab toolbox for optimization over symmetric cones*, Optim. Methods Softw. 11–12 (1999), pp. 625–653.
[19] R. Tibshirani, *Regression shrinkage and selection via the lasso*, J. R. Stat. Soc. 46 (1996), pp. 431–439.
[20] Y. Wang, Z. Chen, and K. Zhang, *A chance-constrained portfolio selection probelm under t-distribution*, Asia-Pac. J. Oper. Res. 24(4) (2007), pp. 535–556.
[21] J.-R. Yu and W.-Y. Lee, *Portfolio rebalancing model using multiple criteria*, Eur. J. Oper. Res. 209 (2011), pp. 166–175.
[22] J. Zhang and C.Xu, *Inverse optimization for linearly constrained convex separable programming problems*, Eur. J. Oper. Res. 200 (2010), pp. 671–679.
[23] X. Zhang, W.-G. Zhang, and R. Cai, *Portfolio adjusting optimization under credibility measures*, J. Comput. Appl. Math. 234 (2010), pp. 1458–1465.
[24] X. Zhang, W.-G. Zhang, and W.-J. Xu, *An optimization model of the portfolio adjusting problem with fuzzy return and a SMO algorithm*, Expert Syst. Appl. 38 (2011), pp. 3069–3074.