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RESEARCH ARTICLE

General H-matrices and their Schur complements

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Abstract The definitions of θ-ray pattern matrix and θ-ray matrix are firstly proposed to establish some new results on nonsingularity/singularity and convergence of general H-matrices. Then some conditions on the matrix $A \in \mathbb{C}^{n\times n}$ and nonempty $\alpha \subset \langle n\rangle = \{1, 2, \ldots, n\}$ are proposed such that A is an invertible H-matrix if $A(\alpha)$ and A/α are both invertible H-matrices. Furthermore, the important results on Schur complement for general H-matrices are presented to give the different necessary and sufficient conditions for the matrix $A \in H_n^M$ and the subset $\alpha \subset \langle n\rangle$ such that the Schur complement matrix $A/\alpha \in H^I_{n-|\alpha|}$ or $A/\alpha \in H^M_{n-|\alpha|}$ or $A/\alpha \in H^S_{n-|\alpha|}$.

Keywords Schur complement, convergence, general H-matrices
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1 Introduction

It is well known that *H-matrices* that closely related to *M-matrices* [2,24] widely arise in numerical linear algebra, numerical solution of partial differential equations, modern control theory, dynamic systems, and so on, see [2,8,16,24].

In the research of the convergence of iterative methods for linear and nonlinear systems and spectral theory, Ostrowski [21] firstly introduced the concept *of nonsingular M-matrix* and *nonsingular H-matrix*. Later, Fiedler and Ptak extended this concept to possible *singular M-matrices* [9] and *singular H-matrices* [10]. Recently, the definition for H-matrices has been extended to encompass a wider set, known as the set of *general H-matrices*. In some recent papers, [3–5], a partition of the $n \times n$ general H-matrix set, H_n, into three mutually exclusive classes was obtained: the invertible class, H_n^I, where

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all general H-matrices are nonsingular, the singular class, H_n^S, formed only by singular H-matrices, and the mixed class, H_n^M, in which singular and nonsingular H-matrices coexist. As is well known, general H-matrices that belong to the classes H_n^I has many beautiful properties such as nonsingularity, eigenvalue distribution, convergence, structure heredity (i.e., properties on Schur complement), and so forth. Furthermore, there are still many researcher to study this class of general H-matrices. For example, [17,23] have proved a theorem that if $A \in H_n^I$, then $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n\rangle$. However, the converse conclusion of this theorem is not true in general. What conditions are such that the converse conclusion of this theorem holds? On the other hand, little attention is paid to the work on the properties of general H-matrices that belong to the classes H_n^M and H_n^S. What properties do the two classes of general H-matrices have?

Aim at the problem above, some further results on the class of general H-matrices are proposed in this paper. In particular, some properties including nonsingularity/singularity, convergence, and Schur complement on general H-matrices that belong to the classes H_n^M and H_n^S are studied and presented to show that general H-matrices that belong to the classes H_n^M still have some beautiful properties.

This paper is organized as follows. Some notations and preliminary results about special matrices are given in Section 2. Based on the result of Kolotilina, the nonsingularity/singularity criteria on general H-matrices is proposed in Section 3. Some convergence results on general H-matrices are then presented in Section 4. The important results on Schur complement for general H-matrices are given in Section 5, where we give the different conditions for the matrix $A \in H_n^M$ and the subset $\alpha \subset N$ such that the Schur complement matrix $A/\alpha \in H_{n-|\alpha|}^I$ or $A/\alpha \in H_{n-|\alpha|}^M$ or $A/\alpha \in H_{n-|\alpha|}^S$. Conclusions are given in Section 6.

2 Preliminaries

In this section, we give some notions and preliminary results about special matrices that are used in this paper.

$\mathbb{C}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$) will be used to denote the set of all $m \times n$ complex (resp. real) matrices. $\mathbb{Z}$ denotes the set of all integers. Let $|\alpha|$ denote the cardinality of the set $\alpha \subseteq \langle n\rangle = \{1, 2, \ldots, n\} \subset \mathbb{Z}$. For nonempty index sets $\alpha, \beta \subseteq \langle n\rangle$, $A(\alpha, \beta)$ is the submatrix of $A \in \mathbb{C}^{n\times n}$ with row indices in α and column indices in β. The submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Let $A \in \mathbb{C}^{n\times n}$, $\alpha \subset \langle n\rangle$, and $\alpha' = \langle n\rangle - \alpha$. If $A(\alpha)$ is nonsingular, the matrix

$$A/\alpha = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') \quad (1)$$

is called the Schur complement with respect to $A(\alpha)$, indices in both α and α' are arranged with increasing order. We shall confine ourselves to the nonsingular $A(\alpha)$ as far as A/α is concerned.

Let

$$A = (a_{ij}) \in \mathbb{C}^{m\times n}, \quad B = (b_{ij}) \in \mathbb{C}^{m\times n}.$$

Denote

$$A \otimes B = (a_{ij}b_{ij}) \in \mathbb{C}^{m\times n}$$

the Hadamard product of the matrices A and B. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *nonnegative* if $a_{ij} \geqslant 0$ for all $i, j \in \langle n\rangle$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a *Z-matrix* if $a_{ij} \leqslant 0$ for all $i \neq j$. We will use Z_n to denote the set of all $n \times n$ Z-matrices. A matrix $A = (a_{ij}) \in Z_n$ is called an *M-matrix* if A can be expressed in the form $A = sI - B$, where $B \geqslant 0$, and $s \geqslant \rho(B)$, the spectral radius of B. If $s > \rho(B)$, A is called a *nonsingular M-matrix*; if $s = \rho(B)$, A is called a *singular M-matrix*. M_n, $M_n^{\bullet}$, and M_n^0 will be used to denote the set of all $n \times n$ M-matrices, the set of all $n \times n$ nonsingular M-matrices, and the set of all $n \times n$ singular M-matrices, respectively. It is easy to see that

$$M = M_n^{\bullet} \cup M_n^0, \quad M_n^{\bullet} \cap M_n^0 = \emptyset. \tag{2}$$

The *comparison matrix* of a given matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

It is clear that $\mu(A) \in Z_n$ for a matrix $A \in \mathbb{C}^{n\times n}$. The set of *equimodular matrices* associated with A, denoted by

$$\omega(A) = \{B \in \mathbb{C}^{n\times n} \colon \mu(B) = \mu(A)\}.$$

Note that both A and $\mu(A)$ are in $\omega(A)$. A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called a *general H-matrix* if $\mu(A) \in M_n$ (see [2]). If $\mu(A) \in M_n^{\bullet}$, A is called an *invertible H-matrix*; if $\mu(A) \in M_n^0$ with $a_{ii} = 0$ for at least one $i \in \langle n\rangle$, A is called a *singular H-matrix*; if $\mu(A) \in M_n^0$ with $a_{ii} \neq 0$ for all $i \in \langle n\rangle$, A is called a *mixed H-matrix*. H_n, H_n^I, H_n^S, and H_n^M will denote the set of all $n \times n$ general H-matrices, the set of all $n \times n$ invertible H-matrices, the set of all $n \times n$ singular H-matrices, and the set of all $n \times n$ mixed H-matrices, respectively (see [3]). Similar to (2), we have

$$H_n = H_n^I \cup H_n^S \cup H_n^M, \quad H_n^I \cap H_n^S \cap H_n^M = \emptyset. \tag{3}$$

For $n \geqslant 2$, an $n \times n$ complex matrix A is *reducible* if there exists an $n \times n$ permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{4}$$

where A_{11} is an $r \times r$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix, where $1 \leqslant r < n$. If no such permutation matrix exists, then A is called *irreducible*. If

A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Definition 2.1 A matrix $A \in \mathbb{C}^{n\times n}$ is called *diagonally dominant by row* if

$$|a_{ii}| \geqslant \sum_{1\leqslant j\leqslant n,\, j\neq i} |a_{ij}| \tag{5}$$

holds for all $i \in \langle n\rangle$. If inequality in (5) holds strictly for all $i \in \langle n\rangle$, A is called *strictly diagonally dominant by row.* If A is irreducible and the inequality in (5) holds strictly for at least one $i \in \langle n\rangle$, A is called *irreducibly diagonally dominant by row.* If (5) holds with equality for all $i \in \langle n\rangle$, A is called *diagonally equipotent by row.*

D_n (resp. SD_n, ID_n) and DE_n will be used to denote the sets of all $n \times n$ (resp. strictly, irreducibly) diagonally dominant matrices and the set of all $n\times n$ diagonally equipotent matrices, respectively.

Definition 2.2 A matrix $A \in \mathbb{C}^{n\times n}$ is called *generalized diagonally dominant* if there exist positive constants α_i, $i \in \langle n\rangle$, such that

$$\alpha_i|a_{ii}| \geqslant \sum_{1\leqslant j\leqslant n,\, j\neq i} \alpha_j|a_{ij}| \tag{6}$$

holds for all $i \in \langle n\rangle$. If inequality in (6) holds strictly for all $i \in \langle n\rangle$, A is called *generalized strictly diagonally dominant.* If (6) holds with equality for all $i \in \langle n\rangle$, A is called *generalized diagonally equipotent.*

We will denote the sets of all $n\times n$ generalized (strictly) diagonally dominant matrices and the set of all $n \times n$ generalized diagonally equipotent matrices by GD_n (GSD_n) and GDE_n, respectively.

Definition 2.3 A matrix A is called *nonstrictly diagonally dominant*, if either (5) or (6) holds with equality for at least one $i \in \langle n\rangle$.

Remark 2.4 Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be nonstrictly diagonally dominant and $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$. If $A(\alpha)$ is a (generalized) diagonally equipotent principal submatrix of A, then the following hold:

- $A(\alpha, \alpha') = 0$, which shows that A is reducible;
- $A(i_1) = (a_{i_1i_1})$ being (generalized) diagonally equipotent implies $a_{i_1i_1} = 0$.

Remark 2.5 Definitions 2.2 and 2.3 show that

$$D_n \subset GD_n, \quad GSD_n \subset GD_n.$$

The following lemma will introduce the relationship of (generalized) diagonally dominant matrices and general H-matrices and some properties of general H-matrices that will be used in the rest of the paper.

Lemma 2.6 [26,28,30,31] *Let $A \in D_n$ (GD_n). Then $A \in H_n^I$ if and only if A has no* (*generalized*) *diagonally equipotent principal submatrices. Furthermore,* *if* $A \in D_n \cap Z_n$ $(GD_n \cap Z_n)$, *then* $A \in M_n^{\bullet}$ *if and only if* A *has no* (*generalized*) *diagonally equipotent principal submatrices.*

Lemma 2.7 [2] $SD_n \cup ID_n \subset H_n^I = GSD_n$.

Lemma 2.8 [3] $GD_n \subset H_n$.

It is interested in whether $H_n \subseteq GD_n$ is true or not. The answer is "NOT". Some counterexamples are given in [3] to show that $H_n \subseteq GD_n$ is not true. But, under the condition "irreducibility", the following conclusion holds.

Lemma 2.9 [3] *Let* $A \in \mathbb{C}^{n\times n}$ *be irreducible. Then* $A \in H_n$ *if and only if* $A \in GD_n$.

More importantly, under the condition "reducibility", we have the following conclusion.

Lemma 2.10 *Let* $A \in \mathbb{C}^{n\times n}$ *be reducible. Then* $A \in H_n$ *if and only if in the Frobenius normal form of* A,

$$PAP^{\mathrm{T}} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1s} \\ & R_{22} & \cdots & R_{2s} \\ & & \ddots & \vdots \\ 0 & & & R_{ss} \end{bmatrix}, \tag{7}$$

each irreducible diagonal square block R_{ii} *is generalized diagonally dominant, where* P *is a permutation matrix,* $R_{ii} = A(\alpha_i)$ *is either* 1×1 *zero matrices or irreducible square matrices,* $R_{ij} = A(\alpha_i, \alpha_j)$, $i \neq j$, $i, j = 1, 2, \ldots, s$, *furthermore,* $\alpha_i \cap \alpha_j = \emptyset$ *for* $i \neq j$, *and* $\cup_{i=1}^{s} \alpha_i = \langle n \rangle$.

The proof of Lemma 10 follows from Lemma 2.9 and [3, Theorem 5].

Lemma 2.11 *A matrix* $A \in H_n^M \cup H_n^S$ *if and only if in the Frobenius normal form* (7) *of* A, *each irreducible diagonal square block* R_{ii} *is generalized diagonally dominant and has at least one generalized diagonally equipotent principal submatrix.*

Proof It follows from (3), Lemmas 2.6 and 2.10 that the conclusion of this lemma is obtained immediately. □

3 Nonsingularity/singularity on general *H*-matrices

As is well known, nonsingularity/singularity of a matrix is a very important property. On the other hand, the relationship between diagonal dominance of a matrix and its nonsingularity attracts researchers' attention. A series of concepts, such as strictly diagonally dominant matrix, irreducibly diagonally dominant matrix, diagonally dominant matrix with nonzero-entry chain, and semi-strictly diagonally dominant matrix, have been proposed for the research of nonsingularity of diagonally dominant matrices (see [2,24]). Later, the concept of "diagonal dominance" is extended to the one of "generalized diagonal dominance". Furthermore, generalized strictly diagonally dominant matrices, equivalent to invertible H-matrices, is nonsingular. In recent years, Kolotilina [13] and Zhang et al. [27–31] have considerable interest in the work on nonsingularity/singularity of nonstrictly diagonally dominant matrices, and obtained a lot of results as follows.

Theorem 3.1 [13] *If an irreducible matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ satisfies that there exist positive constants α_i, $i = 1, 2, \ldots, n$, such that*

$$\alpha_i|a_{ii}| \geqslant \sum_{1\leqslant j\leqslant n, j\neq i} \alpha_j|a_{ij}|, \quad \forall\ i \in \langle n\rangle, \tag{8}$$

then A is singular if and only if all the relations in (8) *are equalities and there exists a unitary diagonal matrix D such that*

$$D^{-1}D_A^{-1}AD = \mu(D_A^{-1}A), \tag{9}$$

where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ and $\mu(D_A^{-1}A)$ is the comparison matrix for $D_A^{-1}A$.

Theorem 3.2 [30,31] *A matrix $A \in D_n(GD_n)$ is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible and* (*generalized*) *diagonally equipotent principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $1 < k \leqslant n$, which satisfies condition that there exists a $k \times k$ unitary diagonal matrix U_k such that*

$$U_k^{-1}D_{A_k}^{-1}A_kU_k = \mu(D_{A_k}^{-1}A_k), \tag{10}$$

where

$$D_{A_k} = \mathrm{diag}(a_{i_1i_1}, a_{i_2i_2}, \ldots, a_{i_ki_k}).$$

Theorem 3.3 [28,30] *Let $A \in D_n$ (GD_n). Then A is singular if and only if A has at least one singular principal submatrix.*

Lemma 2.8 shows that the class of general H-matrices includes the class of generalized diagonally dominant matrices. Conversely, it is NOT. Therefore, it is necessary to study nonsingularity/singularity of general H-matrices. In this section, the definitions of θ-ray pattern matrix and θ-ray matrix are firstly proposed to establish some new results on nonsingularity/singularity of general H-matrices.

Definition 3.4 Let $E^{\mathrm{i}\theta} = (\mathrm{e}^{\mathrm{i}\theta_{rs}}) \in \mathbb{C}^{n\times n}$, where $\mathrm{e}^{\mathrm{i}\theta_{rs}} = \cos\theta_{rs} + i\sin\theta_{rs}$, $i = \sqrt{-1}$ and $\theta_{rs} \in \mathbb{R}$ for all $r, s \in \langle n\rangle$. The matrix $E^{\mathrm{i}\theta} = (\mathrm{e}^{\mathrm{i}\theta_{rs}}) \in \mathbb{C}^{n\times n}$ with $n \geqslant 3$ is called θ-ray pattern matrix if

(i) $\theta_{rs} + \theta_{sr} = 2k\pi$ holds for all $r, s \in \langle n\rangle$, $r \neq s$, where $k \in \mathbb{Z}$;

(ii) both $\theta_{rs} - \theta_{rt} = \theta_{ts} + (2k + 1)\pi$ and $\theta_{sr} - \theta_{tr} = \theta_{st} + (2k + 1)\pi$ hold for all $r, s, t \in \langle n\rangle$ and $r \neq s$, $r \neq t$, $t \neq s$, where $k \in \mathbb{Z}$;

(iii) $\theta_{rr} = \theta$ for all $r \in \langle n \rangle$, $\theta \in [0, 2\pi)$.

The matrix $E^{\mathrm{i}\theta} = (\mathrm{e}^{\mathrm{i}\theta_{rs}}) \in \mathbb{C}^{2\times 2}$ is called θ-ray pattern matrix if the first and third item of three items above both hold.

Remark 3.5 It follows from Definition 3.4 that the matrices with different arguments θ_{sr} of complex entries can belong to the same class of θ-ray pattern matrices because of the periodicity of trigonometric functions (sine functions and cosine functions), i.e., $E^{\mathrm{i}\theta} = E^{\mathrm{i}(\theta+2k\pi)}$ for all $k \in \mathbb{Z}$.

Example 3.6 The matrix

$$A = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi/3} & e^{-\mathrm{i}\pi/4} \\ \mathrm{e}^{\mathrm{i}\pi/4} & \mathrm{e}^{\mathrm{i}\pi/3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\mathrm{i}\sqrt{3} & -\sqrt{2}-\mathrm{i}\sqrt{2} \\ \sqrt{2}+\mathrm{i}\sqrt{2} & 1+\mathrm{i}\sqrt{3} \end{bmatrix}$$

is a θ-ray pattern matrix.

Example 3.7 The matrix

$$\begin{aligned} E^{\mathrm{i}\theta} &= \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi/2} & \mathrm{e}^{-\mathrm{i}\pi/3} & \mathrm{e}^{-\mathrm{i}\pi/6} \\ \mathrm{e}^{\mathrm{i}\pi/3} & \mathrm{e}^{\mathrm{i}\pi/2} & \mathrm{e}^{-5\mathrm{i}\pi/6} \\ \mathrm{e}^{\mathrm{i}\pi/6} & \mathrm{e}^{5\mathrm{i}\pi/6} & \mathrm{e}^{\mathrm{i}\pi/2} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2\,\mathrm{i} & 1-\mathrm{i}\sqrt{3} & \sqrt{3}-\mathrm{i} \\ 1+\mathrm{i}\sqrt{3} & 2\,\mathrm{i} & -\sqrt{3}-\mathrm{i} \\ \sqrt{3}+\mathrm{i} & -\sqrt{3}+\mathrm{i} & 2\,\mathrm{i} \end{bmatrix} \end{aligned}$$

is a θ-ray pattern matrix.

Example 3.8 $n \times n$ matrices

$$A = 2\mathrm{diag}(1, 1, \ldots, 1) - \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

with $\theta_{ii} = 0$, $\theta_{ij} = \pi$, $i > j$ and $\theta_{ij} = -\pi$, $i < j$, $i, j = 1, 2, \ldots, n$ is a θ-ray pattern matrix.

Definition 3.9 Any matrix $A = (a_{rs}) \in \mathbb{C}^{n\times n}$ has the following form:

$$A = \mathrm{e}^{\mathrm{i}\eta} \cdot |A| \otimes E^{\mathrm{i}\theta} = (\mathrm{e}^{\mathrm{i}\eta} \cdot |a_{rs}|\mathrm{e}^{\mathrm{i}\theta_{rs}}) \in \mathbb{C}^{n\times n}, \tag{11}$$

where $\eta \in \mathbb{R}$, $|A| = (|a_{rs}|) \in \mathbb{R}^{n\times n}$ and $E^{\mathrm{i}\theta} = (\mathrm{e}^{\mathrm{i}\theta_{rs}}) \in \mathbb{C}^{n\times n}$, $\theta_{rs} \in \mathbb{R}$ for $r, s \in \langle n \rangle$. The matrix $E^{\mathrm{i}\theta}$ is called *ray pattern matrix* of the matrix A. If the ray pattern matrix $E^{\mathrm{i}\theta}$ of the matrix A is a θ-ray pattern matrix, then A is called a *θ-ray matrix*.

$\mathscr{R}_n^\theta$ denotes the set of all $n\times n$ θ-ray matrices. Obviously, if a matrix $A \in \mathscr{R}_n^\theta$, then $\xi \cdot A \in \mathscr{R}_n^\theta$ for all $\xi \in \mathbb{C}$.

Example 3.10 The matrix

$$\begin{aligned} A = & \begin{bmatrix} 3\,\mathrm{i} & \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} & \sqrt{3} - \mathrm{i} \\ 2 + 2\sqrt{3}\,\mathrm{i} & 2\,\mathrm{i} & -3\sqrt{3} - 3\,\mathrm{i} \\ \sqrt{3} + \mathrm{i} & -\frac{\sqrt{3}}{2} + \frac{\mathrm{i}}{2} & 4\,\mathrm{i} \end{bmatrix} \\ = & \begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} \otimes \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi/2} & \mathrm{e}^{-\mathrm{i}\pi/3} & \mathrm{e}^{-\mathrm{i}\pi/6} \\ \mathrm{e}^{\mathrm{i}\pi/3} & \mathrm{e}^{\mathrm{i}\pi/2} & \mathrm{e}^{-5\mathrm{i}\pi/6} \\ \mathrm{e}^{\mathrm{i}\pi/6} & \mathrm{e}^{5\mathrm{i}\pi/6} & \mathrm{e}^{\mathrm{i}\pi/2} \end{bmatrix} \\ = & \mathrm{e}^{0i} \cdot |A| \otimes E^{\mathrm{i}\theta} \end{aligned}$$

is a θ-ray matrix, where $E^{\mathrm{i}\theta}$ is defined in Example 3.7.

Theorem 3.11 *Let a matrix*

$$A = D_A - B = (a_{rs}) \in \mathbb{C}^{n\times n}$$

with

$$D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn}).$$

Then $A \in \mathscr{R}_n^\theta$ *if and only if there exists an* $n \times n$ *unitary diagonal matrix* D *such that*

$$D^{-1}AD = \mathrm{e}^{\mathrm{i}\eta} \cdot (|D_A|\mathrm{e}^{\mathrm{i}\theta} - |B|), \quad \eta \in \mathbb{R}.$$

Proof According to Definition 3.9,

$$A = \mathrm{e}^{\mathrm{i}\eta} \cdot |A| \otimes E^{\mathrm{i}\theta} = (\mathrm{e}^{\mathrm{i}\eta} \cdot |a_{rs}|\mathrm{e}^{\mathrm{i}\theta_{rs}}).$$

Define a diagonal matrix

$$D_\phi = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_n})$$

with

$$\phi_r = \theta_{1r} + \phi_1 + (2k+1)\pi, \quad \phi_1 \in \mathbb{R},\ r = 2, 3, \ldots, n,\ k \in \mathbb{Z}.$$

By Definition 3.4,

$$D^{-1}AD = \mathrm{e}^{\mathrm{i}\eta} \cdot (|D_A|\mathrm{e}^{\mathrm{i}\theta} - |B|),$$

which shows that the necessity is true.

Now, we prove the sufficiency. Assume that there exists an $n \times n$ unitary diagonal matrix $D_\phi = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_1}, \ldots, \mathrm{e}^{\mathrm{i}\phi_n})$ such that

$$D_\phi^{-1}AD_\phi = \mathrm{e}^{\mathrm{i}\eta} \cdot (|D_A|\mathrm{e}^{\mathrm{i}\theta} - |B|), \quad \eta \in \mathbb{R}.$$

Then the following equalities hold:

$$\begin{aligned} \theta_{rs} &= \phi_s - \phi_r + (2k_1 + 1)\pi, \\ \theta_{sr} &= \phi_r - \phi_s + (2k_2 + 1)\pi, \\ \theta_{rt} &= \phi_t - \phi_r + (2k_3 + 1)\pi, \\ \theta_{tr} &= \phi_r - \phi_t + (2k_4 + 1)\pi, \end{aligned} \tag{12}$$

where $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. In (12),

$$\theta_{rs} + \theta_{sr} = 2(k_1 + k_2 + 1)\pi = 2k\pi$$

with $k = k_1 + k_2 + 1 \in \mathbb{Z}$ and for all $r, s \in \langle n \rangle,\ r \neq s$. Following (12),

$$\theta_{ts} = \phi_s - \phi_t + (2k_5 + 1)\pi.$$

Hence,

$$\phi_s - \phi_t = \theta_{ts} - (2k_5 + 1)\pi.$$

Consequently,

$$\theta_{rs} - \theta_{rt} = \phi_s - \phi_t + 2(k_1 - k_3)\pi = \theta_{ts} + [2(k_1 - k_3 - k_5 - 1) + 1]\pi\theta_{ts} + (2k+1)\pi$$

for all $r, s, t \in \langle n \rangle$ and $r \neq s,\ r \neq t,\ t \neq s$, where $k = k_1 - k_3 - k_5 - 1 \in \mathbb{Z}$. Using the same method, we can prove that

$$\theta_{sr} - \theta_{tr} = \theta_{st} + (2k + 1)\pi$$

hold for all $r, s, t \in \langle n \rangle$ and $r \neq s,\ r \neq t,\ t \neq s$, where $k \in \mathbb{Z}$. Furthermore, it is obvious that $\theta_{rr} = \theta$ for all $r \in \langle n \rangle$. This proves the sufficiency. □

Example 3.12 For the matrix A defined in Example 3.10, there exists an 3×3 unitary diagonal matrix $D = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\pi}, \mathrm{e}^{\mathrm{i}\pi/3}, \mathrm{e}^{\mathrm{i}\pi/6})$ such that

$$D^{-1}AD = \mathrm{e}^{\mathrm{i}0} \cdot (|D_A|\mathrm{e}^{\mathrm{i}\pi/2} - |B|),$$

where

$$D_A = \mathrm{diag}(3\,\mathrm{i}, 2\,\mathrm{i}, 4\,\mathrm{i}), \quad B = D_A - A = -\begin{bmatrix} 0 & \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} & \sqrt{3} - \mathrm{i} \\ 2 + 2\sqrt{3}\,\mathrm{i} & 0 & -3\sqrt{3} - 3\,\mathrm{i} \\ \sqrt{3} + \mathrm{i} & -\frac{\sqrt{3}}{2} + \frac{\mathrm{i}}{2} & 0 \end{bmatrix}.$$

As a result, $A \in \mathscr{R}_3^\theta$.

Corollary 3.13 *Let a matrix*

$$A = D_A - B = (a_{rs}) \in \mathbb{C}^{n \times n}$$

with $a_{ii} \neq 0$ *for all* $i \in \langle n \rangle$. *Then* $D_A^{-1}A \in \mathscr{R}_n^\theta$ *if* $A \in \mathscr{R}_n^\theta$, *where* $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

Proof Theorem 3.11 shows that if $A \in \mathscr{R}_n^\theta$, then there exists an $n \times n$ unitary diagonal matrix D such that

$$A = \mathrm{e}^{\mathrm{i}\eta} \cdot D(|D_A|\mathrm{e}^{\mathrm{i}\theta} - |B|)D^{-1}, \quad \eta \in \mathbb{R}.$$

Hence,

$$D_A^{-1}A = \mathrm{e}^{-\mathrm{i}\theta}\cdot(I\cdot\mathrm{e}^{\mathrm{i}\theta} - D|D_A|^{-1}|B|D^{-1}) = \mathrm{e}^{-\mathrm{i}\theta}\cdot(|I|\cdot\mathrm{e}^{\mathrm{i}\theta} - D|D_A^{-1}B|D^{-1}), \quad (13)$$

where I is the $n \times n$ identity matrix. (13) indicates that there exists an $n \times n$ unitary diagonal matrix D such that

$$D^{-1}D_A^{-1}AD = \mathrm{e}^{-\mathrm{i}\theta}\cdot(|I|\cdot\mathrm{e}^{\mathrm{i}\theta} - |D_A^{-1}B|).$$

This shows that $D_A^{-1}A \in \mathscr{R}_n^{\theta}$. □

In particular, when $\theta = 0$, Corollary 3.13 indicates the following corollary.

Corollary 3.14 *Let a matrix $A = D_A - B = (a_{rs}) \in \mathbb{C}^{n\times n}$ with $a_{ii} \neq 0$ for all $i \in \langle n\rangle$. If $A \in \mathscr{R}_n^0$, then there exists an $n \times n$ unitary diagonal matrix D such that*

$$D^{-1}(D_A^{-1}A)D = \mu(D_A^{-1}A) \in Z_n,$$

where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

Theorems 3.1, 3.2, and Corollary 3.14 yield the following conclusions.

Theorem 3.15 *Let an irreducible matrix $A \in D_n$ (GD_n). Then A is singular if and only if $D_A^{-1}A \in DE_n$ $(GDE_n) \cap \mathscr{R}_n^0$, where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.*

Theorem 3.16 *A matrix $A \in D_n$ (GD_n) is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible principal submatrix $A_k = A(i_1, i_2, \ldots, i_k), 1 < k \leqslant n$, such that $D_{A_k}^{-1}A_k \in DE_k$ $(GDE_k) \cap \mathscr{R}_k^0$, where $D_{A_k} = \mathrm{diag}(a_{i_1i_1}, a_{i_2i_2}, \ldots, a_{i_ki_k})$.*

In the rest of this section, we will propose the main theorems to give some necessary and sufficient conditions on the matrix $A \in H_n$ such that A is singular.

Theorem 3.17 *Let an irreducible matrix $A \in H_n$. Then A is singular if and only if $D_A^{-1}A \in GDE_n \cap \mathscr{R}_n^0$, where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.*

Proof Since $A \in H_n$ is irreducible, Lemma 2.9 shows that $H_n = GD_n$ holds under the condition irreducibility. It then follows from Theorem 3.15 that A is singular if and only if $D_A^{-1}A \in GDE_n \cap \mathscr{R}_n^0$. □

Theorem 3.18 *A matrix $A \in H_n$ is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $1 < k \leqslant n$, such that $D_{A_k}^{-1}A_k \in GDE_k \cap \mathscr{R}_k^0$, where $D_{A_k} = \mathrm{diag}(a_{i_1i_1}, a_{i_2i_2}, \ldots, a_{i_ki_k})$.*

Proof When the matrix $A \in H_n$ is irreducible, Theorem 3.17 shows that the conclusion is true. Otherwise, following from Lemma 2.10, $A \in H_n$ is singular if and only if in the Frobenius normal form of A (7), there exists at least either one zero principal submatrix or one irreducible diagonal square block $R_{kk} \in GD_{|\alpha_k|}$ such that R_{kk} is singular. If there exists an irreducible diagonal square block $R_{kk} \in GD_{|\alpha_k|}$ with $\alpha_k = \{i_1, i_2, \ldots, i_k\}$ such that R_{kk} is singular, it follows from Theorem 3.15 that $R_{kk} = A(\alpha_k) = A_k \in GD_{|\alpha_k|}$ is singular if and only if $D_{A_k}^{-1}A_k \in GDE_k \cap \mathscr{R}_k^0$, $D_{A_k} = \mathrm{diag}(a_{i_1i_1}, a_{i_2i_2}, \ldots, a_{i_ki_k})$. Therefore, $A \in H_n$ is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $1 < k \leqslant n$, such that $D_{A_k}^{-1}A_k \in GDE_k \cap \mathscr{R}_k^0$. This completes the proof. □

Corollary 3.19 *A matrix $A \in H_n^M$ is singular if and only if the matrix A has at least either one irreducible principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $1 < k \leqslant n$, such that $D_{A_k}^{-1}A_k \in GDE_k \cap \mathscr{R}_k^0$, where $D_{A_k} = \mathrm{diag}(a_{i_1i_1}, a_{i_2i_2}, \ldots, a_{i_ki_k})$.*

Theorem 3.20 *A matrix $A \in H_n$ is nonsingular if and only if every successive principal submatrix of A is nonsingular.*

Proof Assume that A is singular. Then there obviously exists a singular successive principal submatrix of A for A is a successive principal submatrix of A.(?????)

Conversely, suppose that $A \in H_n$ has a $k \times k$ singular successive principal submatrix $A_k \in H_k$. Then from the necessity of Theorem 3.18, A_{kk} has at least either one zero principal submatrix or one irreducible principal submatrix $A_{k'} = A(i_1, i_2, \ldots, i_{k'})$, $1 < k' \leqslant n$, such that

$$D_{A_{k'}}^{-1}A_{k'} \in GDE_{k'} \cap \mathscr{R}_{k'}^0.$$

Obviously, $A_{k'}$ is also a singular principal submatrix of A. Thus, A is singular from the sufficiency of Theorem 3.18. □

Corollary 3.21 *A matrix $A \in H_n$ is nonsingular if and only if there exists a triangular decomposition $A = LU$, where L and U are lower and upper triangular matrices, respectively. Furthermore, if L is prescribed as a identity lower triangular matrices, then the triangular decomposition is unique* (*the triangular decomposition is called Doolittle decomposition*).

Proof According to Theorem 3.20 and [11, Theorem 4.2-1], it is easy to get the proof. □

Theorem 3.22 *Let $A \in H_n$. Then A is nonsingular if and only if every principal submatrix of A is nonsingular.*

Proof The proof can be finished by proving the equivalent statement of this theorem that $A \in H_n$ is singular if and only if there exists a singular principal submatrix in A. The necessity of the equivalent statement is obvious.

Now, we prove the sufficiency. Suppose that $A(i_1, i_2, \ldots, i_k)$, a principal submatrix of A, is singular. Then there exists an $n \times n$ permutation matrix P such that the successive principal submatrix of PAP^{T} with k order is $A(i_1, i_2, \ldots, i_k)$. Since the permutation transformation does not change the diagonal dominance of matrices, PAP^{T} is still a diagonally dominant matrix. Thus, PAP^{T} is singular from Theorem 3.20, so is A. □

Corollary 3.23 *Let $A \in H_n$. Then A is nonsingular if and only if A/α exists and is nonsingular for each $\alpha \subset \langle n \rangle$.*

Proof It is similar to the proof of [28, Lemma 3.13]. □

4 Convergence on general *H*-matrices

It is well known that convergence on invertible H-matrices (that belong to H_n^I) widely apply in many classical iterative methods like the Jacobi, Gauss-Seidel, SOR, AOR, etc. (see, e.g., [1,14,24]) for linear and nonlinear systems and linear complementarity problems. However, little attention on convergence of mixed H-matrices and singular H-matrices has been paid. In fact, the Jacobi iterative method for singular H-matrices fails to exist since at least one diagonal entry of a singular H-matrix is zero. Therefore, we mainly study convergence of mixed H-matrices.

In this section, some necessary and sufficient conditions on a mixed H-matrix are proposed such that the associated Jacobi iterative method is convergent.

Let us recall the standard decomposition of the matrix $A \in \mathbb{C}^{n\times n}$,

$$A = D_A - L - U, \tag{14}$$

where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ is a diagonal matrix, L and U are strictly lower and strictly upper triangular matrices, respectively. If $a_{ii} \neq 0$ for all $i \in \langle n\rangle$, then the Jacobi iteration matrix associated with the matrix A,

$$J_A = D_A^{-1}(L + U), \tag{15}$$

and the Jacobi iterative scheme on linear system,

$$Ax = b, \quad A \in \mathbb{C}^{n\times n}, \det A \neq 0, \tag{16}$$

can be described as

$$x^{(i+1)} = J_A x^{(i)} + f, \quad i = 0, 1, 2, \ldots, \tag{17}$$

where $f = D_A^{-1}b$. On the Jacobi iterative method for an invertible H-matrix, we have the following classical result.

Theorem 4.1 [1] *A matrix $A \in H_n^I$ if and only if*

$$\rho(J_A) \leqslant \rho(J_{\mu(A)}) < 1,$$

i.e., the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme (17) *converges to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$, where J_A (resp. $J_{\mu(A)}$) is the Jacobi iteration matrix associated with the matrix A (resp. the comparison matrix $\mu(A)$ of A).*

Bru et al. [3] presented a result on the Jacobi iteration matrix associated with a general H-matrix A as follows.

Theorem 4.2 [3] *Let $A \in \mathbb{C}^{n\times n}$ with $a_{ii} \neq 0$ for all $i \in \langle n\rangle$. Then the following statements are equivalent*:

(i) $A \in H_n$;

(ii) $\rho(J_{\mu(A)}) \leqslant 1$;

(iii) *for any matrix* $B \in \omega(A)$, $\rho(J_B) \leqslant 1$.

As is shown in [3], if $A \in H_n^M$, then

$$\rho(J_A) \leqslant \rho(J_{\mu(A)}) = 1.$$

This shows that one does not know whether the matrix A is convergent or not for the Jacobi iterative method. For example, the matrices

$$B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2\mathrm{e}^{\mathrm{i}\cdot(\pi/2)} & 1 & -1 \\ 1 & 2\mathrm{e}^{\mathrm{i}\cdot(\pi/2)} & 1 \\ -1 & 1 & 2\mathrm{e}^{\mathrm{i}\cdot(\pi/2)} \end{bmatrix}$$

are both mixed H-matrices and also nonsingular. However, direct computations show that

$$\rho(J_B) < \rho(J_{\mu(B)}) = 1,$$

but

$$\rho(J_C) = \rho(J_{\mu(C)}) = 1.$$

Without direct computations, how do we judge the convergence of Jacobi iterative method for mixed H-matrices?

The following theorem considers the case when A is a (generalized) diagonally equipotent matrix.

Theorem 4.3 *A 2×2 irreducible matrix $A = (a_{ij}) \in GDE_2$ if and only if $\rho(J_A) = 1$. Therefore, the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme* (17) *does not converge to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$.*

Proof Assume

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GDE_2.$$

By Definition 2.2,

$$\alpha_1|a_{11}| = \alpha_2|a_{12}|, \quad \alpha_2|a_{22}| = \alpha_1|a_{21}|,$$

with $a_{ij} \neq 0$ and $\alpha_i > 0$ for all $i,j = 1,2$. Consequently, $A \in GDE_2$ if and only if

$$\frac{|a_{12}a_{21}|}{|a_{11}a_{11}|} = 1.$$

The Jacobi iteration matrix associated with the matrix A is

$$J_A = \begin{bmatrix} 0 & -\dfrac{a_{12}}{a_{11}} \\ -\dfrac{a_{21}}{a_{22}} & 0 \end{bmatrix}.$$

Let λ be any eigenvalue of J_A. Direct computation gives that

$$\lambda^2 = \frac{a_{12}a_{21}}{a_{11}a_{22}},$$

and consequently,

$$\rho(J_A) = \sqrt{\frac{|a_{12}a_{21}|}{|a_{11}a_{11}|}}.$$

Thus, the Jacobi iterative method fails to converge, i.e., $\rho(J_A) = 1$ if and only if a 2×2 irreducible matrix $A = (a_{ij}) \in GDE_2$. $\square$

Lemma 4.4 *Let* $A = (a_{ij}) \in DE_n$ $(n \geqslant 3)$ *be irreducible. Then* $\mathrm{e}^{-\mathrm{i}\theta}$ *is an eigenvalue of* J_A *if and only if* $D_A^{-1}A \in \mathscr{R}_n^{\theta}$*, where* $\theta \in \mathbb{R}$.

Proof We prove the sufficiency first. Since $A = (a_{ij}) \in DE_n$ is irreducible, $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Thus, D_A^{-1} exists, where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. Assume $D_A^{-1}A \in \mathscr{R}_n^{\theta}$. Theorem 3.11 shows that there exists a unitary diagonal matrix D such that

$$D^{-1}(D_A^{-1}A)D = I - \mathrm{e}^{-\mathrm{i}\theta} \cdot |D_A^{-1}B|.$$

Hence,

$$J_A = D_A^{-1}B = \mathrm{e}^{-\mathrm{i}\theta} D(|D_A^{-1}B|)D^{-1} = \mathrm{e}^{-\mathrm{i}\theta} D J_{\mu(D_A^{-1}A)} D^{-1}. \tag{18}$$

Using (18), we have

$$\begin{aligned}
\det(\mathrm{e}^{-\mathrm{i}\theta} I - J_{D_A^{-1}A}) &= \det(\mathrm{e}^{-\mathrm{i}\theta} I - \mathrm{e}^{-\mathrm{i}\theta} D J_{\mu(D_A^{-1}A)} D^{-1}) \\
&= \mathrm{e}^{-\mathrm{i}\theta} \det(I - J_{\mu(D_A^{-1}A)}) \\
&= \mathrm{e}^{-\mathrm{i}\theta} \det \mu(D_A^{-1}A).
\end{aligned} \tag{19}$$

Since $A \in DE_n$ is irreducible, so is $D_A^{-1}A$. Again, $\mu(D_A^{-1}A) \in \mathscr{R}_n^0$, and Theorem 3.15 shows that $\mu(D_A^{-1}A)$ is singular. As a result, (19) gives $\det(\mathrm{e}^{-\mathrm{i}\theta} I - J_A) = 0$ to reveal that $\mathrm{e}^{-\mathrm{i}\theta}$ is an eigenvalue of J_A. This proves the sufficiency.

Now, we prove the necessity. Let $\mathrm{e}^{-\mathrm{i}\theta}$ is an eigenvalue of J_A. Then

$$\det(\mathrm{e}^{-\mathrm{i}\theta} I - J_A) = \det(\mathrm{e}^{-\mathrm{i}\theta} I - D_A^{-1}B) = 0.$$

Thus, $\mathrm{e}^{-\mathrm{i}\theta} I - D_A^{-1}B$ is singular. Since $\mathrm{e}^{-\mathrm{i}\theta} I - D_A^{-1}B \in DE_n$ and irreducible for $A = D_A - B \in DE_n$ and irreducible, Theorem 3.15 shows that

$$I - \mathrm{e}^{\mathrm{i}\theta} D_A^{-1}B \in \mathscr{R}_n^0.$$

It follows from Corollary 3.14 that there exists a unitary diagonal matrix D such that

$$D^{-1}(I - \mathrm{e}^{\mathrm{i}\theta} D_A^{-1}B)D = I - \mathrm{e}^{\mathrm{i}\theta} D^{-1}(D_A^{-1}B)D = I - |D_A^{-1}B|. \tag{20}$$

Equality (20) shows

$$D^{-1}(D_A^{-1}B)D = \mathrm{e}^{-\mathrm{i}\theta}|D_A^{-1}B|.$$

Therefore,

$$D^{-1}(D_A^{-1}A)D = I - D^{-1}(D_A^{-1}B)D = I - \mathrm{e}^{-\mathrm{i}\theta}|D_A^{-1}B|,$$

that is, there exists a unitary diagonal matrix D such that

$$D^{-1}(D_A^{-1}A)D^{-1} = \mathrm{e}^{-\mathrm{i}\theta}(\mathrm{e}^{\mathrm{i}\theta}I - |D_A^{-1}B|).$$

Hence, $D_A^{-1}A \in \mathscr{R}_n^{\theta}$, and we prove the necessity. □

Theorem 4.5 *Let $A \in DE_n$ $(n \geqslant 3)$ be irreducible. Then $\rho(J_A) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme* (17) *converges to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{R}_n^{\theta}$.*

Proof Since $A \in DE_n$, $A \in H_n$. It follows from Theorem 4.2 that $\rho(J_A) \leqslant 1$ for $A \in \omega(A)$. Therefore, the Jacobi iterative method converges, i.e., $\rho(J_A) < 1$ if and only if $\rho(J_A) \neq 1$. Since $\rho(J_A) \neq 1$ is equivalent to that $\mathrm{e}^{-\mathrm{i}\theta}$ is not an eigenvalue of J_A. According to Lemma 4.4, The Jacobi iterative method converges, i.e., $\rho(J_A) < 1$ if and only if $D_A^{-1}A \notin \mathscr{R}_n^{\theta}$. □

According to Definitions 2.1, 2.2, and Theorem 4.5, it is easy to generalize the conclusion of Theorem 4.5 to irreducible generalized diagonally equipotent matrices.

Theorem 4.6 *Let $A = (a_{ij}) \in GDE_n$ $(n \geqslant 3)$ be irreducible. Then $\rho(J_A) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme* (17) *converges to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{R}_n^{\theta}$.*

Proof According to Definition 2.2, there exists a diagonal matrix

$$E = \mathrm{diag}(e_1, e_2, \ldots, e_n),$$

with $e_k > 0$ for all $k \in \langle n \rangle$, such that

$$AE = (a_{ij}e_j) \in DE_n.$$

Let

$$AE = F = (f_{ij}), \quad f_{ij} = a_{ij}e_j, \ \forall\, i, j \in \langle n \rangle.$$

Then

$$J_F = E^{-1}J_AE, \quad D_F^{-1}F = E^{-1}(D_A^{-1}A)E,$$

with $D_F = D_AE$. Theorem 4.5 yields that $\rho(J_F) < 1$ if and only if $D_F^{-1}F \notin \mathscr{R}_n^{\theta}$. Since $\rho(J_F) = \rho(J_A)$ and $D_A^{-1}A \notin \mathscr{R}_n^{\theta}$ for $D_F^{-1}F = E^{-1}(D_A^{-1}A)E \notin \mathscr{R}_n^{\theta}$ and $E = \mathrm{diag}(e_1, e_2, \ldots, e_n)$ with $e_k > 0$ for all $k \in \langle n \rangle$, the Jacobi iterative method converges, i.e., $\rho(J_A) < 1$ if and only if $D_A^{-1}A \notin \mathscr{R}_n^{\theta}$. □

Theorem 4.7 *Let $A \in GD_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then $\rho(J_A) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme* (17) *converges to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leqslant k \leqslant n$, such that*

$$D_{A_k}^{-1} A_k \in GDE_k \cap \mathscr{R}_k^{\theta},$$

where $D_{A_k} = \mathrm{diag}(a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_k i_k})$.

Proof The necessity will be proved first. When

$$\rho(J_A) \leqslant \rho(J_{\mu(A)}) < 1,$$

it follows from Theorem 4.1 that $A \in H_n^I$. Furthermore, Lemma 2.6 indicates that A does not have any irreducibly generalized diagonally equipotent principal submatrix. When

$$\rho(J_A) < \rho(J_{\mu(A)}) = 1,$$

we have $A \notin H_n^I$, but $A \in GD_n$. Lemma 2.6 shows that $A \in GD_n$ has at least one irreducibly generalized diagonally equipotent principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $2 \leqslant k \leqslant n$. Theorems 4.3 and 4.6 reveal that A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leqslant k \leqslant n$, such that

$$D_{A_k}^{-1} A_k \in GDE_k \cap \mathscr{R}_k^{\theta}.$$

Now, we prove the sufficiency. If A does not have any irreducibly generalized diagonally equipotent principal submatrix, then Theorem 4.1 gives $A \in H_n^I$. As a result, $\rho(J_A) < 1$ follows from Theorem 4.1. If A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leqslant k \leqslant n$, such that $D_{A_k}^{-1} A_k \in GDE_k \cap \mathscr{R}_k^{\theta}$, it follows from Theorems 4.3 and 4.6 that $\rho(J_A) < 1$. □

Theorem 4.8 *Let $A \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then $\rho(J_A) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme* (17) *converges to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leqslant k \leqslant n$, such that*

$$D_{A_k}^{-1} A_k \in GDE_k \cap \mathscr{R}_k^{\theta},$$

where $D_{A_k} = \mathrm{diag}(a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_k i_k})$.

Proof When $A \in H_n$ is irreducible, Lemma 2.9 shows that $A \in GD_n$. Theorems 4.3 and 4.6 indicate that the Jacobi iterative method converges, i.e., $\rho(J_A) < 1$ if and only if A is neither a 2×2 irreducibly generalized diagonally equipotent matrix nor an irreducibly matrix such that

$$D_A^{-1} A \in GDE_n \cap \widehat{\mathscr{R}}_n^{\theta}.$$

Otherwise, $A \in H_n$ is reducible. Since $A \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n\rangle$, it follows from Theorem 2.10 that in the Frobenius normal form (7) of A, each diagonal square block R_{ii} is irreducible generalized diagonally dominant for all $i = 1, 2, \ldots, s$. Let $J_{R_{ii}}$ be the iteration matrix associated with the diagonal square block R_{ii} for all $i = 1, 2, \ldots, s$. Then

$$\rho(J_A) = \max_i \{\rho(J_{R_{ii}})\colon i = 1, 2, \ldots, s\}.$$

Therefore, the Jacobi iterative method converges, i.e., $\rho(J_A) < 1$ if and only if A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leqslant k \leqslant n$, such that $D_{A_k}^{-1} A_k \in GDE_k \cap \mathscr{R}_k^{\theta}$. □

Corollary 4.9 *Let $A \in H_n(GD_n)$ $(n \geqslant 3)$ be irreducible. Then $\rho(J_A) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by the Jacobi iterative scheme* (17) *converges to the unique solution of* (16) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1} A \notin GDE_n \cap \mathscr{R}_k^{\theta}$, where $D_A = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.*

The research in this section shows that the Jacobi iterative method associated with the irreducible matrix $A \in H_n^M \cap R_n^{\theta}$ fails to converge. It is natural to consider the cases of the block Jacobi iterative method, Gauss-Seidel iterative method, SOR iterative method, and so on. As some open problems, these cases need to be studied further.

5 Schur complement of general H-matrices

Recently, considerable interest appears in the work on the Schur complements of some families of matrices and several significant results are proposed. As is shown in [4,6,7,15,17–20,27–30,32], the Schur complements of positive semidefinite matrices are positive semidefinite (see, e.g., [6]); the same is true for M-matrices, inverse M-matrices (see, e.g., [12]), invertible H-matrices (see, e.g., [17]), diagonally dominant matrices (see, e.g., [6,15]), Dashnic-Zusmanovich matrices (see, e.g., [7]), and generalized doubly diagonally dominant matrices (see, e.g., [18]).

Since M-matrices, Dashnic-Zusmanovich matrices, strictly generalized doubly diagonally dominant matrices, and strictly or irreducibly diagonally dominant matrices are all invertible H-matrices (see, e.g., [2, pp. 132–161], [7,18], and [24, p. 92]), so are their Schur complements. This very property has been repeatedly used for the convergence of the Gauss-Seidel iterations and stability of Gaussian elimination in numerical analysis (see, e.g., [14, p. 58], [11, p. 508], and [14, pp. 122, 123]). Lately, Zhang et al. [28] and Bru et al. [4] extended this property to nonstrictly diagonally dominant matrices and general H-matrices that are not necessarily invertible H-matrices.

Continuing in this direction, in the rest of this paper, we establish new results on the Schur complements of general H-matrices. These results will not only propose some conditions such that $A \in H_n^I$ if $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n \rangle$, but also give some different conditions for the matrix $A \in H_n^M$ and the subset $\alpha \subseteq N$ such that the Schur complement matrix $A/\alpha \in H_{n-|\alpha|}^I$ or $A/\alpha \in H_{n-|\alpha|}^M$ or $A/\alpha \in H_{n-|\alpha|}^S$.

Following, we will improve and complement some classical result on the Schur complement of invertible H-matrices.

Theorem 5.1 [6,25] *Let $A \in Z_n$. Then $A \in M_n^\bullet$ if and only if $A(\alpha) \in M_{|\alpha|}^\bullet$ and $A/\alpha \in M_{n-|\alpha|}^\bullet$ for all nonempty subset $\alpha \subset \langle n \rangle$.*

Theorem 5.2 [17,23] *Given a matrix $A \in \mathbb{C}^{n\times n}$, if $A \in H_n^I$, then $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n \rangle$.*

It is noted that unlike the conclusion in Theorem 5.1, the condition of Theorem 5.2 "$A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n \rangle$" in general does not get the conclusion "$A \in H_n^I$".

Example 5.3 Let

$$A = \begin{bmatrix} 3 & -1 & -1 & -2 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}. \tag{21}$$

Then $\langle 4 \rangle = \{1, 2, 3, 4\}$. Direct computations show that for all nonempty $\alpha \subset \langle 4 \rangle$, $A(\alpha)$ and A/α are both invertible H-matrices. But, it is verified that $A \notin H_4^I$, and thus, the converse proposition of Theorem 5.2 is not true.

The following will propose some conditions such that $A \in H_n^I$ if $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n \rangle$. A sufficient condition will be proposed first.

Theorem 5.4 *Give a matrix $A \in \mathbb{C}^{n\times n}$ with $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n \rangle$. Then $A \in H_n^I$ if $A \in \mathscr{R}_n^0$.*

Proof Since $A \in \mathscr{R}_n^0$, it follows from Theorem 3.11 that there exists an $n \times n$ unitary diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{C}^{n\times n}$ such that $D^{-1}AD = \mathrm{e}^{\mathrm{i}\eta} \cdot \mu(A)$ for $\eta \in \mathbb{R}$. Furthermore, there exists a permutation matrix P_α such that

$$A_\alpha = P_\alpha^{\mathrm{T}} A P_\alpha = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix}$$

and

$$P_\alpha^{\mathrm{T}} D P_\alpha = \begin{bmatrix} D(\alpha) & 0 \\ 0 & D(\alpha') \end{bmatrix},$$

where $\alpha' = \langle n\rangle - \alpha \subset \langle n\rangle$. Then

$$\begin{aligned}
&P_\alpha^{\mathrm{T}}(D^{-1}AD)P_\alpha \\
&\quad= (P_\alpha^{\mathrm{T}}DP_\alpha)^{-1}(P_\alpha^{\mathrm{T}}AP_\alpha)(P_\alpha^{\mathrm{T}}DP_\alpha) \\
&\quad= \begin{bmatrix} [D(\alpha)]^{-1} & 0 \\ 0 & [D(\alpha')]^{-1} \end{bmatrix} \begin{bmatrix} A(\alpha) & A(\alpha,\alpha') \\ A(\alpha',\alpha) & A(\alpha') \end{bmatrix} \begin{bmatrix} D(\alpha) & 0 \\ 0 & D(\alpha') \end{bmatrix} \\
&\quad= \mathrm{e}^{\mathrm{i}\eta} \begin{bmatrix} \mu[A(\alpha)] & -|A(\alpha,\alpha')| \\ -|A(\alpha',\alpha)| & \mu[A(\alpha')] \end{bmatrix}.
\end{aligned} \tag{22}$$

(22) implies that

$$\begin{aligned}
\mu[A(\alpha)] &= \mathrm{e}^{-\mathrm{i}\eta}[D(\alpha)]^{-1}A(\alpha)D(\alpha), \\
|A(\alpha,\alpha')| &= -\mathrm{e}^{-\mathrm{i}\eta}[D(\alpha)]^{-1}A(\alpha,\alpha')D(\alpha'), \\
|A(\alpha',\alpha)| &= -\mathrm{e}^{-\mathrm{i}\eta}[D(\alpha')]^{-1}A(\alpha',\alpha)D(\alpha), \\
\mu[A(\alpha')] &= \mathrm{e}^{-\mathrm{i}\eta}[D(\alpha')]^{-1}A(\alpha')D(\alpha').
\end{aligned} \tag{23}$$

Thus,

$$\begin{aligned}
\mu(A)/\alpha &= |A(\alpha',\alpha)[\mu[A(\alpha)]]^{-1}|A(\alpha,\alpha') \\
&= \mathrm{e}^{-\mathrm{i}\eta}[D(\alpha')]^{-1}A(\alpha')D(\alpha') - \mathrm{e}^{-\mathrm{i}\eta}[D(\alpha')]^{-1}A(\alpha',\alpha)D(\alpha) \\
&\quad \times\{[D(\alpha)]^{-1}A(\alpha)D(\alpha)\}^{-1}[D(\alpha)]^{-1}A(\alpha,\alpha')D(\alpha') \\
&= \mathrm{e}^{-\mathrm{i}\eta}[D(\alpha')]^{-1}\,[A(\alpha') - A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha')]D(\alpha') \\
&= \mathrm{e}^{-\mathrm{i}\eta}[D(\alpha')]^{-1}\,[A/\alpha]D(\alpha') \\
&= \mu[A/\alpha].
\end{aligned} \tag{24}$$

Since $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$, we have

$$\mu[A(\alpha)] \in M_{|\alpha|}^\bullet, \quad \mu[A/\alpha] \in M_{n-|\alpha|}^\bullet.$$

It then follows from Theorem 5.1 that $\mu(A) \in M_n^\bullet$. Therefore, $A \in H_n^I$. This completes the proof. □

Now, we propose a necessary and sufficient condition such that $A \in H_n^I$ if $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n\rangle$. A lemma will be used in this section.

Lemma 5.5 [22,25] *Let $A \in Z_n$. Then $A \in M_n^\bullet$ if and only if there exists a matrix $B \in Z_n$ such that*

$$B^{-1} \geqslant 0, \quad B \geqslant A, \quad B^{-1}A \in M_n^\bullet.$$

Theorem 5.6 *Give a matrix $A \in \mathbb{C}^{n\times n}$ with $A(\alpha) \in H_{|\alpha|}^I$ and $A/\alpha \in H_{n-|\alpha|}^I$ for all nonempty $\alpha \subset \langle n\rangle$. Then $A \in H_n^I$ if and only if*

$$[\mu(A/\alpha)]^{-1}[\mu(A)/\alpha] \in M_{n-|\alpha|}^\bullet.$$

Proof Assume that there exists a permutation matrix P_α such that

$$A_\alpha = P_\alpha^{\mathrm{T}} A P_\alpha = \begin{bmatrix} A(\alpha) & A(\alpha,\alpha') \\ A(\alpha',\alpha) & A(\alpha') \end{bmatrix},$$

where $\alpha' = \langle n\rangle - \alpha \subset \langle n\rangle$. Let

$$\begin{aligned} \mathscr{L}_\alpha &= \begin{bmatrix} I_{|\alpha|} & 0 \\ -|A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1} & I_{|\alpha'|} \end{bmatrix}, \\ \mathscr{U}_\alpha &= \begin{bmatrix} I_{|\alpha|} & -[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| \\ 0 & I_{|\alpha'|} \end{bmatrix}. \end{aligned} \tag{25}$$

Then

$$B_\alpha = \mathscr{L}_\alpha \begin{bmatrix} \mu(A(\alpha)) & 0 \\ 0 & \mu(A/\alpha) \end{bmatrix} \mathscr{U}_\alpha = \begin{bmatrix} \mu(A(\alpha)) & -|A(\alpha,\alpha')| \\ -|A(\alpha',\alpha)| & \mathscr{Z} \end{bmatrix}, \tag{26}$$

where

$$\mathscr{Z} = \mu(A/\alpha) + |A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')|.$$

Since $A(\alpha) \in H_{|\alpha|}^I$, we have

$$\mu(A(\alpha)) \in M_{|\alpha|}^{\bullet}, \quad [\mu(A(\alpha))]^{-1} \geqslant 0.$$

The same argument shows that $[\mu(A/\alpha)]^{-1} \geqslant 0$ since $A/\alpha \in H_{n-|\alpha|}^I$. Hence,

$$|A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1} \geqslant 0, \quad [\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| \geqslant 0.$$

It is easy to get that

$$\mathscr{L}_\alpha \in M_n^{\bullet}, \quad \mathscr{U}_\alpha \in M_n^{\bullet},$$

and consequently,

$$\mathscr{L}_\alpha^{-1} \geqslant 0, \quad \mathscr{U}_\alpha^{-1} \geqslant 0.$$

As a result,

$$B_\alpha^{-1} = \mathscr{U}_\alpha^{-1} \begin{bmatrix} [\mu(A(\alpha))]^{-1} & 0 \\ 0 & [\mu(A/\alpha)]^{-1} \end{bmatrix} \mathscr{L}_\alpha^{-1} \geqslant 0,$$

which shows that B_α is inverse-positive. Again,

$$\begin{aligned} \mathscr{Z} &= \mu(A/\alpha) + |A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| \\ &= \mu[A(\alpha') - A(\alpha',\alpha)(A(\alpha))^{-1}A(\alpha,\alpha')] \\ &\quad + |A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| \\ &\geqslant \mu[A(\alpha')] - |A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| \\ &\quad + |A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| \\ &= \mu[A(\alpha')]. \end{aligned} \tag{27}$$

Following (26) and (27), we have

$$B_\alpha = \begin{bmatrix} \mu(A(\alpha)) & -|A(\alpha,\alpha')| \\ -|A(\alpha',\alpha)| & \mathscr{Z} \end{bmatrix} \geqslant \begin{bmatrix} \mu(A(\alpha)) & -|A(\alpha,\alpha')| \\ -|A(\alpha',\alpha)| & \mu[A(\alpha')] \end{bmatrix} \geqslant \mu(A). \quad (28)$$

Furthermore,

$$\begin{aligned} B_\alpha^{-1}\mu(A) &= I - \begin{bmatrix} \mu(A(\alpha)) & -|A(\alpha,\alpha')| \\ -|A(\alpha',\alpha)| & \mathscr{Z} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \mathscr{Z} - \mu[A(\alpha')] \end{bmatrix} \\ &= \begin{bmatrix} I & -[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')|[\mu(A/\alpha)]^{-1}(\mathscr{Z} - \mu[A(\alpha')]) \\ 0 & I - [\mu(A/\alpha)]^{-1}(\mathscr{Z} - \mu[A(\alpha')]) \end{bmatrix}. \end{aligned} \quad (29)$$

Since

$$\begin{aligned} &[\mu(A/\alpha)]^{-1}(\mathscr{Z} - \mu[A(\alpha')]) \\ &\quad = [\mu(A/\alpha)]^{-1}(\mu(A/\alpha) + |A(\alpha',\alpha)|[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')| - \mu[A(\alpha')]) \\ &\quad = [\mu(A/\alpha)]^{-1}(\mu(A/\alpha) - \mu(A)/\alpha) \\ &\quad = I - [\mu(A/\alpha)]^{-1}[\mu(A)/\alpha], \end{aligned} \quad (30)$$

we have

$$B_\alpha^{-1}\mu(A) = \begin{bmatrix} I & -[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')|[\mu(A/\alpha)]^{-1}(\mu(A/\alpha) - \mu(A)/\alpha) \\ 0 & [\mu(A/\alpha)]^{-1}[\mu(A)/\alpha] \end{bmatrix}. \quad (31)$$

From (27), we get

$$\mu(A/\alpha) \geqslant \mu(A)/\alpha.$$

Thus,

$$-[\mu(A(\alpha))]^{-1}|A(\alpha,\alpha')|[\mu(A/\alpha)]^{-1}(\mu(A/\alpha) - \mu(A)/\alpha) \leqslant 0. \quad (32)$$

Assume that

$$[\mu(A/\alpha)]^{-1}[\mu(A)/\alpha] \in M^\bullet_{n-|\alpha|}.$$

Then (31) and (32) give $B_\alpha^{-1}\mu(A) \in M^\bullet_n$. With (28) and Lemma 5.5, $\mu(A) \in M^\bullet_n$, and thus, $A \in H^I_n$. Conversely, if $A \in H^I_n$, then $\mu(A) \in M^\bullet_n$ and Theorem 5.1 gives $\mu(A)/\alpha \in M^\bullet_{n-|\alpha|}$. Since $A/\alpha \in H^I_{n-|\alpha|}$, $\mu(A/\alpha) \in M^\bullet_{n-|\alpha|}$. Again, (26) implies

$$\mu(A/\alpha) \geqslant \mu(A)/\alpha.$$

Lemma 5.5 shows that

$$[\mu(A/\alpha)]^{-1}[\mu(A)/\alpha] \in M^\bullet_{n-|\alpha|}. \qquad \square$$

Now, we consider the matrix A in Example 5.3. Although $A(\alpha) \in H^I_2$ and $A/\alpha \in H^I_2$ for all nonempty $\alpha \subset \langle n\rangle$, Theorem 5.6 still shows that A is not an H-matrix since direct computations yield that $[\mu(A/\alpha)]^{-1}[\mu(A)/\alpha]$ is not a nonsingular M-matrix. In fact, Example 5.3 has verified that A is not an H-matrix. This shows that Theorem 5.6 is effective.

Example 5.7 Let

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \tag{33}$$

and $\langle 4 \rangle = \{1, 2, 3, 4\}$. Then for all nonempty $\alpha \subset \langle 4 \rangle$, both $A(\alpha)$ and A/α are invertible H-matrices. Furthermore, $[\mu(A/\alpha)]^{-1}[\mu(A)/\alpha]$ is nonsingular M-matrices. Therefore, it follows from Theorem 5.6 that $A \in H_4^I$. In fact, direct computations have verified $A \in ID_4 \subset H_4^I$. This also shows that Theorem 5.6 is valid.

Bru et al. [4] extend the result of Theorem 5.2 to general H-matrices.

Theorem 5.8 [4] *Let $A \in H_n$, and let $\alpha \subset \langle n \rangle$ such that $A(\alpha) \in H^I_{|\alpha|}$. Then the Schur complement matrix $A/\alpha \in H_{n-|\alpha|}$.*

In fact, Theorem 5.8 still holds if the condition "$A(\alpha) \in H^I_{|\alpha|}$" is weakened to "$A(\alpha)$ is nonsingular".

Theorem 5.9 *Let $A \in H_n$, and let $\alpha \subset \langle n \rangle$ such that $A(\alpha)$ is nonsingular. Then the Schur complement matrix $A/\alpha \in H_{n-|\alpha|}$.*

Proof It follows from Theorem 5.8 that the conclusion of this theorem holds if $A(\alpha) \in H^I_{|\alpha|}$. We need only to consider the case that $A(\alpha)$ is nonsingular but $A(\alpha) \notin H^I_{|\alpha|}$. In this case, $A(\alpha) \in H^M_{|\alpha|}$ since $A(\alpha) \in H_{|\alpha|}$ for $A \in H_n$. By Lemma 2.11, it is easy to get that $A(\alpha)$ is either an irreducible diagonal block in the Frobenius normal form (7) of A if $A(\alpha)$ is irreducible or a block triangular matrix whose diagonal blocks come from some irreducible diagonal blocks of the Frobenius normal form (7) if $A(\alpha)$ is reducible. If $A(\alpha)$ is an irreducible diagonal block in the Frobenius normal form (7) of A, it follows from the Frobenius normal form (7) of A that $A/\alpha = A(\alpha') \in H_{n-|\alpha|}$ for $A(\alpha')$ is a principal submatrix of the matrix $A \in H_n$, where $\alpha' = \langle n \rangle - \alpha$. Otherwise, $A(\alpha)$ is a block triangular matrix whose diagonal blocks come from some irreducible diagonal blocks of the Frobenius normal form (7). Let

$$\alpha = \bigcup_{j=1}^{s} \beta_j, \quad \emptyset \subseteq \beta_j \subseteq \alpha_j,\ j = 1, 2, \ldots, s,$$

and let the number of nonempty set β_j be at least equal to 2, such that

$$A(\alpha) = \begin{bmatrix} A(\beta_1) & A(\beta_1, \beta_2) & \cdots & A(\beta_1, \beta_s) \\ & \ddots & \ddots & \vdots \\ & & \ddots & A(\beta_{s-1}, \beta_s) \\ 0 & & & A(\beta_s) \end{bmatrix}, \tag{34}$$

where $A(\beta_i)$ is irreducible generalized diagonally dominant for $i = 1, 2, \ldots, s$. Let $\gamma_j = \alpha_j - \beta_j$ for $j = 1, 2, \ldots, s$. Then

$$\alpha' = \langle n\rangle - \alpha = \bigcup_{j=1}^{s} \alpha_j - \bigcup_{j=1}^{s} \beta_j = \bigcup_{j=1}^{s} \gamma_j.$$

Furthermore, there exists an $n \times n$ permutation matrix P_1 such that

$$\begin{aligned} C &= P_1 P A P^{\mathrm{T}} P_1^{\mathrm{T}} \\ &= P_1 \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ & \ddots & \ddots & \vdots \\ & & \ddots & A_{(s-1)s} \\ 0 & & & A_{ss} \end{bmatrix} P_1^{\mathrm{T}} \\ &= \begin{bmatrix} A'(\alpha_1) & A'(\alpha_1, \alpha_2) & \cdots & A'(\alpha_1, \alpha_s) \\ & \ddots & \ddots & \vdots \\ & & \ddots & A'(\alpha_{s-1}, \alpha_s) \\ 0 & & & A'(\alpha_s) \end{bmatrix}, \end{aligned} \tag{35}$$

where

$$A'(\alpha_i) = \begin{bmatrix} A(\gamma_i) & A(\gamma_i, \beta_i) \\ A(\beta_i, \gamma_i) & A(\beta_i) \end{bmatrix}, \quad A'(\alpha_i, \alpha_j) = \begin{bmatrix} A(\gamma_i, \gamma_j) & A(\gamma_i, \beta_j) \\ A(\beta_i, \gamma_j) & A(\beta_i, \beta_j) \end{bmatrix},$$

for $1 \leqslant i < j \leqslant s$. Therefore, there exists an $n \times n$ permutation matrix Q such that

$$QCQ^{\mathrm{T}} = QP_1 P A P^{\mathrm{T}} P_1^{\mathrm{T}} Q^{\mathrm{T}} = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix}, \tag{36}$$

where $A(\alpha)$ is given in (34), and

$$A(\alpha') = \begin{bmatrix} A(\gamma_1) & A(\gamma_1, \gamma_2) & \cdots & A(\gamma_1, \gamma_s) \\ & \ddots & \ddots & \vdots \\ & & \ddots & A(\gamma_{s-1}, \gamma_s) \\ 0 & & & A(\gamma_s) \end{bmatrix}, \tag{37}$$

$$\begin{aligned} A(\alpha, \alpha') &= \begin{bmatrix} A(\beta_1, \gamma_1) & A(\beta_1, \gamma_2) & \cdots & A(\beta_1, \gamma_s) \\ & \ddots & \ddots & \vdots \\ & & \ddots & A(\beta_{s-1}, \gamma_s) \\ 0 & & & A(\beta_s, \gamma_s) \end{bmatrix}, \\ A(\alpha', \alpha) &= \begin{bmatrix} A(\gamma_1, \beta_1) & A(\gamma_1, \beta_2) & \cdots & A(\gamma_1, \beta_s) \\ & \ddots & \ddots & \vdots \\ & & \ddots & A(\gamma_{s-1}, \beta_s) \\ 0 & & & A(\gamma_s, \beta_s) \end{bmatrix}. \end{aligned} \tag{38}$$

Direct computation yields

$$\begin{aligned} A/\alpha &= A(\alpha') - A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha') \\ &= \operatorname{diag}\Big(\frac{[A(\alpha_1)]}{\beta_1},\ldots,\frac{[A(\alpha_s)]}{\beta_s}\Big) + *, \end{aligned} \tag{39}$$

where $*$ denotes some unknown strictly upper triangular matrix. By Lemma 2.10, $A(\alpha_i) = R_{ii}$ is either 1×1 zero matrices or irreducible generalized diagonally dominant matrices for $i = 1,2,\ldots,s$. Furthermore, $\beta_i \subseteq \alpha_i$. As a result, $[A(\alpha_i)]/\beta_i$ is either 1×1 zero matrices or generalized diagonally dominant matrices for $i = 1,2,\ldots,s$. Again, Lemma 2.10 shows that $A/\alpha \in H_{n-|\alpha|}$. We complete the proof. □

On the Schur complement of an irreducible matrix $A \in H_n^M$, some equivalent conditions will be revealed such that the Schur complement of an irreducible mixed H-matrix still is an invertible H-matrix.

Theorem 5.10 *Let $A \in H_n^M$ be an irreducible matrix. Then, for all $\alpha \subset \langle n\rangle$, the following conclusions are equivalent*:

(i) *A is nonsingular*;

(ii) *$A/\alpha \in H_{n-|\alpha|}^I$*;

(iii) *$[\mu(A)]/\alpha < \mu(A/\alpha)$*;

(iv) *A/α is nonsingular.*

Proof Similar to the proof Lemma 3.12 or Theorem 4.1 in [28], it is obvious to get that (i) $\Longleftrightarrow$ (ii). Corollary 3.23 shows that (i) $\Longleftrightarrow$ (iv). [4, Corollary 3] shows (iii) $\Longleftrightarrow$ (i). As a result, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv). □

The following corollary is a direct consequence of Theorem 5.10.

Corollary 5.11 *Let $A \in H_n^M$ be an irreducible matrix. Then, for all $\alpha \subset \langle n\rangle$, the following conclusions are equivalent*:

(i) *A is singular*;

(ii) *$A/\alpha \in H_{n-|\alpha|}^M$ if $1 \leqslant |\alpha| \leqslant n-2$ and $A/\alpha = [0] \in H_{n-|\alpha|}^S$ if $|\alpha| = n-1$*;

(iii) *$[\mu(A)]/\alpha = \mu(A/\alpha)$*;

(iv) *A/α is singular.*

Note that Theorem 3, Corollaries 3 and 5 in [4] are some corollaries of Theorem 5.10 and Corollary 5.11.

It follows that we need only to consider the reducible H_n^M matrix. We will propose a theorem that is much easier to judge $A/\alpha \in H_{n-|\alpha|}^I$ than the one in [4].

Theorem 5.12 *Let $A \in H_n^M$ be a reducible matrix. If A is nonsingular, then $A/\alpha \in H_{n-|\alpha|}^I$ if and only if $A(\alpha') \in H_{n-|\alpha|}^I$ for nonempty $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$.*

Proof Assume that A is nonsingular. Theorems 3.22 and 2.10 show that in the Frobenius normal form (7) of A, each diagonal square block R_{ii} is irreducible and nonsihgular generalized diagonally dominant for all $i = 1, 2, \ldots, s$. Let

$$\alpha' = \bigcup_{j=1}^{s} \gamma_j, \quad \emptyset \subseteq \gamma_j \subseteq \alpha_j,\ j = 1, 2, \ldots, s,$$

and the number of nonempty set γ_j is at least equal to 2, such that $A(\alpha') \in H^I_{n-|\alpha|}$. Again, let $\beta_j = \alpha_j - \gamma_j$ for $j = 1, 2, \ldots, s$. Then

$$\alpha = N - \alpha' = \bigcup_{j=1}^{s} \alpha_j - \bigcup_{j=1}^{s} \gamma_j = \bigcup_{j=1}^{s} \beta_j. \tag{40}$$

Then there exists an $n \times n$ permutation matrix P_1 such that (35) holds. What is more, there exists an $n \times n$ permutation matrix Q such that (36) holds, where $A(\alpha)$ and $A(\alpha')$ are given in (34) and (37), respectively, $A(\alpha, \alpha')$ and $A(\alpha', \alpha)$ are given in (38). Direct computation yields (39). Since $A(\alpha_i)$ is irreducible generalized and nonsingular diagonally dominant for $i = 1, 2, \ldots, s$, it follows from Theorems 5.2 and 5.10 that

$$[A(\alpha_i)]/\beta_i \in H^I_{|\gamma_i|}, \quad i = 1, 2, \ldots, s.$$

As a consequence, $A/\alpha \in H^I_{n-|\alpha|}$ for all $\alpha = \langle n \rangle - \alpha' \subset \langle n \rangle$. This proves the sufficiency.

Now, we prove the necessity by contradiction. Assume that $A(\alpha') \notin H^I_{n-|\alpha|}$. Since $A \in H_n$, $A(\alpha') \in H_{n-|\alpha|}$. If $A(\alpha')$ is irreducible, Lemmas 2.6, 2.9, and 2.10 indicate that $A(\alpha') \in GDE_{n-|\alpha|}$ is an irreducible diagonal square block in the Frobenius normal form (7) of A. It is easy to get

$$A/\alpha = A(\alpha') \notin H^I_{n-|\alpha|},$$

which contradicts $A/\alpha \in H_{n-|\alpha|}$. A contradiction arises to illustrate $A(\alpha') \in H^I_{n-|\alpha|}$. If $A(\alpha') \notin H_{n-|\alpha|}$ is reducible, Lemma 2.11 indicates that $A(\alpha')$ has at least one irreducible generalized diagonally equipotent principal submatrix, say $A(\theta)$ for $\theta \subset \alpha'$. Furthermore, Lemma 2.10 shows that $A(\theta)$ is an irreducible diagonal square block in the Frobenius normal form (7) of A. As a result, assume $\theta = \alpha_k = \gamma_k$, where α_k and γ_k are in (40) for $1 \leqslant k \leqslant s$. Then $\beta_k = \alpha_k - \gamma_k = \emptyset$. Therefore,

$$[A(\alpha_k)]/\beta_k = [A(\alpha_k)]/\emptyset = A(\alpha_k) = A(\theta)$$

is irreducible generalized diagonally equipotent. This shows that A/α in (39) has at least an irreducible diagonal square block that is generalized diagonally equipotent. It follows from Lemma 2.11 that

$$A/\alpha \in H^M_{n-|\alpha|} \cup H^S_{n-|\alpha|},$$

but $A/\alpha \notin H^I_{n-|\alpha|}$. This contradicts $A/\alpha \in H^I_{n-|\alpha|}$ which demonstrate that the assumption is incorrect. Therefore, $A(\alpha') \in H_{n-|\alpha|}$. This completes the proof. □

Theorem 5.13 *Let $A \in H_n^M$ be a reducible matrix, and let $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$ such that $A(\alpha)$ is nonsingular. Then $A/\alpha \in H_{n-|\alpha|}^M$ if and only if one of the following two conclusions holds:*

(i) *$A(\alpha') \in H_{|\alpha'|}^M$ when A is nonsingular;*

(ii) *A does not have any irreducible submatrix $A(\beta)$ such that $A(\beta)$ is singular with $|\beta \cap \alpha| = |\beta| - 1$ when A is singular.*

Proof When A is nonsingular, Theorem 5.12 shows that the conclusion of this theorem is true.

When A is singular, since $A(\alpha)$ is nonsingular, it follows form Corollary 3.23 that A/α is singular, and hence, $A/\alpha \notin H_{n-|\alpha|}^I$. Again, since A does not have any irreducible submatrix $A(\beta)$ such that $A(\beta)$ is singular with $|\beta \cap \alpha| = |\beta| - 1$, the 2(a) conclusion of [4, Theorem 7] demonstrate that $A/\alpha \notin H_{n-|\alpha|}^S$. However, Theorem ????? gives $A/\alpha \in H_{n-|\alpha|}$. As a result, $A/\alpha \in H_{n-|\alpha|}^M$. We proves the sufficiency. Using the 2(a) conclusion of [4, Theorem 7], the necessity is obvious. □

Theorem 5.14 *Let $A \in H_n^M$ be a reducible matrix, and let $\alpha \subset \langle n\rangle$ such that $A(\alpha)$ is nonsingular. Then $A/\alpha \in H_{|\alpha|}^S$ if and only if A has at least one irreducible submatrix $A(\beta)$ such that $A(\beta)$ is singular with $|\beta \cap \alpha| = |\beta| - 1$.*

Proof It is obvious from the second conclusion of Theorem 5.13 that the conclusion is true. □

In the end, a result on the Schur complement for a singular H-matrix is given. This result is similar to [4, Theorem 8].

Theorem 5.15 *Let $A \in H_n^S$ be a reducible matrix, and let $\alpha \subset \langle n\rangle$ such that $A(\alpha)$ is nonsingular. Then $A/\alpha \in H_{|\alpha|}^S$.*

6 Conclusions

This paper studies some properties on general H-matrices and their Schur complements. Above all, the definitions of θ-ray pattern matrix and θ-ray matrix are firstly proposed to establish some new results on nonsingularity/singularity and convergence of general H-matrices. Following, some conditions on the matrix $A \in \mathbb{C}^{n\times n}$ and nonempty $\alpha \subset \langle 1, 2, \ldots, n\rangle$ are proposed such that A is an invertible H-matrix if $A(\alpha)$ and A/α are both invertible H-matrices. In the end, the important results on Schur complement for general H-matrices are presented to give the different necessary and sufficient conditions for the matrix $A \in H_n^M$ and the subset $\alpha \subseteq \langle n\rangle$ such that the Schur complement matrix $A/\alpha \in H_{n-|\alpha|}^I$ or $A/\alpha \in H_{n-|\alpha|}^M$ or $A/\alpha \in H_{n-|\alpha|}^S$.

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