A brief survey on Bounded Index Property

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Nielsen Theory and Related Topics
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Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

$$\mathrm{Fix} f := \{x \in X | f(x) = x\} = \bigsqcup_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathbf{F}$$

Definition

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

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Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Rank of a fixed point class

Definition

For a fixed point $x \in \mathbf{F}$, let

$$\mathrm{Stab}(f, x) := \{\gamma \in \pi_1(X, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(X, x),$$

where $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ is the induced endomorphism. It is independent of the choice of $x \in F$, up to isomorphism. For a fixed point class $\mathbf{F}$ of f , define the rank to be

$$\mathrm{rk}(\mathbf{F}) := \mathrm{rk}(f, x) := \mathrm{rk}\mathrm{Stab}(f, x), \quad \forall x \in \mathbf{F}.$$

For an empty fixed point class $\mathbf{F}$, we can make it nonempty by deforming f.

Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \quad \mathrm{rk}(f_0, \mathbf{F}_0) = \mathrm{rk}(f_1, \mathbf{F}_1).$$

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Commutation invariance

Suppose $\phi : X \to Y$ and $\psi : Y \to X$ are maps. Then $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are said to differ by a commutation. The map ϕ sets up a natural one-one correspondence

$$\mathbf{F}_X \to \mathbf{F}_Y$$

from the fixed point classes of $\psi \circ \phi$ to the fixed point classes of $\phi \circ \psi$.

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Mutation invariance

Among selfmaps of compact polyhedra, homotopy and commutation generates an equivalence relation:

Definition

A sequence $\{f_i : X_i \to X_i | i = 0, \cdots, k\}$ of self-maps is a mutation if for each i, either

1. $X_{i+1} = X_i$ and $f_{i+1} \simeq f_i$, or
2. f_{i+1} is obtained from f_i by commutation.

A mutation sets up a one-one correspondence between fixed point classes of the end maps.

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The index $\mathrm{ind}(\mathbf{F})$ *and the rank* $\mathrm{rk}(\mathbf{F})$ *are mutation invariants.*

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Characteristic of a fixed point class

From now on, unless otherwise stated, we always assume X to be a graph, a surface or a Seifert manifold, and $f : X \to X$ is a selfmap. For convenience, we define another term.

Definition

The characteristic of a fixed point class $\mathbf{F}$ is defined as

$$\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rk}(\mathbf{F}).$$

with the exception is when $\mathrm{Stab}(f, \mathbf{F}) = \pi_1(S)$ for some closed hyperbolic surface $S \subset X$, in this case

$$\mathrm{chr}(\mathbf{F}) := \chi(S) = 2 - \mathrm{rk}(\mathbf{F}).$$

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Definition

- A fixed point class $\mathbf{F}$ of f is essential if $\mathrm{ind}(f, \mathbf{F}) \neq 0$.
- Nielsen number $N(f) := \#\{\textit{essential fixed point classes of } f\}$.
- Lefschetz number

$$L(f) := \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X;\mathbb{Q}) \to H_q(X;\mathbb{Q})).$$

Lefschetz Fixed Point Theorem

$$\sum_{\mathbf{F}\in \mathrm{Fpc}(f)} \mathrm{ind}(f, \mathbf{F}) = \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X;\mathbb{Q}) \to H_q(X;\mathbb{Q})).$$

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Bounds for graphs and surfaces

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite **graph** *or a connected compact hyperbolic* **surface**, *and $f : X \to X$ is a* **selfmap**. *Then*

(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of f;*

(B) *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X),$$

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Fixed subgroups: free groups

As corollaries, we have

Theorem (Bestvina-Handel, 1992)

Let ϕ be an **automorphism** *of F_n. Then* $\mathrm{rkFix}\phi \leq \mathrm{rk}F_n$.

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Let ϕ be an **injective endomorphism** *of F_n. Then*

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Corollary (Jiang, 1998)

Let X be either a connected finite **graph** *(not a tree) or a connected compact hyperbolic* **surface**, *and $f : X \to X$ a* **selfmap**. *Then*

1. $\mathrm{ind}(\mathbf{F}) \leq 1$, $\forall \mathbf{F} \in \mathrm{Fpc}(f)$;
2. *Almost all fixed point classes have index ≥ -1, in the sense*
$$\sum_{\mathrm{ind}(\mathbf{F})<-1} \{\mathrm{ind}(\mathbf{F})+1\} \geq 2\chi(X).$$
3. $|L(f) - \chi(X)| \leq N(f) - \chi(X)$.

Theorem (Jiang-Wang-Z., 2011)

Let ϕ be an **endomorphism** *of a surface group G. Then*

1. $\mathrm{rkFix}\phi \leq \mathrm{rk}G$, *with equality if and only if $\phi = id$;*
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Fixed subgroups: surface groups

Theorem (Wu-Z.,2014)

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Bounds for 3-manifolds

Theorem (Jiang-Wang, 1992)

Suppose a closed **aspherical 3-manifold** *M is finitely covered by an orientable 3-manifold which is either a Seifert manifold, or a hyperbolic 3-manifold, or admits a non-trivial JSJ-decomposition. Let $f : M \to M$ is a homeomorphism. Then*

1. $\mathrm{ind}(\mathbf{F}) \leq 1$, $\forall \mathbf{F} \in \mathrm{Fpc}(f)$, *hence* $L(f) \leq N(f)$;
2. *If M is orientable and f is* **orientation-preserving**, *then*
$$\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}, \quad \forall\ \mathbf{F} \in \mathrm{Fpc}(f),$$
hence $|L(f)| \leq N(f)$.
3. $\forall\ n > 3$, $\exists f$ *on a closed aspherical n-manifold such that*
$$L(f) > N(f).$$

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Bounds for Seifert 3-manifolds, I

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Suppose M is a compact orientable **Seifert** *3-manifold with hyperbolic orbifold, and $f : M \to M$ is a homeomorphism. Then*

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where the sum is taken over all **essential** *fixed point classes* $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$, *and*

$$\mathcal{B} = \begin{cases} 4(3 - \mathrm{rk}\pi_1(M)) & M \text{ is a closed surface } F \times S^1 \\ 4(2 - \mathrm{rk}\pi_1(M)) & \text{others} \end{cases}.$$

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As a corollary, we have

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2. $\sum_{\mathrm{ind}(\mathbf{F})<-1}\{\mathrm{ind}(\mathbf{F})+1\} \geq \mathcal{B}$.
3. $|L(f)-\mathcal{B}/2| \leq N(f)-\mathcal{B}/2$.

The bound above is analogous to the one on graphs and surfaces. For f **orient.-preserving**, [Jiang-Wang, 1992]: $\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}$.

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The bound above is analogous to the one on graphs and surfaces.
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Bounds for Seifert 3-manifolds, II

As a corollary, we have

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Proposition (Z., 2012)

Suppose $f : M \to M$ is a homeomorphism of a compact orientable **Seifert** *3-manifold with hyperbolic orbifold. Let $f_\pi : \pi_1(M, x) \to \pi_1(M, x)$ be the induced automorphism and $\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(M, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(M, x)$, where x is in an* **essential** *fixed point class. Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

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Suppose M is a compact orientable **Seifert** *3-manifold, and $f_\pi : \pi_1(M) \to \pi_1(M)$ is an automorphism induced by an* **orientation-reversing** *homeomorphism $f : M \to M$. Then*

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Bounds for hyperbolic 3-manifolds

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Theorem (Z., 2013)

Let M^3 be a compact hyperbolic 3-manifold (orientable or nonorientable). Then for any homeomorphism $f : M \to M$,

1. $\mathrm{ind}(f, \mathbf{F}) \leq 1$ *for every fixed point class* $\mathbf{F}$ *of* f*;*
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As a corollary, we have a bound for hyperbolic 3-manifolds

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Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a compact orientable **hyperbolic** *3-manifold with finite volume. Then*

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Theorem (Z., 2015)

Let M^4 be a hyperbolic 4-manifold. Then for any homeomorphism $f: M \to M$, we have

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}(M),$$

where $\mathcal{B}(M) = \max\{\dim H_*(M; \mathbb{Z}_p) | p \text{ is a prime}\}$. In particular, if f is not homotopic to the identity, then

$$\mathrm{ind}(f, \mathbf{F}) \leq 1, \qquad L(f) \leq N(f).$$

Bounds for hyperbolic n-manifolds

Theorem (Z., 2015)

Let M^n be a hyperbolic n-manifold ($n \geq 5$). If the isometry group $\mathrm{Isom}(M)$ is a **p-group** *($|\mathrm{Isom}(M)|$ is a power of some prime p), then for any homeomorphism $f : M \to M$, we have*

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \dim H_*(M; \mathbb{Z}_p),$$

where $\dim H_*(M; \mathbb{Z}_p)$ denotes the dimension of the $\mathbb{Z}_p$-linear space

$$H_*(M; \mathbb{Z}_p) = \bigcup_{r \geq 0} H_r(M; \mathbb{Z}_p).$$

Question

Is there an analogous explicit bound for any compact hyperbolic n-manifold with $n \geq 5$?

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Key points of Proofs of the three Theorems above

- $n \geq 3$, Mostow Rigidity Thm $\Longrightarrow$ f can be homotopied to a unique **isometry** g of finite order.
- $\mathbf{F}$: a compact hyperbolic submanifold, $|\mathrm{ind}(\mathbf{F})| = |\chi(\mathbf{F})| < \infty$.
- P.A. Smith Theory: Let X be a compact topological space and $t : X \to X$ a transformation of order a prime p. Suppose X has a triangulation in which t is simplicial. Let F denote the set of fixed points of t, and X' be the quotient space $X/(x = tx)$. The projection $X \to X'$ maps F homeomorphically onto a subset of X', which we again denote by F. Then for any q,

$$\dim H_q(X', F; \mathbb{Z}_p) + \sum_{r=q}^{\infty} \dim H_r(F; \mathbb{Z}_p) \leq \sum_{r=q}^{\infty} \dim H_r(X; \mathbb{Z}_p).$$

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Bounded Index Property: BIP $\implies$ BIPHE $\implies$ BIPH

A compact polyhedron X is said to have the *Bounded Index Property (BIP)*(resp. *Bounded Index Property for Homeomorphisms (BIPH)*, *Bounded Index Property for Homotopy Equivalences (BIPHE)*), if $\exists\ \mathcal{B} > 0$ s.t. for any map (resp. homeomorphism, homotopy equivalence) $f : X \to X$,

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Open problem

Question (Jiang, 1998)

Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

Positive examples:

- [McCord, 1992]: Infra-solvmanifolds have BIP;
- [Jiang-Wang, 1992]: Closed aspherical 3-manifolds have BIPH for orientation preserving self-homeomorphisms;
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Bounds for products of hyperbolic surfaces

Suppose S_1 and S_2 are two connected compact surfaces with Euler characteristics $\chi_1 := \chi(S_1) \leq \chi_2 := \chi(S_2) < 0$, then $S_1 \times S_2$ has BIPH. More precisely,

Theorem (Z.-Zhao, 2017)

Let $f : S_1 \times S_2 \to S_1 \times S_2$ be a homeomorphism. Then the indices of the Nielsen fixed point classes of f are bounded:

1. *For every fixed point class* $\mathbf{F}$ *of f, we have*
$$2\chi_1 - 1 \leq \mathrm{ind}(f, \mathbf{F}) \leq (2\chi_1 - 1)(2\chi_2 - 1);$$
2. $|L(f) - 2\chi_1\chi_2| \leq (1 - 2\chi_1)N(f) + 2(\chi_1\chi_2 - \chi_1).$

To prove the above Theorem, we first consider two good forms of selfmaps called **fiber-preserving maps** and **alternating homeomorphisms**, and then show that any homeomorphism f can be homotoped to one of the two good forms.

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Case 1: Fiber-preserving maps

Definition

A selfmap $f : S_1 \times S_2 \to S_1 \times S_2$ is called a fiber-preserving map, if

$$f = f_1 \times f_2 : S_1 \times S_2 \to S_1 \times S_2, \quad (a, b) \mapsto (f_1(a), f_2(b)),$$

where f_i is a selfmap of $S_i (i = 1, 2)$.

For any fiber-preserving map f, we have $\mathrm{Fix} f = \mathrm{Fix} f_1 \times \mathrm{Fix} f_2$, and each fixed point class $\mathbf{F}$ of f splits into a product of some fixed point classes of f_i, i.e.,

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1. $\mathbf{F} = \mathbf{F}_1 \times \mathbf{F}_2, \qquad \mathrm{ind}(f, \mathbf{F}) = \mathrm{ind}(f_1, \mathbf{F}_1) \cdot \mathrm{ind}(f_2, \mathbf{F}_2),$
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Case 1: Fiber-preserving maps

By the Lemma above, we can show that the product $S_1 \times S_2$ has BIP for fiber-preserving maps.

Proposition (BIP for fiber-preserving maps)

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If $f : S_1 \times S_2 \to S_1 \times S_2$ is a **fiber-preserving map**, *then*

1. *For every fixed point class* $\mathbf{F}$ *of f, we have*

$$2\chi_1 - 1 \leq \mathrm{ind}(f, \mathbf{F}) \leq (2\chi_1 - 1)(2\chi_2 - 1);$$

2. $|L(f) - 2\chi_1\chi_2| \leq (1 - 2\chi_1)N(f) + 2(\chi_1\chi_2 - \chi_1).$

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Case 2: Alternating homeomorphisms

Let $S_1 = S_2$ be two copies of a connected compact hyperbolic surface S, and hence, their Euler characteristics $\chi_1 = \chi_2 = \chi(S) < 0$.

Definition

A self-homeomorphism $f : S_1 \times S_2 \to S_1 \times S_2$ is called an alternating homeomorphism, if

$$f = \tau \circ (f_1 \times f_2) : S_1 \times S_2 \to S_1 \times S_2$$

$$(a, b) \mapsto (f_2(b), f_1(a)),$$

where f_1, f_2 are two self-homeomorphisms of S, and τ is a transposition.

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Lemma

If $f : S_1 \times S_2 \to S_1 \times S_2$ is an alternating homeomorphism, then the nature map

$$\rho : S_1 \to S_1 \times S_2, \quad a \mapsto (a, f_1(a))$$

induces an index-preserving one-to-one corresponding between the set $\mathrm{Fpc}(f_2 \circ f_1)$ of fixed point classes of $f_2 \circ f_1$ and the set $\mathrm{Fpc}(f)$ of fixed point classes of f.

Proof: Let $M = Df_1(a)$ and $N = Df_2(b)$ for $b = f_1(a)$. Then the differential $Df(a,b)$ of f at (a,b) is $\begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}$. Hence

$$\begin{aligned} \mathrm{ind}(f,(a,b)) &= sgn \det(I_4 - \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}) \\ &= sgn \det(I_2 - NM) \\ &= \mathrm{ind}(f_2 \circ f_1, a). \end{aligned}$$

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If $f = \tau \circ (f_1 \times f_2) : S_1 \times S_2 \to S_1 \times S_2$ is an alternating homeomorphism, then by the previous lemma, we have

Lemma

$$N(f) = N(f_2 \circ f_1) = N(f_1 \circ f_2), \quad L(f) = L(f_2 \circ f_1) = L(f_1 \circ f_2).$$

Proposition (BIPH for alternating homeomorphisms)

1. $2\chi_1 - 1 \leq \mathrm{ind}(f, \mathbf{F}) \leq 1$, $\forall \mathbf{F} \in Fpc(f)$;
2. *Almost all fixed point classes have index ≥ -1, in the sense*
$$\sum_{\mathrm{ind}(f,\mathbf{F})<-1} \{\mathrm{ind}(f, \mathbf{F}) + 1\} \geq 2\chi_1;$$
3. $|L(f) - \chi_1| \leq N(f) - \chi_1$.

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Good forms of self-homeomorphisms of $S_1 \times S_2$

Recall that S_1 and S_2 be two compact hyperbolic surfaces.

Lemma (Z.-Ventura-Wu, 2015)

Let $G = \pi_1(S_1) \times \pi_1(S_2)$ and $\phi \in \mathrm{Aut}(G)$ be an automorphism. Then there exist automorphisms $\phi_i \in \mathrm{Aut}(\pi_1(S_i))$ such that ϕ must have one of the following forms:

1. *if $S_1 \ncong S_2$, then $\phi = \phi_1 \times \phi_2$;*
2. *if $S_1 \cong S_2$, then $\phi = \begin{cases} \phi_1 \times \phi_2 \\ \tau \circ (\phi_1 \times \phi_2) \end{cases}$,*

   where τ is a transposition.

Good forms of self-homeomorphisms of $S_1 \times S_2$

Proposition

Let $f : S_1 \times S_2 \to S_1 \times S_2$ be a homeomorphism, where S_1, S_2 are two compact hyperbolic surfaces. Then

1. *if $S_1 \ncong S_2$, then f can be homotoped to a fiber-preserving homeomorphism $f_1 \times f_2$;*
2. *if $S_1 \cong S_2$, then f can be homotoped to either a fiber-preserving homeomorphism or an alternating homeomorphism.*

Proof: f homeomorphism $\Longrightarrow f_\pi = \phi_1 \times \phi_2$ or $f_\pi = \tau \circ (\phi_1 \times \phi_2)$, where $\phi_i \in \mathrm{Aut}(\pi_1 S_i)$. By Dehn-Nielsen-Bar Thm for hyperbolic surfaces, ϕ_i can be induced by a self-homeomorphism f_i of S_i. Hence

$$f_\pi = (f_1 \times f_2)_\pi \quad or \quad f_\pi = \left(\tau \circ (f_1 \times f_2)\right)_\pi.$$

S_i hyperbolic $\Longrightarrow S_1 \times S_2$ aspherical $\Longrightarrow f \simeq \begin{cases} f_1 \times f_2 \\ \tau \circ (f_1 \times f_2) \end{cases}.$ $\square$

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New progress on BIPHE & BIPH

Fact: For a compact aspherical polyhedron X, then

$$\mathrm{Out}(\pi_1(X)) \textit{ finite} \Longrightarrow X \textit{ has BIPHE} \Longrightarrow X \textit{ has BIPH}.$$

Theorem (Ye-Z., 2019)

A closed Riemannian $n-$manifold M^n with **negative sectional curvature** *everywhere has BIPHE (and hence has BIPH).*

Proof.

- $n = 2$, M^2 is a closed hyperbolic surface, and hyperbolic surfaces have BIP;
- $n \geq 3$, $\mathrm{Out}(\pi_1(M^n))$ is finite $\Longrightarrow$ M has BIPHE.

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Suppose $X_1, \ldots, X_n$ are compact aspherical polyhedra satisfying the following two conditions:
(1) $\pi_1(X_i) \ncong \pi_1(X_j)$ for $i \neq j$, and all of them are **centerless** *and* **indecomposable***;*
(2) all of $X_1, \ldots, X_n$ have BIPHE.
Then the product $X_1 \times \cdots \times X_n$ also has BIPHE (and hence has BIPH).

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Let $M = M_1 \times \cdots \times M_n$ be the product of finitely many closed Riemannian manifolds, each with **negative sectional curvature** *everywhere but not necessarily with the same dimensions (in particular hyperbolic manifolds). Then M has BIPHE.*

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Thanks！　　谢　谢！