The fixed subgroups of homeomorphisms of Seifert manifolds

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Fixed subgroup: definition

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For a free group, Bestvina and Handel solved the well-known Scott's conjecture:

Theorem (Bestvina-Handel, 1992)

Let ϕ be an automorphism of a finitely generated free group G. Then $\mathrm{rkFix}(\phi) \leq \mathrm{rk}G$.

Fixed subgroup: surface and hyperbolic 3-manifold groups

For the fundamental group of a compact surface, B. Jiang, S. D. Wang and Q. Zhang proved that

Theorem (Jiang-Wang-Z., 2011)

Let S be a compact surface and ϕ be an endomorphism of $\pi_1(S)$. Then

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Later, J. Lin and S. C. Wang showed that

Theorem (Lin-Wang, 2012)

Let M be a compact orientable hyperbolic 3-manifold with finite volume and ϕ be an automorphism of $\pi_1(M)$. Then

$$\mathrm{rkFix}(\phi) < 2\mathrm{rk}\pi_1(M).$$

Seifert 3-manifold: definitions

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A compact orientable 3-manifold M is called a Seifert manifold, if M possesses a Seifert fibration which is a decomposition of M into disjoint simple closed curves, called fibers, such that each fiber has a solid torus neighborhood consisting of a union of fibers. Identifying each fiber of M to a point, we get a set B_M, called the orbifold of M, which has a natural 2-orbifold structure with singular points consisting of cone points.

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A Seifert manifold can be think as a circle bundle over an orbifold.

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- A map f on a Seifert manifold M is called fiber-preserving if it maps fibers to fibers. If f is fiber-preserving, then it induces a map $f' : B_M \to B_M$ on the orbifold B_M.
- Suppose M is a compact orientable Seifert manifold and $p : M \to B_M$ is a Seifert fibration with hyperbolic orbifold B_M. Then any homeomorphism on M is isotopic to a fiber-preserving homeomorphism with respect to this fibration.

Our main result

Theorem

Let M be a compact orientable Seifert manifold with hyperbolic orbifold B_M, and f_π an automorphism of $\pi_1(M)$ induced by an **orientation-reversing** *homeomorphism $f : M \to M$. Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

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- The condition that f is **orientation-reversing** is necessary: If f is orientation-preserving, then the fixed subgroup $\mathrm{Fix}(f_\pi)$ can be infinitely generated.
- The constant 2 is sharp: $\forall \varepsilon > 0$, $\exists$ a Seifert manifold M_n and an orientation-reversing homeomorphism f of M_n, such that

$$\frac{\mathrm{rkFix}(f_\pi)}{\mathrm{rk}\pi_1(M_n)} = \frac{4n-2}{2n+1} > 2 - \varepsilon.$$

Fixed point class: classical definition

Suppose $f : X \to X$ is a selfmap of a path-connected space, and $\mathrm{Fix} f := \{x \in X | f(x) = x\}$ is the set of fixed points.

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Definition

Two fixed points $x, x' \in \mathrm{Fix} f$ are in the same fixed point class $\Longleftrightarrow$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

A fixed point class $\mathbf{F}$ is essential if $\mathrm{ind}(\mathbf{F}) \neq 0$. Otherwise, it is called inessential.

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The more subtle definition of fixed point class which includes **empty** ones will be given below in this talk. Of course their $\mathrm{ind} = 0$.

Fixed point class: alternative definition

Definition

An *f-route* is a path $w : I \to X$ such that $w(1) = f(w(0))$.

Two f-routes w, w' are *conjugate* if $\exists$ a path $q : I \to X$ from $w(0)$ to $w'(0)$ such that $qw' \simeq w(f \circ q)$ rel endpoints. An *f-route class* is an conjugacy class of f-routes.

For each f-route w, a *fixed point class* $\mathbf{F}_w$ is associated by the rule: a fixed point $x \in \mathrm{Fix} f$ belongs to $\mathbf{F}_w \iff$ the constant f-route at x is conjugate to w.

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- Fixed point classes provide a disjoint decomposition of $\mathrm{Fix} f$.
- A fixed point class could be empty.
- This definition is clearly equivalent to the classical one.

Fixed point class: rank

Definition

An f-route w gives rise to an endomorphism

$$f_w : \pi_1(X, w(0)) \to \pi_1(X, w(0)), \quad [a] \mapsto [w(f \circ a)\bar{w}].$$

The rank of a fixed point class $\mathbf{F}_w$ is

$$\mathrm{rk}(f, \mathbf{F}_w) := \mathrm{rkFix}(f_w)$$

which is well defined because conjugate f-routes induce isomorphic fixed subgroups.

Fixed point class: homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

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Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \quad \mathrm{rk}(f_0, \mathbf{F}_0) = \mathrm{rk}(f_1, \mathbf{F}_1).$$

Fixed point class: lifting

Let $p : \widetilde{M} \to M$ be a finite covering of a compact manifold M, and $f : M \to M$ be a homeomorphism.

Lemma

If $\tilde{f} : \widetilde{M} \to \widetilde{M}$ is a lifting of f, and the $\tilde{f}$-route $\tilde{w}$ is a lifting of the f-route w. Then the f-fixed point class $\mathbf{F}_w$ is essential if and only if the $\tilde{f}$-fixed point class $\mathbf{F}_{\tilde{w}}$ is essential, moreover,

$$\mathrm{ind}(\tilde{f}, \mathbf{F}_{\tilde{w}}) = n \times \mathrm{ind}(f, \mathbf{F}_w)$$

where n is a positive integer.

Fixed subgroups of inessential fixed point classes

For a compact orientable Seifert manifold M with hyperbolic orbifold, let $f : M \to M$ be a homeomorphism, w an f-route, and $f_w : \pi_1(M, w(0)) \to \pi_1(M, w(0)$ the automorphism induced by f.

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Proposition (Z., 2012)

If the fixed point class $\mathbf{F}_w$ *of* f *is* **essential**, *then*

$$\mathrm{rkFix}(f_w) < 2\mathrm{rk}\pi_1(M).$$

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Below we will show that

Proposition (Fixed subgroups of inessential fixed point classes)

If the fixed point class $\mathbf{F}_w$ *of* f *is* **inessential**, *and* f *reverses the orientation of* M, *then* $\mathrm{rkFix}(f_w) \leq 3$.

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The two propositions above $\Longrightarrow$ MAIN THEOREM, i.e, $\forall$ orientation-reversing f, we have $\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M)$.

Fixed subgroups on circle bundles

To prove Proposition for fixed subgroups of inessential fixed point class, we consider fixed subgroups of circle bundles at first.

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Proposition (Fixed subgroups on circle bundles)

Let $p : M \to S$ be a compact orientable circle bundle over an orientable hyperbolic surface S, $f : M \to M$ an orientation-reversing fiber-preserving homeomorphism of M, and $f' : S \to S$ the induced homeomorphism of f. If an f-route w corresponds to an **inessential** *fixed point class $\mathbf{F}_w$. Then*

1. $\mathrm{Fix}(f'_{p\circ w})$ *is trivial or the free cyclic group $\mathbb{Z}$;*
2. $\mathrm{Fix}(f_w)$ *is trivial or a free abelian group of rank ≤ 2.*

Proof of Proposition for circle bundles

1. Some results of fixed subgroups on surface group in [Jiang-Wang-Z., 2011] $\Longrightarrow \mathrm{Fix}(f'_{p\circ w})$ is trivial or $\mathbb{Z}$.

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2. Let $x = w(0)$. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_1(M,x) & \xrightarrow{f_w} & \pi_1(M,x) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\
\pi_1(S,p(x)) & \xrightarrow{f'_{p\circ w}} & \pi_1(S,p(x))
\end{array}$$

where

$$p_* : \pi_1(M,x) \to \pi_1(S,p(x)) \cong \pi_1(M,x)/\langle t\rangle$$

is the quotient map and $\langle t\rangle$ generated by a fiber of M is the center of $\pi_1(M,x)$. Hence

$$\mathrm{Fix}(f_w) \le p_*^{-1}\mathrm{Fix}(f'_{p\circ w}) \cong \mathrm{Fix}(f'_{p\circ w}) \times \langle t\rangle \le \mathbb{Z}\oplus\mathbb{Z}.$$

Therefore, $\mathrm{Fix}(f_w)$ is trivial or free abelian of rank ≤ 2.

Proof of Prop. for inessential fixed point classes, I

Let $f : M \to M$ be a homeomorphism of a compact orientable Seifert manifold M, and $p : M \to B$ the Seifert fibration with hyperbolic orbifold B.

- Isotopy f to a fiber-preserving homeomorphism.
- B is hyperbolic $\Longrightarrow \exists$ a finite covering $q : S \to B$ with S an orientable hyperbolic surface.
- By pull back via the finite covering $q : S \to B$, we have a commutative diagram:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "q'", "deg-d\ \ cover"'] \arrow[d, "circle\ \ bundle"', "p'"] & M \arrow[d, "p"] \\ S \arrow[r, "q"] & B \end{tikzcd}$$

Proof of Prop. for inessential fixed point classes, II

- Let $H = p_*\mathrm{Fix}(f_w) \leq \pi_1(B)$ and $H^d = \{h^d | h \in H\}$.
  Then H^d is contained in a free cyclic subgroup of the Fuchsian group $\pi_1(B)$.
  Therefore, by group theory, H is a **metacyclic group** (i.e., an extension of a cyclic group by a cyclic group). Thus
  $$\mathrm{rk} H \leq 2.$$

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- Consider the quotient map

$$p_* : \pi_1(M, x) \to \pi_1(M, x)/\langle t \rangle = \pi_1(B)$$

  we have

$$\mathrm{Fix}(f_w) \leq p_*^{-1}(H)$$

  which is an extension of the metacyclic group H by $\langle t \rangle \cong \mathbb{Z}$. Therefore,

$$\mathrm{rkFix}(f_w) \leq 3.$$

Example

Let S_n be a closed orientable surface of genus $n \geq 2$. Define an orientation-reversing homeomorphism f as follows:

$$f = f_1 \times f_2 : \; S_n \times S^1 \to S_n \times S^1,$$

where $f_1 : S_n \to S_n$ is a reflection on a simple closed curve γ, and $f_2 : S^1 \to S^1$ is a rotation. Then all the fixed point classes of f are inessential, and f induces an automorphism f_π of $\pi_1(S_n \times S^1)$ such that

$$\mathrm{Fix}(f_\pi) = \pi_1(\gamma \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Namely, there is an inessential fixed point class which has

$$\mathrm{rkFix}(f_\pi) = 2.$$

Question

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Is there an orientation-reversing homeomorphism f of a Seifert manifold M whose inessential fixed point class has

$$\mathrm{rkFix}(f_\pi) = 3\ ?$$

Namely, is the bound 3 in Proposition for inessential fixed point classes sharp?

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Thanks！