The group fixed by a family of endomorphisms of a surface group

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Intersection of subgroups of free groups

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For any **free group**, Mineyev and Friedman proved the following theorem conjectured by [Neumann, 1957] independently.

Theorem (Mineyev, Friedman, 2011)

Let F_n be a f.g. free group, and A, B any two f.g. subgroups of F_n. Then

$$\mathrm{rk}(A \cap B) - 1 \leq (\mathrm{rk}A - 1)(\mathrm{rk}B - 1).$$

Intersection of subgroups of one-relator groups

Let $G = \langle X | r \rangle$ be a **one-relator group** where r is a cyclically reduced word in the free group on the generating set X.

A subset $Y \subset X$ is called a *Magnus subset* if Y omits a generator which appears in the relator r. A subgroup H of G is called a *Magnus subgroup* if $H = \langle Y \rangle$ for some Magnus subset Y of X, and hence by the Magnus Freiheitssatz, H is free of rank $|Y|$.

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Theorem (Collins, 2004)

The intersection $\langle Y \rangle \cap \langle Z \rangle$ of two Magnus subgroups of the one-relator group G is either $\langle Y \cap Z \rangle$ or the free product of $\langle Y \cap Z \rangle$ with an infinite cyclic group and thus of rank $|Y \cap Z| + 1$.

Definition of fixed subgroups

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For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

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Let G be a finitely generated free group, and $\mathcal{B}$ a family of injective endomorphisms of G. Then $\mathrm{rkFix}\mathcal{B} \leq \mathrm{rk}G$.

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Definition

A subgroup A is inert in G if for every subgroup $B \leq G$,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk}B.$$

Fixed subgroups of free groups

Theorem (Bergman, 1999)

The Dicks-Ventura Theorem also holds for any family of endomorphisms (not necessarily injective).

Question 3 (Bergman, 1999)

Is the fixed subgroup of a family of endomorphisms of a finitely generated free group inert?

Fixed subgroups of surface groups

Let G be a surface group, namely, $G \cong \pi_1(S)$ where S is a closed surface with $\chi(S) < 0$.

Theorem (Jiang-Wang-Z., 2011)

Let ϕ be an endomorphism of G. Then

1. $\mathrm{rkFix}\phi \leq \mathrm{rk}G$ *if ϕ is epimorphic, with equality if and only if $\phi = id$;*
2. $\mathrm{rkFix}\phi \leq \frac{1}{2}\mathrm{rk}G$ *if ϕ is not epimorphic.*

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We generalize this result to any family of endomorphisms:

Theorem A (Wu-Z.)

Let $\mathcal{B}$ be a family of endomorphisms of G. Then

1. $\mathrm{rkFix}\mathcal{B} \leq \mathrm{rk}G$, *with equality if and only if $\mathcal{B} = \{id\}$;*
2. $\mathrm{rkFix}\mathcal{B} \leq \frac{1}{2}\mathrm{rk}G$, *if $\mathcal{B}$ contains a non-epimorphic endomorphism*

Inertia of geometric subgroups

A connected subsurface F of a connected surface S is called incompressible if the natural homomorphism $\pi_1(F) \to \pi_1(S)$ induced by the inclusion $F \hookrightarrow S$ is injective. If F is incompressible in S, then we can think of $\pi_1(F)$ as a subgroup of $\pi_1(S)$. Subgroups which arise in this way are called geometric.

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For geometric subgroups of a surface group, we prove that

Theorem B (Wu-Z.)

If A is a geometric subgroup of a surface group G, then A is inert in G, i.e., for any subgroup B of G, we have $\mathrm{rk}(A \cap B) \leq \mathrm{rk} B$.

Corollary (Wu-Z.)

The fixed subgroup of any family of epimorphisms of a surface group G is inert in G.

Sketch of the proof of Theorem B

Using covering theory:

- A is a geometric subgroup of $G \iff \exists$ incomp. subsurface F of a closed surface S, s.t. $A = \pi_1(F, *) \leq \pi_1(S, *) = G$.

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- Consider the commutative diagram

$$\begin{array}{ccc} F_0 \subset \tilde{F} & \xrightarrow{i'} & K \\ p\Big\downarrow & & k\Big\downarrow \\ F & \xrightarrow{i} & S \end{array}$$

where $p : \tilde{F} \to S$ is the **pull back** map of k via i, and F_0 is the component of $\tilde{F}$ containing the base point.

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  where $p : \tilde{F} \to S$ is the **pull back** map of k via i, and F_0 is the component of $\tilde{F}$ containing the base point.
- $i_* p_*(\pi_1(F_0)) = A \cap B \quad \Longrightarrow \quad \mathrm{rk}(A \cap B) = \mathrm{rk}\pi_1(F_0) \leq \mathrm{rk}B.$

Definitions of equalizers and retracts

- Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A section of ϕ is a homomorphism $\sigma : H \to G$ such that
$$\phi\sigma = id : H \to H.$$
For any family $\mathcal{B}$ of sections of ϕ, the equalizer of $\mathcal{B}$ is
$$\mathrm{Eq}(\mathcal{B}) := \{h \in H | \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\} \leq H.$$

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- Suppose H is a subgroup of a group G. If there is a homomorphism $\pi : G \to G$ such that $\pi(G) \leq H$ and
$$\pi|_H = id : H \to H,$$
we say that H is a retract of G. If $H \neq G$, it is called a proper retract.

Results on equalizers and retracts

We have the following relation between equalizers and retracts:

Lemma

Let G, H be two groups, and $\phi : G \to H$ an epimorphism. If $\mathcal{B}$ is a family of sections of ϕ, then for any section $\sigma \in \mathcal{B}$, $\sigma(H)$ is a retract of G, and

$$\sigma|_{\mathrm{Eq}(\mathcal{B})} : \mathrm{Eq}(\mathcal{B}) \to \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$$

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For free groups, Bergman showed

Proposition (Bergman, 1999)

1. *Any intersection of retracts of a f.g. free group is also a retract;*
2. *If $\phi : G \to H$ is an epimorphism of free groups with H f.g., then the equalizer of any family of sections of ϕ is a free factor in H.*

Retracts on surface groups: a key lemma

For a surface group G, we have

Key Lemma

1. *Any proper retract of* G *is free of rank* $\leq \frac{1}{2}\mathrm{rk}G$.

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1. *Any proper retract of G is free of rank $\leq \frac{1}{2}\mathrm{rk}G$.*
2. *If H_1, H_2 are two proper retracts of G, and $H = \langle H_1, H_2 \rangle \leq G$, the subgroup generated by H_1 and H_2, then*
   (1) *If $H < G$, then $H_1 \cap H_2$ is a retract of both H_1 and H_2, and*
   $$\mathrm{rk}(H_1 \cap H_2) \leq \min\{\mathrm{rk}H_1, \mathrm{rk}H_2\}.$$
   (2) *If $H = G$, then $H_1 \cap H_2$ is cyclic (possibly trivial).*

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   (2) *If $H = G$, then $H_1 \cap H_2$ is cyclic (possibly trivial).*
3. *If $\mathcal{R}$ is a family retracts of G, then*
   $$\mathrm{rk}(\bigcap_{H \in \mathcal{R}} H) \leq \min\{\mathrm{rk}H | H \in \mathcal{R}\} \leq \begin{cases} \mathrm{rk}G, & \mathcal{R} = \{G\} \\ \frac{1}{2}\mathrm{rk}G, & \mathcal{R} \neq \{G\} \end{cases}.$$

Proof of Key Lemma

The proof is based on some facts below:

1. The inner rank $\mathrm{Ir}(G)$ of a group G defined as the maximal rank of free homomorphic images of G.
   [Zieschang, 1961]: If G is a surface group, then
   $\mathrm{Ir}(G) = [\frac{1}{2}\mathrm{rk}G]$ (i.e., the greatest integer not more than $\frac{1}{2}\mathrm{rk}G$).

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   [Zieschang, 1961]: If G is a surface group, then $\mathrm{Ir}(G) = [\frac{1}{2}\mathrm{rk}G]$ (i.e., the greatest integer not more than $\frac{1}{2}\mathrm{rk}G$).
2. [Louder, 2010]: Let S be a closed surface, and $n = \mathrm{rk}\pi_1(S)$. If $X = \{x_1, x_2, \ldots, x_n\}$ is any generating set of $\pi_1(S)$, then $\pi_1(S)$ has a new one-relator presentation $\pi_1(S) = \langle x_1, x_2, \ldots, x_n | r\rangle$, where r is a cyclically reduced word in the free group on the generating set X.

Equalizers on surface groups

According to Key Lemma, we have

Proposition (Equalizers on surface groups)

Let $\mathcal{B}$ be a family of sections of an epimorphism $\phi : G \to F$, where F is a f.g. free group. Then

$$\mathrm{rkEq}(\mathcal{B}) \leq \mathrm{rk}F \leq \frac{1}{2}\mathrm{rk}G.$$

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- Choose $\beta \in \mathcal{B}$ such that $\beta(G)$ is free, and

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- Let $\beta\mathcal{B} = \{\beta\gamma | \gamma \in \mathcal{B}\}$. Then $\beta\mathcal{B}|_{\beta(G)}$ is a family of injective endomorphisms of the free group $\beta(G)$.

Sketch of the proof of Theorem A: II

- Note that $\mathrm{Fix}(\beta\mathcal{B}) = \mathrm{Fix}(\beta\mathcal{B}|_{\beta(G)}) \leq \beta(G)$. Therefore, the Dicks-Ventura Theorem implies

$$\mathrm{rk}\mathrm{Fix}(\beta\mathcal{B}) \leq \mathrm{rk}(\beta(G)) \leq \frac{1}{2}\mathrm{rk}\, G. \tag{0.1}$$

Sketch of the proof of Theorem A: II

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- Now we **Claim**:

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Therefor,
Equation (0.1) + (0.2)$\Longrightarrow \mathrm{rkFix}\mathcal{B} \leq \frac{1}{2}\mathrm{rk}G$ (i.e., Theorem A holds).

To complete the proof, we only need to prove the Claim.

Sketch of the proof of Theorem A: III

Proof of Claim: $\mathrm{rkFix}\mathcal{B} \leq \mathrm{rkFix}(\beta\mathcal{B})$.

- $E := \beta^{-1}(\mathrm{Fix}(\beta\mathcal{B})) \leq G$: a free group or a surface group
  $\mathcal{B}|_{\mathrm{Fix}(\beta\mathcal{B})}$: a family of sections of β

Sketch of the proof of Theorem A: III

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  $\mathcal{B}|_{\mathrm{Fix}(\beta\mathcal{B})}$: a family of sections of β
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$$\mathrm{Fix}\mathcal{B} = \mathrm{Fix}(\mathcal{B}|_{\mathrm{Fix}(\beta\mathcal{B})}) = \mathrm{Eq}(\mathcal{B}|_{\mathrm{Fix}(\beta\mathcal{B})}).$$

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- If E is a free group, then **Bergman's Proposition** $\Longrightarrow$ $\mathrm{rkEq}(\mathcal{B}|_{\mathrm{Fix}(\beta\mathcal{B})}) \leq \mathrm{rkFix}(\beta\mathcal{B})$;
  if E is a surface group, then **Proposition of equalizers on surface groups** $\Longrightarrow$ $\mathrm{rkEq}(\mathcal{B}|_{\mathrm{Fix}(\beta\mathcal{B})}) \leq \mathrm{rkFix}(\beta\mathcal{B})$.

  Therefore, $\quad \mathrm{rkFix}\mathcal{B} \leq \mathrm{rkFix}(\beta\mathcal{B})$.

$\square$

Further Questions

For a surface group, we have

Question 2

Is the fixed subgroup of a family of (not necessarily epimorphic) endomorphisms inert?

Question 2 is an analog of Question 1 (inertia of the fixed subgroups of free groups), and can be deduced to the following

Question 3

Is every retract H of a surface group G inert in G? Namely, is

$$\mathrm{rk}(H \cap K) \leq \mathrm{rk} K$$

for any subgroup $K \leq G$?

Main references

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Thanks !