Boundary Slopes of Immersed Surfaces in Haken Manifolds

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Abstract

Abstract

We give a bound for the number of boundary slopes of orientable immersed proper π_1-injective surfaces of given genus g in an orientable Haken 3-manifold M with a torus boundary, where the bound is independent of M, and a function of g and m, the number of the Jaco-Shalen-Johannson decomposition tori of M.

Introduction

In this talk, suppose M is a compact, orientable, connected, irreducible 3-manifold with ∂M a torus.
An immersed, proper, π_1-injective surface in M with non-empty boundary is *essential* if it can not be properly homotoped into ∂M. Let c be a homotopically non-trivial simple loop in ∂M. If there is an essential surface in M such that each component of ∂F is homotopic to a multiple of c, then we call c a *boundary slope* of M.

Introduction

The question we investigate is a problem of P. Shalen [4, Question 1]: Does the set of essential surfaces with bounded genus in a simple knot complement given rise to at most finitely many boundary slopes?

Introduction

M. Baker has given examples to show that if the bounded genus assumption is dropped, then infinitely many boundary slopes can be realized [2], and U. Oertel has found examples of manifolds in which every slope is realized by the boundary of an immersed essential surfaces [3], see also [2]. On the other hand, A. Hatcher has shown that there are only finitely many boundary slopes for embedded essential surfaces, without a genus restriction [5].

Introduction

In the paper [4], J. Hass, J. Rubinstein & S. Wang have shown:

Theorem ([4])

Suppose M is a compact orientable 3-manifold with ∂M a torus, and $g \geq 0$ is a given integer.
If int(M) admits a complete hyperbolic metric of finite volume, then the number of boundary slopes of an essential immersed surface of genus at most g is bounded by a function $N(1)$ if $g \leq 1$ and $N(g) + 1$ if $g > 1$.
If M is a Haken 3-manifold, then there are only finitely many boundary slopes realized by orientable essential proper surfaces of genus at most g.

Introduction

In this talk, we give explicit bounds, independent of the Haken manifold:

Theorem (main Theorem)

Suppose M is an orientable Haken 3-manifold with ∂M a torus. Let $m \geq 1$ be the number of Jaco-Shalen-Johannson decomposition tori of M. Then, given $g \geq 0$, the number of boundary slopes of an essential orientable immersed surface of genus at most g is bounded by a function $F(g,m)$, independent of M, where

$$F(g,m) = \begin{cases} 128\pi m + 5 & g = 0 \\ 128\pi m g^3 + 4g + 2 & g \geq 1 \end{cases}.$$

Outline of the Proof of the Main Theorem

Outline of the proof of the main Theorem:
Let M be an orientable Haken 3-manifold and Γ be the Jaco-Shalen-Johannson decomposition tori of M. Let $m = \#\Gamma \geq 1$. Call each component of $\overline{M - N(\Gamma)}$ a *vertex manifold*, where $N(\Gamma)$ is a regular neighborhood of Γ. Let M^* be the vertex manifold containing ∂M, which plays a key role of the proof. Let $\{B_n,\ n = 1, 2, \ldots\}$ be the boundary slopes for essential immersed surfaces of genus at most g. Then for each B_n, there is an essential surface F_n of genus at most g such that ∂F_n has l_n components, each with slope B_n. Deform F_n so that the number of components of $F_n \cap \partial N(\Gamma)$ is a minimum.

Outline of the Proof of the Main Theorem

Let F_n^* be the union of the components of $F_n \cap M^*$ with at least one boundary component in ∂M. Let l_n^* be the number of components of ∂F_n^* in $\partial M^* - \partial M$, and let $\#F_n^*$ be the number of components of F_n^*. Let $g(F_n^*)$ denote the sum of the genera of the components of F_n^*. Then we have

Lemma (1)

$$l_n^* \leq 2(g + \#F_n^* - g(F_n^*) - 1) \leq 2(g + l_n - 1).$$

Outline of the Proof of the Main Theorem

We have two cases.

Case (1): M^* is hyperbolic.

Let $|B_n|$ denote the length of B_n. We can prove

$$|B_n| l_n \leq \sum_{c \in F_n^* \cap \partial M^*} L(c) \leq -2\pi\chi(F_n^*) \leq 2\pi(2g-2+l_n+l_n^*).$$

By Lemma (1), $l_n^* \leq 2(g+l_n-1)$, So when $l_n \geq 2$, we have

$$|B_n| \leq \begin{cases} 6\pi & g=0 \\ 2\pi(2g+1) & g \geq 1 \end{cases}.$$

Outline of the Proof of the Main Theorem

So by Lemma 2.3 in [4], $\#\{B_n,\ n = 1, 2, \ldots\} \leq N(2g+1)$, if $g \geq 1$; $\#\{B_n,\ n = 1, 2, \ldots\} \leq N(3) = 198$, if $g = 0$. So, we can easily know that the main Theorem holds for Case (1).

Outline of the Proof of the Main Theorem

Case (2): M^* is a Seifert manifold.
Let $\partial M^* = \{T_1, \ldots, T_h\}$, and $\partial M = T_h$. Let C_{nj} denote the union of all components of ∂F_n^* in T_j. For convenience, the coordinates of a closed curve $c \subset T(\mu, \lambda)$ will be denoted by (u_c, v_c).
When $l_n = 1$, B_n is homologically zero, and there is at most one such boundary slope in ∂M. So we assume $l_n \geq 2$ below. Let $B_n = (u_n, v_n)$ and let $O(M^*)$ be the Seifert orbifold of M^*. Let χ^* denote the Euler characteristic of $O(M^*)$. Since ∂M^* has at least two components, we have $\chi^* \leq -1/2$. First, we prove two Lemmas:

Outline of the Proof of the Main Theorem

Lemma (2)

$$|u_n| \leq U(g) = \begin{cases} 2 & g = 0 \\ 2g & g \geq 1 \end{cases}.$$

Lemma (3)

If $u_n \neq 0$, *then*

$$\#\{v_n, n = 1, 2, \ldots\} \leq \begin{cases} 32\pi m + 1 & g = 0 \\ 32\pi m g^2 + 1 & g \geq 1 \end{cases}.$$

Proof of Lemma (2): We only to consider $u_n \neq 0$. So, we can assume that F_n^* is horizontal relative to the Seifert fibering. Since F_n^* has l_n boundary curves in ∂M, each of which has a coordinates a non zero multiple of (u_n, v_n), we know that the projection $p : F_n^* \to O(M^*)$ is an orbifold branched covering of degree at least $l_n|u_n|$. By the estimate of the degree of p, we have

$$l_n|u_n|\chi^* \geq \chi(F_n^*) = 2\#F_n^* - 2g(F_n^*) - \#\partial F_n^* .$$

We have $\#\partial F_n^* = l_n^* + l_n$ and by Lemma (1), $2\#F_n^* - 2g(F_n^*) \geq l_n^* - 2g + 2$, so

$$l_n|u_n|\chi^* \geq -(l_n + 2g - 2).$$

Outline of the Proof of the Main Theorem

Since $\chi^* \leq -1/2$ and $l_n \geq 2$, we have

$$|u_n| \leq \frac{l_n + 2g - 2}{l_n|\chi^*|} \leq \frac{1}{|\chi^*|} + \frac{2g-2}{l_n|\chi^*|}\,.$$

So

$$|u_n| \leq U(g) := \begin{cases} 2 & g = 0 \\ 2g & g \geq 1 \end{cases},$$

i.e., Lemma (2) holds.

Outline of the Proof of the Main Theorem

Proof of Lemma (3): Let $M_i,\ i=1,2,\ldots,h-1$, be the vertex manifold of M sharing the torus T_i with M^* (it is possible that $M_i=M_j$, for some $i\neq j$). Denote the copy of T_i on M_i by T_i' and the gluing map by $g_i: T_i\to T_i'$. Let $F_{ni}^*\subset M_i\bigcup_{g_i} M^*$ be a subsurface of F_n, which is obtained from gluing F_n^* and the components of $F_n\cap M_i$ having $g_i(c)$ as boundary components for some $c=(u_c,v_c)\in C_{ni}$, by the map $g_i|_{C_{ni}}: C_{ni}\to g_i(C_{ni}),\ c\mapsto g_i(c)$.

Outline of the Proof of the Main Theorem

If M_i is a Seifert manifold, the gluing map $g_i : T_i(\mu_i, \lambda_i) \to T'_i(\mu'_i, \lambda'_i)$ is determined by a matrix $A_i = \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix}$, where $r_i \neq 0$, and $p_i s_i - q_i r_i = -1$, $p_i, q_i, r_i, s_i \in \mathbb{Z}$. Let g_{i*} be the induced map on homology, so that

$$g_{i*}(u_i\mu_i + v_i\lambda_i) = (u'_i\mu'_i + v'_i\lambda'_i)\,.$$

Then $u'_i = p_i u_i + r_i v_i$ and $v'_i = q_i u_i + s_i v_i$. So for any $c = (u_c, v_c) \in C_{ni}$, we have $u'_{g_i(c)} = p_i u_c + r_i v_c$. So

$$\sum_{c \in C_{ni}} |u'_{g_i(c)}| = \sum_{c \in C_{ni}} |p_i u_c + r_i v_c|.$$

Outline of the Proof of the Main Theorem

If M_i is hyperbolic, then we can give $T_i' \subset \partial M_i$ a Euclidean metric (possibly nonunique) induced by the hyperbolic structure of $\operatorname{int}(M_i)$ and a Euclidean coordinate system $T_i'(\mu_i', \lambda_i')$ on T_i' such that $L(c') \geq |u'_{c'}|$ and $L(c') \geq |v'_{c'}|$, $\forall c' = (u'_{c'}, v'_{c'}) \subset T_i'(\mu_i', \lambda_i')$. Like the case M_i being a Seifert manifold, we also have (If $r_i = 0$, then $s_i \neq 0$. For convenience, we still call the non-zero one r_i)

$$\sum_{c \in C_{ni}} |u'_{g_i(c)}| = \sum_{c \in C_{ni}} |p_i u_c + r_i v_c|.$$

Outline of the Proof of the Main Theorem

In the notation above, we have

Assertion

$$\sum_{c\in C_{ni}} |p_i u_c + r_i v_c| = \sum_{c\in C_{ni}} |u'_{g_i(c)}| \leq f(g, l_n),$$

where $f(g, l_n) := 2\pi\big(4g - 4 + l_n(U(g) + 2)\big)$.

Proof: If M_i is a Seifert manifold, then the proof is similar to the one of Lemma (2) and if M_i is hyperbolic, then the proof is similar to the one of Case (1).

Outline of the Proof of the Main Theorem

Recall we have assume $u_n \neq 0$. So F_n^* is horizontal in M^* and $u_c \neq 0$. So $|r_i u_c| \geq 1$, and

$$\sum_{c \in C_{ni}} \frac{|p_i u_c + r_i v_c|}{|r_i u_c|} \leq f(g, l_n)\,,$$

i.e.,

$$\sum_{c \in C_{ni}} \left|\frac{p_i}{r_i} + \frac{v_c}{u_c}\right| \leq f(g, l_n) \tag{2.1}$$

Since F_n^* is horizontal in M^*, by Lemma 2.3 in [4], we have

$$\sum_{i=1}^{h-1} \sum_{c \in C_{ni}} \frac{v_c}{u_c} = -l_n \frac{v_n}{u_n} - u \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i}\,, \tag{2.2}$$

where $u = l_n u_n$ and $\sum_{i=1}^{k} \frac{\beta_i}{\alpha_i}$ is the Euler number of the fibering of M^*.

Outline of the Proof of the Main Theorem

So, by (2.1) and (2.2), we have

$$-(h-1)f(g,l_n)-P\leq -l_n\frac{v_n}{u_n}\leq (h-1)f(g,l_n)-P\,,$$

where $P=\sum_{i=1}^{h-1}\sum_{c\in C_{ni}}\frac{p_i}{r_i}-u\sum_{i=1}^{k}\frac{\beta_i}{\alpha_i}$ is a constant. So

$$\#\{v_n;\, n=1,2,\ldots\}\leq 2|u_n(h-1)f(g,l_n)|/l_n+1.$$

Outline of the Proof of the Main Theorem

Since $h-1 \leq m$, by Lemma (2) and Assertion, we have

$$\#\{v_n; n=1,2,\ldots\} \leq \begin{cases} 32\pi m+1 & g=0 \\ 32\pi m g^2+1 & g\geq 1 \end{cases},$$

where $l_n \geq 2$. So Lemma (3) holds.

Outline of the Proof of the Main Theorem

If $u_n = 0$, then $|v_n| = 1$. So when $l_n \geq 2$, by Lemma (2) and (3), we have

$$\begin{aligned} \#\{B_n,\ n=1,2,\ldots\} &= \#\{(u_n,v_n),\ n=1,2,\ldots\} \\ &\leq 2|u_n| \times \#\{v_n,\ n=1,2,\ldots\}+1 \\ &\leq \begin{cases} 128\pi m+5 & g=0 \\ 128\pi m g^3+4g+1 & g\geq 1 \end{cases}. \end{aligned}$$

Recall that when $l_n = 1$, there is at most one boundary slope if $g \geq 1$. So

$$\#\{B_n,\ n=1,2,\ldots\} \leq \begin{cases} 128\pi m+5 & g=0 \\ 128\pi m g^3+4g+2 & g\geq 1 \end{cases},$$

i.e., the main Theorem also holds.

Question

Since (u_n, v_n) is a boundary slope, u_n and v_n must be coprime. So we can know that $F(g, m)$ may be far larger than $\#\{B_n,\ n = 1, 2, \ldots\}$ from the calculation of $\#\{B_n,\ n = 1, 2, \ldots\}$.
Since the number of boundary slopes is bounded by $N(2g+1)+1$ which is an asymptotic quadratic function of g in Case (1), I guess it is also bounded by a quadratic function of g in Case (2), but I can not give a proof.
Question: Is the number of boundary slopes bounded by a quadratic function of g?

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Thank you!