Bounds for fixed points on Seifert manifolds

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Abstract

In this talk, we consider homeomorphisms of compact connected orientable Seifert manifolds, and give some bounds involving the rank and the index of fixed point classes. One consequence is an index bound for fixed point classes. We rely on the classification of 2-orbifolds homeomorphisms and the similar bound on surfaces.

Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map.

Definition

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x).$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

Rank of a fixed point class

Definition

For a fixed point $x \in \mathbf{F}$, let

$$\mathrm{Stab}(f,x) := \{\gamma \in \pi_1(X,x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(X,x),$$

where $f_\pi : \pi_1(X,x) \to \pi_1(X,x)$ is the induced endomorphism. It is independent of the choice of $x \in F$, up to isomorphism. For a fixed point class $\mathbf{F}$ of f , define the rank to be

$$\mathrm{rank}(\mathbf{F}) := \mathrm{rank}(f,x) := \mathrm{rank}\mathrm{Stab}(f,x), \quad \forall x \in \mathbf{F}.$$

For an empty fixed point class $\mathbf{F}$, we can make it nonempty by deforming f.

Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.
Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \quad \mathrm{rank}(f_0, \mathbf{F}_0) = \mathrm{rank}(f_1, \mathbf{F}_1).$$

Characteristic of a fixed point class

From now on, unless otherwise stated, we always assume X to be a graph, a surface or a Seifert manifold, and $f : X \to X$ is a selfmap. For convenience, we define another term.

Definition

The characteristic of a fixed point class $\mathbf{F}$ is defined as

$$\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rank}(\mathbf{F}).$$

with the exception is when $\mathrm{Stab}(f, \mathbf{F}) = \pi_1(S)$ for some closed hyperbolic surface $S \subset X$, in this case

$$\mathrm{chr}(\mathbf{F}) := \chi(S) = 2 - \mathrm{rank}(\mathbf{F}).$$

Bounds for Seifert manifolds

Our main result is

Theorem (Main Theorem)

Suppose M is a compact connected orientable Seifert manifold (closed or with boundary) with hyperbolic orbifold $X(M)$, and $f: M \to M$ is a homeomorphism. Then
(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f*;*
(B)
$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq \mathcal{B},$$
where the sum is taken over all essential fixed point classes $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$, *and*

$$\mathcal{B} = \begin{cases} 4\big(3 - \mathrm{rank}\pi_1(M)\big) & M \textit{ is a closed surface } F \times S^1 \\ 4\big(2 - \mathrm{rank}\pi_1(M)\big) & \textit{others} \end{cases}.$$

Bounds for Seifert manifolds

Theorem (Jiang-Wang-Zhang)

Suppose X is either a connected finite graph or a connected compact hyperbolic surface, and $f : X \to X$ is a selfmap. Then
(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f*;*
(B) *when* X *is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X),$$

where the sum is taken over all fixed point classes $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

B. Jiang, S.D. Wang, Q. Zhang, *Bounds for fixed points and fixed subgroups on surfaces and graphs*, Alg. Geom. Topology, 11(2011), 2297–2318.

Bounds on index

As a corollary, we have

Theorem (Bounds on index)

Suppose M is a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$, and $f : M \to M$ is a homeomorphism. Then the index of each fixed point class of f has bounds. More precisely,

$$\sum_{\mathrm{ind}(\mathbf{F})+1<0} \{\mathrm{ind}(\mathbf{F}) + 1\} \geq \mathcal{B}.$$

This bound is similar to the one on graphs and surfaces. For f is orientation preserving, B. Jiang and S. Wang proved that the index of each essential fixed point class of f is ± 1.

Bounds on rank

Moreover, we can get an immature bounds on the rank of fixed subgroups.

Proposition (Bounds on rank)

Suppose $f : M \to M$ is a homeomorphism of a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$. Let $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ is the induced automorphism and $\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(X, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(X, x)$, *where x is in an essential fixed point class. Then*

$$\mathrm{rank}\mathrm{Fix}(f_\pi) < 2\mathrm{rank}\pi_1(M).$$

Remark. If M is a Seifert manifold and $\phi : \pi_1(M) \to \pi_1(M)$ is an automorphism, the Proposition dose not holds, and S. Wang give an counter example such that the rank of the fixed subgroup is infinite,

An example

The example bellow shows that the bound in the Main Theorem is sharp.

Example.
Let $M = F_2 \times S^1$, where F_2 is an orientable closed surface of genus 2 and $S^1 = \{e^{i\theta} | \theta \in (-\pi, \pi]\}$ is a circle. Give coordinates of F_2 as following.

$$F_2 = (e^{i\varphi} \times e^{i\psi} \setminus \mathrm{int}D_1) \bigcup_g (\partial D_2 \times [0,1]) \bigcup_h (e^{i\alpha} \times e^{i\beta} \setminus \mathrm{int}D_2),$$

where $\varphi, \psi, \alpha, \beta \in (-\pi, \pi]$, and $D_1 = \{e^{i\varphi} \times e^{i\psi} | \varphi, \psi \in [-\pi/4, \pi/4]\}$, $D_2 = \{e^{i\alpha} \times e^{i\beta} | \alpha, \beta \in [-\pi/4, \pi/4]\}$ are two disks, and

$$g : \partial D_1 \to \partial D_2 \times [0,1], \qquad g(e^{i\varphi}, e^{i\psi}) = (e^{i\alpha}, e^{i\beta}, 0),$$

$$h : \partial D_2 \times \{1\} \to \partial D_2, \qquad h(e^{i\alpha}, e^{i\beta}, 1) = (e^{i\alpha}, e^{i\beta}).$$

An example

Let $f = f_L \bigcup f_M \bigcup f_R : M \to M$, where

$$f_L : (e^{i\varphi} \times e^{i\psi} \setminus \mathrm{int} D_1) \times S^1 \to (e^{i\varphi} \times e^{i\psi} \setminus \mathrm{int} D_1) \times S^1,$$

$$f_L(e^{i\varphi}, e^{i\psi}, e^{i\theta}) = (e^{i\varphi}, e^{i\psi}, e^{-i\theta}),$$

$$f_R : (e^{i\alpha} \times e^{i\beta} \setminus \mathrm{int} D_2) \times S^1 \to (e^{i\alpha} \times e^{i\beta} \setminus \mathrm{int} D_2) \times S^1,$$

$$f_R(e^{i\alpha}, e^{i\beta}, e^{i\theta}) = (e^{i\alpha}, e^{i\beta}, e^{i(\alpha-\theta)}),$$

and

$$f_M : (\partial D_2 \times [0,1]) \times S^1 \to (\partial D_2 \times [0,1]) \times S^1,$$

$$f_M(e^{i\alpha}, e^{i\beta}, t, e^{i\theta}) = (e^{i\alpha}, e^{i\beta}, t, e^{i(t\alpha-\theta)}).$$

So

$$\mathrm{Fix}(f) = \mathrm{Fix}(f_L) \bigcup \mathrm{Fix}(f_M) \bigcup \mathrm{Fix}(f_R) \cong F_3,$$

an orientable closed surface of genus 3.

An example

Hence f has unique essential fixed point class $\mathbf{F} \cong F_3$,

$$\mathrm{ind}(\mathbf{F}) = \mathrm{chr}(\mathbf{F}) = -4,$$

$$\mathrm{rankFix}(f_\pi) = 6 > 5 = \mathrm{rank}\pi_1(M),$$

and

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} = 4\big(3 - \mathrm{rank}\pi_1(M)\big).$$

Outline of the proof of Main Theorem

We first consider the circle bundle over an orientable surface.

Lemma

Suppose $q : M \to F$ is an orientable circle bundle over an orientable compact surface F with $\chi(F) < 0$. If $f : M \to M$ is a fiber preserving homeomorphism that reverses the fiber orientation, then

(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f;

(B)

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 4\chi(F),$$

where the sum is taken over all essential fixed point class $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

Outline of the proof of Main Theorem

Proof.

Via a fiber preserving homotopy, we can suppose the induced map $f' : F \to F$ is in "standard form". Let C' be a component of $\mathrm{Fix}(f')$, and let $C = q^{-1}(C') \cap \mathrm{Fix}(f)$. Then we have the fact that C' is doubly covered by C and the fixed point classes of f are connected. The conclusions follow easily form the bounds on hyperbolic surfaces in [JWZ]:

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(F).$$

$\square$

Outline of the proof of Main Theorem

Now we consider Seifert manifolds.

Proposition

Suppose M is a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$, Then
(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f;
(B)
$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 4\chi(X(M))$$
where the sum is taken over all essential fixed point class $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

Sketch of proof:
Since $\mathrm{ind}(\mathbf{F})$ and $\mathrm{chr}(\mathbf{F})$ are homotopy invariants, up to a isotopy and a fiber preserving isotopy, we may assume f is fiber preserving with respect to this fibration and reverses the orientation of any fiber which contains an essential fixed point of f.

Since $X(M)$ is hyperbolic, there is a finite covering $q : X(\widetilde{M}) \to X(M)$ of orbifold such that $X(\widetilde{M})$ is an orientable surface. The pull-back of the Seifert fibration $p : M \to X(M)$ via q gives a covering manifold $q' : \widetilde{M} \to M$ with fibration $p' : \widetilde{M} \to X(\widetilde{M})$. Since $X(\widetilde{M})$ is an orientable surface, the fibration p' is an orientable circle bundle over the orientable surface $X(\widetilde{M})$. After passing to further finite covering if necessary, we may assume that q' is characteristic.

Outline of the proof of Main Theorem

Let $\tilde{f} : \widetilde{M} \to \widetilde{M}$ be a lifting of f which fixes a point A' with $q'(A') = A$. Since A is a fixed point of f, by the above assumption f reverses the orientation of the fiber containing A. As a lifting of f, the map $\tilde{f}$ reverses the orientation of the fiber containing A', and hence the orientation of all fibers because the fibers of $\widetilde{M}$ can be coherently oriented. Since the orbifold homeomorphism $\tilde{f}' : X(\widetilde{M}) \to X(\widetilde{M})$ is a lifting of the standard map f', it is also standard. So $\tilde{f}$ and f both have the FR-property. Hence, a essential fixed point class of f (resp. $\tilde{f}$) is a connected component of $\mathrm{Fix}(f)$ (resp. $\mathrm{Fix}(\tilde{f})$).

Let $\mathcal{F} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_d\}$ be all the liftings of f, and $\mathcal{F}' = \{\tilde{f}'_1, \tilde{f}'_2, \ldots, \tilde{f}'_d\}$ be all the liftings of f', where $\tilde{f}'_i$ is the induced homeomorphism on orbifold of $\tilde{f}_i$. For any fixed point class $\mathbf{F}$ of f and any fixed point class $\widetilde{\mathbf{F}}_i$ of $\tilde{f} \in \mathcal{F}$ which is a k_i-fold cover of $\mathbf{F}$, we have

$$\mathrm{chr}(\tilde{f}, \widetilde{\mathbf{F}}_i) = k_i \mathrm{chr}(f, \mathbf{F}), \mathrm{ind}(\tilde{f}, \widetilde{\mathbf{F}}_i) = k_i \mathrm{ind}(f, \mathbf{F}).$$

So equality (A) follows from $\mathrm{ind}(f, \mathbf{F}) \leq \mathrm{chr}(f, \mathbf{F})$ on circle bundle.

Outline of the proof of Main Theorem

Now we consider equality (B). By the discussion above, we have

$$\sum_{\tilde{f}\in\mathcal{F}} \sum_{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)<0} \{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)\}$$
$$= \sum k_i \sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})\}$$
$$= d \sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})\}$$

where the sum is taken over all fixed classes $\widetilde{\mathbf{F}}_i$ and all lifting $\tilde{f}$ of f with $\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)<0$, and

$$\sum_{\tilde{f}\in\mathcal{F}} \sum_{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)<0} \{\mathrm{ind}(\tilde{f},\widetilde{\mathbf{F}}_i)+\mathrm{chr}(\tilde{f},\widetilde{\mathbf{F}}_i)\}$$
$$\geq 2 \sum_{\tilde{f}'\in\mathcal{F}'} \sum_{\mathrm{ind}(\tilde{f}',\widetilde{\mathbf{F}}'_i)+\mathrm{chr}(\tilde{f}',\widetilde{\mathbf{F}}'_i)<0} \{\mathrm{ind}(\tilde{f}',\widetilde{\mathbf{F}}'_i)+\mathrm{chr}(\tilde{f}',\widetilde{\mathbf{F}}'_i)\}$$
$$\geq 4d\chi(X(M)).$$

Outline of the proof of Main Theorem

Hence

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 4\chi(X(M)),$$

where the sum runs over all essential fixed point class $\mathbf{F}$ with $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$. So equality (B) holds.

Outline of the proof of Main Theorem

The proof is finished by estimating rank$\pi_1(M)$ and $X(M)$ from the following theorem.

Theorem (Zieschang, Peczynski, Rosenberger)

Let G be a Fuchsian group with orbifold $\mathcal{O} = F(\alpha_1, \ldots, \alpha_k)$, where the closed surface F denotes the underlying space of $\mathcal{O}$ and $\alpha_i > 1$ denotes the order of the cone points. Then the following hold.

(1) *If $k = 0$ then* $\mathrm{rank}(G) = -\chi(F) + 2$;

(2) *If $\chi(F) = 2$, that is, $F = S^2$, $k \geq 4$ is even, α_i is odd for some i, and $\alpha_j = 2$ for $j \neq i$, then* $\mathrm{rank}(G) = -\chi(F) + k = k - 2$;

(3) *In all other cases* $\mathrm{rank}(G) = -\chi(F) + k + 1$.

Question

Suppose $f : M \to M$ is an orientation reversing homeomorphism of a compact orientable Seifert manifold M with hyperbolic orbiford $X(M)$, and

$$f_\pi : \pi_1(X, x) \to \pi_1(X, x)$$

is the induced automorphism. Dose the rank of $\mathrm{Fix}(f_\pi)$ have a bound ?

谢　谢！