Fixed subgroups in low-dimensional manifold groups

张 强

西安交通大学

西交利物浦大学“流形与群研讨会”
苏州 · 西交利物浦大学，2016.1.24

Hanna Neumann Conjecture

For a **f.g.** (finitely generated) group G, let $\mathrm{rk}(G)$ denote the rank (i.e., the minimal number of generators) of G. There are lots of research on the intersection of subgroups in the literature.

Hanna Neumann Conjecture

For a **f.g.** (finitely generated) group G, let $\mathrm{rk}(G)$ denote the rank (i.e., the minimal number of generators) of G. There are lots of research on the intersection of subgroups in the literature.

For any **free group**, Mineyev and Friedman proved the following theorem conjectured by [Neumann, 1957] independently. Dicks gave two versions of simplified proofs. A. Jaikin gave another new proof recently.

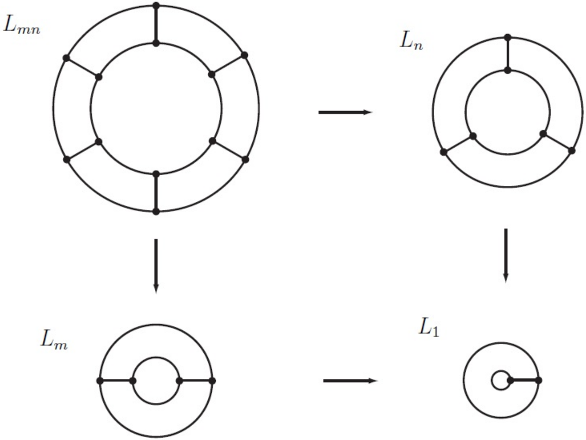
Let $\overline{\mathrm{rk}} := \max\{0, \mathrm{rk}(G) - 1\}$.

Theorem (Mineyev, Friedman, 2011)

Let F_n be a f.g. **free group**, *and H, K any two f.g. subgroups of F_n. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

H. N. Conjecture: a special case



m and n are relatively prime

Intersection of subgroups: surface groups

Let G be a surface group, namely, $G \cong \pi_1(S)$ for a closed (possibly non-orientable) surface S with $\chi(S) < 0$.

Intersection of subgroups: surface groups

Let G be a surface group, namely, $G \cong \pi_1(S)$ for a closed (possibly non-orientable) surface S with $\chi(S) < 0$.

Theorem (Soma, 1991)

Let G be a f.g. **surface group**, *and H, K any two f.g. subgroups of G. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq 1161 \cdot \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

Intersection of subgroups: surface groups

Let G be a surface group, namely, $G \cong \pi_1(S)$ for a closed (possibly non-orientable) surface S with $\chi(S) < 0$.

Theorem (Soma, 1991)

Let G be a f.g. **surface group**, *and H, K any two f.g. subgroups of G. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq 1161 \cdot \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

Question 1

For any subgroups H, K of a **surface group**, does

$$\overline{\mathrm{rk}}(H \cap K) \leq \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K) \ ?$$

Mineyev claimed that the answer of the question above is affirmative.

Intersection of subgroups: one-relator groups

Let $G = \langle X | r \rangle$ be a **one-relator group** where r is a cyclically reduced word in the free group on the generating set X.

A subset $Y \subset X$ is called a *Magnus subset* if Y omits a generator which appears in the relator r. A subgroup H of G is called a *Magnus subgroup* if $H = \langle Y \rangle$ for some Magnus subset Y of X, and hence by the Magnus Freiheitssatz, H is free of rank $|Y|$.

Intersection of subgroups: one-relator groups

Let $G = \langle X | r \rangle$ be a **one-relator group** where r is a cyclically reduced word in the free group on the generating set X.

A subset $Y \subset X$ is called a *Magnus subset* if Y omits a generator which appears in the relator r. A subgroup H of G is called a *Magnus subgroup* if $H = \langle Y \rangle$ for some Magnus subset Y of X, and hence by the Magnus Freiheitssatz, H is free of rank $|Y|$.

Theorem (Collins, 2004)

The intersection $\langle Y \rangle \cap \langle Z \rangle$ of two Magnus subgroups of the one-relator group G is either $\langle Y \cap Z \rangle$ or the free product of $\langle Y \cap Z \rangle$ with an infinite cyclic group and thus of rank $|Y \cap Z| + 1$.

Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Fixed subgroups: free groups

Theorem (Dyer-Scott, 1975)

Let $\phi \in \mathrm{Aut}(F_n)$ be an automorphism with **finite order** *of F_n. Then*

$$\mathrm{rkFix}\phi \leq \mathrm{rk}F_n.$$

Fixed subgroups: free groups

Theorem (Dyer-Scott, 1975)

Let $\phi \in \mathrm{Aut}(F_n)$ be an automorphism with **finite order** *of F_n. Then*

$$\mathrm{rkFix}\phi \leq \mathrm{rk}F_n.$$

Theorem (Bestvina-Handel, 1992)

Let ϕ be an **automorphism** *of F_n. Then $\mathrm{rkFix}\phi \leq \mathrm{rk}F_n$.*

Other alternative proofs (Sela, Paulin, Gaboriau-Jaeger-Levitt-Lustig,...)

Fixed subgroups: free groups

Theorem (Dicks-Ventura, 1996)

Let $\mathcal{B}$ be a family of **injective** *endomorphisms of F_n, then*

$$\mathrm{rkFix}\mathcal{B} \leq \mathrm{rk}F_n.$$

Theorem (Dicks-Ventura, 1996)

Let $\mathcal{B}$ be a family of **injective** *endomorphisms of F_n, then*

$$\mathrm{rk}\mathrm{Fix}\mathcal{B} \leq \mathrm{rk}F_n.$$

They also showed that $\mathrm{Fix}\mathcal{B}$ is **inert** in F_n.

Definition

A subgroup A is inert in G if for every subgroup $B \leqslant G$,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk}B.$$

Fixed subgroups & fixed points on graphs & surfaces

Let $\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ (or $2 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ for some cases).

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite graph or a connected compact hyperbolic surface, and $f : X \to X$ is a **selfmap**. *Then*

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f*;*
2. *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X).$$

Fixed subgroups & fixed points on graphs & surfaces

Let $\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ (or $2 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ for some cases).

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite graph or a connected compact hyperbolic surface, and $f : X \to X$ is a **selfmap**. *Then*

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f*;*
2. *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X).$$

Corollary

Bestvina-Handel results for free groups (Scott Conjecture).

Fixed subgroups: surface groups

Let G be a f.g. **surface group**.

Theorem (Jiang-Wang-Z., 2011)

Let ϕ be an **endomorphism** *of G. Then*

1. $\mathrm{rkFix}\phi \leq \mathrm{rk}G$, *with equality if and only if $\phi = id$;*
2. $\mathrm{rkFix}\phi \leq \frac{1}{2}\mathrm{rk}G$ *if ϕ is not epimorphic.*

[Nielsen,1929]: For any closed **orientable** surface S and **automorphism** ϕ of $\pi_1(S)$, $\mathrm{rkFix}\phi \leq \mathrm{rk}G$.

Let G be a f.g. **surface group**.

Theorem (Jiang-Wang-Z., 2011)

Let ϕ be an **endomorphism** *of G. Then*

1. $\mathrm{rkFix}\phi \leq \mathrm{rk}G$, *with equality if and only if $\phi = id$;*
2. $\mathrm{rkFix}\phi \leq \frac{1}{2}\mathrm{rk}G$ *if ϕ is not epimorphic.*

[Nielsen,1929]: For any closed **orientable** surface S and **automorphism** ϕ of $\pi_1(S)$, $\mathrm{rkFix}\phi \leq \mathrm{rk}G$.

Theorem (Wu-Z.,2014)

Let $\mathcal{B}$ be a family of **endomorphisms** *of G. Then*

1. $\mathrm{rkFix}\mathcal{B} \leq \mathrm{rk}G$, *with equality if and only if $\mathcal{B} = \{id\}$;*
2. $\mathrm{rkFix}\mathcal{B} \leq \frac{1}{2}\mathrm{rk}G$, *if $\mathcal{B}$ contains a non-epimorphic endomorphism*

Geometric subgroups of surface groups

A connected subsurface F of a connected surface S is called incompressible if the natural homomorphism $\pi_1(F) \to \pi_1(S)$ induced by the inclusion $F \hookrightarrow S$ is injective. We can think of $\pi_1(F)$ as a subgroup of $\pi_1(S)$. Subgroups which arise in this way are called geometric.

Geometric subgroups of surface groups

A connected subsurface F of a connected surface S is called incompressible if the natural homomorphism $\pi_1(F) \to \pi_1(S)$ induced by the inclusion $F \hookrightarrow S$ is injective. We can think of $\pi_1(F)$ as a subgroup of $\pi_1(S)$. Subgroups which arise in this way are called geometric.

Geometric subgroups: inertia

Theorem (Nielsen, Jaco-Shalen)

The fixed subgroup of an **automorphism** *of a surface group is either cyclic or geometric.*

Geometric subgroups: inertia

Theorem (Nielsen, Jaco-Shalen)

The fixed subgroup of an **automorphism** *of a surface group is either cyclic or geometric.*

For geometric subgroups of a surface group, we prove that

Theorem* (Wu-Z., 2014)

If A is a geometric subgroup of a surface group G, then A is inert in G, i.e., for any subgroup B of G, we have $\mathrm{rk}(A \cap B) \leq \mathrm{rk} B$.

Corollary (Wu-Z., 2014)

The fixed subgroup of any family of epimorphisms of a surface group G is inert in G.

Sketch of the proof of Theorem*

Using covering theory:

- A is a geometric subgroup of $G \iff \exists$ incomp. subsurface F of a closed surface S, s.t. $A = \pi_1(F,*) \leq \pi_1(S,*) = G$.

Sketch of the proof of Theorem*

Using covering theory:

- A is a geometric subgroup of $G \iff \exists$ incomp. subsurface F of a closed surface S, s.t. $A = \pi_1(F, *) \leq \pi_1(S, *) = G$.
- We have two maps: the inclusion $i : F \hookrightarrow S$, and the covering $k : K \to S$ associated to B (i,e., $k_*(\pi_1(K, \tilde{*})) = B$).

Sketch of the proof of Theorem*

Using covering theory:

- A is a geometric subgroup of $G \iff \exists$ incomp. subsurface F of a closed surface S, s.t. $A = \pi_1(F, *) \leq \pi_1(S, *) = G$.
- We have two maps: the inclusion $i : F \hookrightarrow S$, and the covering $k : K \to S$ associated to B (i,e., $k_*(\pi_1(K, \tilde{*})) = B$).
- Consider the commutative diagram

$$\begin{array}{ccc} F_0 \subset \tilde{F} & \xrightarrow{i'} & K \\ p \downarrow & & k \downarrow \\ F & \xrightarrow{i} & S \end{array}$$

where $p : \tilde{F} \to F$ is the **pull back** map of k via i, and F_0 is the component of $\tilde{F}$ containing the base point.

Sketch of the proof of Theorem*

Using covering theory:

- A is a geometric subgroup of $G \iff \exists$ incomp. subsurface F of a closed surface S, s.t. $A = \pi_1(F, *) \leq \pi_1(S, *) = G$.
- We have two maps: the inclusion $i : F \hookrightarrow S$, and the covering $k : K \to S$ associated to B (i,e., $k_*(\pi_1(K, \tilde{*})) = B$).
- Consider the commutative diagram

$$\begin{array}{ccc} F_0 \subset \tilde{F} & \xrightarrow{i'} & K \\ p \Big\downarrow & & k \Big\downarrow \\ F & \xrightarrow{i} & S \end{array}$$

where $p : \tilde{F} \to F$ is the **pull back** map of k via i, and F_0 is the component of $\tilde{F}$ containing the base point.

- $i_* p_*(\pi_1(F_0)) = A \cap B \quad \Longrightarrow \quad \mathrm{rk}(A \cap B) = \mathrm{rk}\pi_1(F_0) \leq \mathrm{rk}B.$

Equalizers and Retracts: definitions

- Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A section of ϕ is a homomorphism $\sigma : H \to G$ such that
$$\phi\sigma = id : H \to H.$$
For any family $\mathcal{B}$ of sections of ϕ, the equalizer of $\mathcal{B}$ is
$$\mathrm{Eq}(\mathcal{B}) := \{h \in H | \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\} \leq H.$$

Equalizers and Retracts: definitions

- Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A section of ϕ is a homomorphism $\sigma : H \to G$ such that
$$\phi\sigma = id : H \to H.$$
For any family $\mathcal{B}$ of sections of ϕ, the equalizer of $\mathcal{B}$ is
$$\mathrm{Eq}(\mathcal{B}) := \{h \in H | \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\} \leq H.$$

- $id \in \mathcal{B} \Longrightarrow \mathrm{Eq}(\mathcal{B}) = \mathrm{Fix}(\mathcal{B})$.

Equalizers and Retracts: definitions

- Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A section of ϕ is a homomorphism $\sigma : H \to G$ such that
$$\phi\sigma = id : H \to H.$$
For any family $\mathcal{B}$ of sections of ϕ, the equalizer of $\mathcal{B}$ is
$$\mathrm{Eq}(\mathcal{B}) := \{h \in H | \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\} \leq H.$$

- $id \in \mathcal{B} \Longrightarrow \mathrm{Eq}(\mathcal{B}) = \mathrm{Fix}(\mathcal{B})$.
- Suppose H is a subgroup of a group G. If there is a homomorphism $\pi : G \to G$ such that $\pi(G) \leq H$ and
$$\pi|_H = id : H \to H,$$
we say that H is a retract of G. If $H \neq G$, it is called a proper retract.

Equalizers and Retracts: results

We have the following relation between equalizers and retracts:

Lemma

Let G, H be two groups, and $\phi : G \to H$ an epimorphism. If $\mathcal{B}$ is a family of sections of ϕ, then for any section $\sigma \in \mathcal{B}$, $\sigma(H)$ is a retract of G, and

$$\sigma|_{\mathrm{Eq}(\mathcal{B})} : \mathrm{Eq}(\mathcal{B}) \to \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$$

is an isomorphism.

Equalizers and Retracts: results

We have the following relation between equalizers and retracts:

Lemma

Let G, H be two groups, and $\phi : G \to H$ an epimorphism. If $\mathcal{B}$ is a family of sections of ϕ, then for any section $\sigma \in \mathcal{B}$, $\sigma(H)$ is a retract of G, and

$$\sigma|_{\mathrm{Eq}(\mathcal{B})} : \mathrm{Eq}(\mathcal{B}) \to \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$$

is an isomorphism.

For **free groups**, Bergman showed

Proposition (Bergman, 1999)

1. *Any intersection of retracts of a f.g. free group is also a retract;*
2. *If $\phi : G \to H$ is an epimorphism of free groups with H f.g., then the equalizer of any family of sections of ϕ is a free factor in H.*

Retracts on surface groups

For a **surface group** G, we have

Proposition (Wu-Z., 2014)

1. *Any proper retract of* G *is free of rank* $\leq \frac{1}{2}\mathrm{rk}G$.

Retracts on surface groups

For a **surface group** G, we have

Proposition (Wu-Z., 2014)

1. *Any proper retract of G is free of rank $\leq \frac{1}{2}\mathrm{rk}G$.*
2. *If H_1, H_2 are two proper retracts of G, and $H = \langle H_1, H_2\rangle \leq G$, the subgroup generated by H_1 and H_2, then*
   (1) *If $H < G$, then $H_1 \cap H_2$ is a retract of both H_1 and H_2, and*
   $$\mathrm{rk}(H_1 \cap H_2) \leq \min\{\mathrm{rk}H_1, \mathrm{rk}H_2\}.$$
   (2) *If $H = G$, then $H_1 \cap H_2$ is cyclic (possibly trivial).*

Retracts on surface groups

For a **surface group** G, we have

Proposition (Wu-Z., 2014)

1. *Any proper retract of G is free of rank $\leq \frac{1}{2}\mathrm{rk}G$.*
2. *If H_1, H_2 are two proper retracts of G, and $H = \langle H_1, H_2 \rangle \leq G$, the subgroup generated by H_1 and H_2, then*
   (1) *If $H < G$, then $H_1 \cap H_2$ is a retract of both H_1 and H_2, and*
   $$\mathrm{rk}(H_1 \cap H_2) \leq \min\{\mathrm{rk}H_1, \mathrm{rk}H_2\}.$$
   (2) *If $H = G$, then $H_1 \cap H_2$ is cyclic (possibly trivial).*
3. *If $\mathcal{R}$ is a family retracts of G, then*
   $$\mathrm{rk}(\bigcap_{H \in \mathcal{R}} H) \leq \min\{\mathrm{rk}H | H \in \mathcal{R}\} \leq \begin{cases} \mathrm{rk}G, & \mathcal{R} = \{G\} \\ \frac{1}{2}\mathrm{rk}G, & \mathcal{R} \neq \{G\} \end{cases}.$$

Retracts: further questions

Question 2

Is every retract H of a surface/free group G inert in G? Namely, is

$$\mathrm{rk}(H \cap K) \leq \mathrm{rk}(K)$$

for any subgroup $K \leq G$?

Fixed subgroups & fixed points on Seifert manifolds

M: a comp. orient. **Seifert 3-manifold** with hyperbolic orbifold,

Theorem (Z., 2012)

Suppose $f : M \to M$ is a homeomorphism. Then

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f;
2. $\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0}\{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})\} \geq \mathcal{B}$,
   where $\mathcal{B} = 4\big(2 - \mathrm{rk}\pi_1(M)\big)$.

Fixed subgroups & fixed points on Seifert manifolds

M: a comp. orient. **Seifert 3-manifold** with hyperbolic orbifold,

Theorem (Z., 2012)

Suppose $f : M \to M$ is a homeomorphism. Then

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f*;*
2. $\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0}\{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq \mathcal{B}$,
   where $\mathcal{B} = 4\big(2 - \mathrm{rk}\pi_1(M)\big)$.

Corollary (Z., 2012)

Let $f_\pi : \pi_1(M, x) \to \pi_1(M, x)$ is the induced automorphism and

$$\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(M, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(M, x),$$

where x is in an **essential fixed point class**. *Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

Fixed subgroups: Seifert manifold groups

Theorem (Z., 2013)

Suppose $f_\pi : \pi_1(M) \to \pi_1(M)$ *is an automorphism induced by an* **orientation-reversing** *homeomorphism* $f : M \to M$. *Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

Fixed subgroups: Seifert manifold groups

Theorem (Z., 2013)

Suppose $f_\pi : \pi_1(M) \to \pi_1(M)$ *is an automorphism induced by an* **orientation-reversing** *homeomorphism* $f : M \to M$*. Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

However, an analogue as above does **not hold** for a generic automorphism of Seifert manifold groups.

Example

Let $G = F_2 \times \mathbb{Z} = \langle a, b\rangle \times \langle t \rangle$ and

$$\phi \in \mathrm{Aut}(G) : a \mapsto at, b \mapsto b, t \mapsto t.$$

Then $\mathrm{Fix}\phi = \langle t, a^{-m}ba^{m} | m \in \mathbb{Z}\rangle$.

Fixed subgroups: hyperbolic 3-manifold groups

Theorem (Lin-Wang, 2012)

Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a compact orientable **hyperbolic 3-manifold** *with finite volume. Then*

$$\mathrm{rkFix}\phi < 2\mathrm{rk}G.$$

Fixed subgroups: hyperbolic 3-manifold groups

Theorem (Lin-Wang, 2012)

Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a compact orientable **hyperbolic 3-manifold** *with finite volume. Then*

$$\mathrm{rkFix}\phi < 2\mathrm{rk}G.$$

"$<$" is sharp. They also proved

Theorem (Lin-Wang, 2012)

There exists a sequence of automorphism $\phi_n : \pi_1(M_n) \to \pi_1(M_n)$ of closed hyperbolic 3-manifolds M_n such that $\mathrm{Fix}\phi_n$ is the group of a closed surface and

$$\frac{\mathrm{rkFix}\phi_n}{\mathrm{rk}\pi_1(M_n)} > 2 - \varepsilon \quad \textit{as } n \to +\infty$$

for any $\varepsilon > 0$.

Inert, compressed and BH

Definition

A subgroup A is inert in G if for every subgroup $B \leqslant G$,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk} B.$$

Definition

A subgroup A is compressed in G if for every subgroup $A \leqslant B \leqslant G$,

$$\mathrm{rk} A \leq \mathrm{rk} B.$$

Definition (For this talk)

A subgroup A is BH in G if $\mathrm{rk} A \leq \mathrm{rk} G$.

Remark

Inert $\Longrightarrow$ Compressed $\Longrightarrow$ BH.

Fixed subgroups of endomorphisms

For any family $B \subseteq \mathrm{End}(G)$,

Theorem (Bergman, 1999)

$\mathrm{Fix}\mathcal{B}$ *is* **BH** *in* F_n.

Question 3 (Bergman, 1999)

Is $\mathrm{Fix}\mathcal{B}$ of F_n **inert**?

Fixed subgroups of endomorphisms

For any family $B \subseteq \mathrm{End}(G)$,

Theorem (Bergman, 1999)

$\mathrm{Fix}\mathcal{B}$ *is* **BH** *in* F_n.

Question 3 (Bergman, 1999)

Is $\mathrm{Fix}\mathcal{B}$ of F_n **inert**?

Theorem (Martino-Ventura, 2004)

$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in* F_n.

Fixed subgroups of endomorphisms

For any family $B \subseteq \mathrm{End}(G)$,

Theorem (Bergman, 1999)

$\mathrm{Fix}\mathcal{B}$ *is* **BH** *in* F_n.

Question 3 (Bergman, 1999)

Is $\mathrm{Fix}\mathcal{B}$ of F_n **inert**?

Theorem (Martino-Ventura, 2004)

$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in* F_n.

Theorem (Z.-Ventura-Wu, 2015)

$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in any* **surface group**.

Fixed subgroups in product groups: most are BH

Let $G = G_1 \times G_2 \times \cdots \times G_n$, each G_i is a f.g. free group or $\pi_1(S)$ for a closed surface S (maybe $\mathbb{R}P^2, 2\mathbb{R}P^2$ or a torus). We call it a product group.

Fixed subgroups in product groups: most are BH

Let $G = G_1 \times G_2 \times \cdots \times G_n$, each G_i is a f.g. free group or $\pi_1(S)$ for a closed surface S (maybe $\mathbb{R}P^2, 2\mathbb{R}P^2$ or a torus). We call it a product group.

Theorem A (Z.-Ventura-Wu, 2015)

$\mathrm{rkFix}\phi \leq \mathrm{rk}G$ for every $\phi \in \mathrm{Aut}(G)$
$\Longleftrightarrow$ All G_i are of the same type (Euclidean or hyperbolic).

Euclidean type: $\mathbb{Z}$, $\pi_1(S)$ for $\chi(S) \geq 0$.
Hyperbolic type: F_n $(n > 1)$, $\pi_1(S)$ for $\chi(S) < 0$.

Fixed subgroups in product groups: most are BH

Let $G = G_1 \times G_2 \times \cdots \times G_n$, each G_i is a f.g. free group or $\pi_1(S)$ for a closed surface S (maybe $\mathbb{R}P^2, 2\mathbb{R}P^2$ or a torus). We call it a product group.

Theorem A (Z.-Ventura-Wu, 2015)

$\mathrm{rkFix}\phi \leq \mathrm{rk}G$ for every $\phi \in \mathrm{Aut}(G)$
$\Longleftrightarrow$ All G_i are of the same type (Euclidean or hyperbolic).

Euclidean type: $\mathbb{Z}$, $\pi_1(S)$ for $\chi(S) \geq 0$.
Hyperbolic type: F_n $(n > 1)$, $\pi_1(S)$ for $\chi(S) < 0$.

Example (NOT satisfying the conditions of Theorem A)

Let $G = F_2 \times \mathbb{Z} = \langle a, b\rangle \times \langle t\rangle$ and

$$\phi \in \mathrm{Aut}(G) : a \mapsto at, b \mapsto b, t \mapsto t.$$

Then $\mathrm{Fix}\phi = \langle t, a^{-m}ba^m | m \in \mathbb{Z}\rangle$.

Fixed subgroups in product groups: few are compressed

Theorem B (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **compressed** in G for every $\phi \in \mathrm{Aut}(G)$, then G must be of one of the following forms:

($euc1$) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or
($euc2$) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or
($euc3$) $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for some $p \geqslant 1$; or
($euc4$) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$; or

($hyp1$) $G = F_r \times NS_3^\ell$ for some $r \geqslant 2$, $\ell \geqslant 0$; or
($hyp2$) $G = S_g \times NS_3^\ell$ for some $g \geqslant 2$, $\ell \geqslant 0$; or
($hyp3$) $G = NS_k \times NS_3^\ell$ for some $k \geqslant 3$, $\ell \geqslant 0$.

Fixed subgroups in product groups: few are compressed

Theorem B (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **compressed** in G for every $\phi \in \mathrm{Aut}(G)$, then G must be of one of the following forms:

(*euc1*) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or
(*euc2*) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or
(*euc3*) $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for some $p \geqslant 1$; or
(*euc4*) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$; or

(*hyp1*) $G = F_r \times NS_3^\ell$ for some $r \geqslant 2$, $\ell \geqslant 0$; or
(*hyp2*) $G = S_g \times NS_3^\ell$ for some $g \geqslant 2$, $\ell \geqslant 0$; or
(*hyp3*) $G = NS_k \times NS_3^\ell$ for some $k \geqslant 3$, $\ell \geqslant 0$.

Question 4

Is the implication in Theorem B an equivalence?

Examples: fixed subgroups NOT compressed

Example 1:

Let $G = F_2 \times F_2 = \langle t, u\rangle \times \langle a, b\rangle$, and

$$\phi \in \mathrm{Aut}(G) : t \mapsto t, u \mapsto tu, a \mapsto a, b \mapsto ab.$$

Then

$$\mathrm{Fix}\phi = \langle t, u^{-1}tu, a, b^{-1}ab\rangle \leqslant \langle t, bu, a\rangle.$$

Examples: fixed subgroups NOT compressed

Example 1:

Let $G = F_2 \times F_2 = \langle t, u \rangle \times \langle a, b \rangle$, and

$$\phi \in \mathrm{Aut}(G) : t \mapsto t, u \mapsto tu, a \mapsto a, b \mapsto ab.$$

Then

$$\mathrm{Fix}\phi = \langle t, u^{-1}tu, a, b^{-1}ab \rangle \leqslant \langle t, bu, a \rangle.$$

Example 2:

Let $G = F_2 \times NS_4 = \langle t, u \rangle \times \langle a, b, c, d | aba^{-1}bcdc^{-1}d \rangle$, and

$$\phi \in \mathrm{Aut}(G) : t \mapsto t, u \mapsto tu, a \mapsto ab, b \mapsto b, c \mapsto cd, d \mapsto d.$$

Then

$$\mathrm{Fix}\phi = \langle t, u^{-1}tu, b, aba^{-1}, d, cdc^{-1} \rangle = \langle t, u^{-1}tu \rangle \times \langle b, aba^{-1}, d \rangle \cong F_5$$

but $\qquad \mathrm{Fix}\phi \leqslant \langle t, au, b, d \rangle \cong F_4.$

Examples: fixed subgroups NOT compressed

Example 3:

Let $G = NS_2 \times NS_2 \times \mathbb{Z}_2 = \langle a, b | aba^{-1}b \rangle \times \langle c, d | cdc^{-1}d \rangle \times \langle e | e^2 \rangle$,

$$\phi \in \mathrm{Aut}(G) : a \mapsto a, b \mapsto be, c \mapsto cd, d \mapsto d, e \mapsto e.$$

Then

$$\mathrm{Fix}\phi = \langle a, b^2, c^2, d, e \rangle \leqslant \langle a, bc, d, e \rangle$$

since $c^2 = a \cdot bc \cdot a^{-1} \cdot bc$ and $b^2 = bc \cdot bc \cdot c^{-2}$.

Examples: fixed subgroups NOT inert

Example 4:

Let $G = F_2 \times NS_3 = \langle t, u\rangle \times \langle a, b, c | aba^{-1}bc^2\rangle$, and

$$\phi \in \mathrm{Aut}(G) : t \mapsto t, u \mapsto u, a \mapsto ab, b \mapsto b, c \mapsto c.$$

Then

$$\mathrm{Fix}\phi = \langle t, u, aba^{-1}, b, c\rangle = \langle t, u\rangle \times \langle b, c\rangle.$$

Let $K = \langle at, u\rangle$. Then

$$\mathrm{Fix}\phi \cap K = \langle t^{-m}ut^m | m \in \mathbb{Z}\rangle$$

is infinite generated.

Fixed subgroups in product groups: less are inert

Theorem C (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **inert** in G for every $\phi \in \mathrm{Aut}(G)$, then G is of one of the forms (euc1), or (euc2), or (euc3), or (euc4), or

- $G = F_r$ for some $r \geqslant 2$; or
- $G = \pi_1(S)$ for some closed surface $\chi(S) < 0$

Open problems

Inertia Conjecture (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, the following are equivalent:

1. Every $\phi \in \mathrm{End}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
2. Every $\phi \in \mathrm{Aut}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
3. G is of the form (euc1), or (euc2), or (euc3), or (euc4), or a free group F_n for $n \geq 2$, or a surface group $\pi_1(S)$ for $\chi(S) < 0$.

Open problems

Inertia Conjecture (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, the following are equivalent:

1. Every $\phi \in \mathrm{End}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
2. Every $\phi \in \mathrm{Aut}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
3. G is of the form (euc1), or (euc2), or (euc3), or (euc4), or a free group F_n for $n \geq 2$, or a surface group $\pi_1(S)$ for $\chi(S) < 0$.

Remark

$$(1) \xrightarrow{trivial} (2) \xrightarrow{Theorem \quad C} (3) \xrightarrow{???} (1)$$

$$\{(3) - (euc4)\} \xrightarrow[Wu-Z.,2014]{Dicks-Ventura,1996} (2)$$

Main references I

Z.-Ventura-Wu, *Fixed subgroups are compressed in surface groups*, Int. J. Algebra Comput., 25(5) (2015), 865-887.

Z., *The fixed subgroups of homeomorphisms of Seifert manifolds*, Acta Math. Sin. (Engl. Ser.), 31(5) (2015), 797-810.

Wu-Z., *The group fixed by a family of endomorphisms of a surface group*, J. Algebra 417(2014), 412 – 432.

Lin-Wang, *Fixed subgroups of automorphisms of hyperbolic 3-manifold groups*, Topology Appl., 173 (2014), 209–218.

Friedman, *Sheaves on Graphs, Their Homological Invariants, and a Proof of the Hanna Neumann Conjecture: with an Appendix by Warren Dicks*, Mero. AMS, 2014.

Z., *Bounds for fixed points on Seifert manifolds*, Topology Appl., 159(15)(2012), 3263-3273.

Mineyev, *Submultiplicativity and the Hanna Neumann conjecture*, Ann. of Math., 175 (2012) 393-414.

Main references II

Jiang-Wang-Z., *Bounds for fixed points and fixed subgroups on surfaces and graphs*, Alge. Geom. Topol. 11(2011), 2297-2318.

Collins, *Intersections of Magnus subgroups of one-relator groups*, in: T.W. M ü ller (Ed.), Groups: Topological, Combinatorial and Arithmetic Aspects, in: London Math. Soc. Lecture Note Ser., vol.311, 2004, pp.255 – 296.

Martino-Ventura, *Fixed subgroups are compressed in free groups*, Commun. Algebra 32(10) (2004), 3921 – 3935.

Bergman, *Supports of derivations, free factorizations and ranks of fixed subgroups in free groups*, Trans. AMS. 351 (1999), 1531-1550.

Dicks-Ventura, *The group fixed by a family of injective endomorphisms of a free group*, Contemporary Mathematics vol. 195, (1996)

Bestvina-Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. 135 (1992), 1-51.

Soma, *Intersection of finitely generated surface groups II*, Bull. Kyushu Inst. Technol. (Math. Natur. Sci.) 38 (1991), 13 – 22.

谢 谢！