Bounds for fixed points on hyperbolic manifolds

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Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

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The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

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Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1).$$

Commutation invariance

Suppose $\phi : X \to Y$ and $\psi : Y \to X$ are maps. Then $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are said to differ by a commutation. The map ϕ sets up a natural one-one correspondence

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Under the correspondence via commutation,

$$\mathrm{ind}(\psi \circ \phi; \mathbf{F}_X) = \mathrm{ind}(\phi \circ \psi; \mathbf{F}_Y).$$

Nielsen number & Lefschetz number

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- A fixed point class $\mathbf{F}$ of f is essential if $\mathrm{ind}(f, \mathbf{F}) \neq 0$.

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$$L(f) := \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X;\mathbb{Q}) \to H_q(X;\mathbb{Q})).$$

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Lefschetz Fixed Point Theorem

$$\sum_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathrm{ind}(f, \mathbf{F}) = \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

Bounds for graphs and surfaces

Theorem (Jiang, 1998, Jiang-Wang-Z., 2011)

Suppose X is either a connected **finite graph** *or a connected compact* **hyperbolic surface**, *and $f : X \to X$ is a* **selfmap**. *Then*

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When f is a self-**homeomorphism** of a hyperbolic surface, the same bound was given by [Jiang-Guo, 1993].

Bounded Index Property: 3-manifolds

Theorem (Jiang-Wang, 1992)

Suppose a closed **aspherical 3-manifold** *M is finitely covered by an orientable 3-manifold which is either a Seifert manifold, or a hyperbolic 3-manifold, or admits a non-trivial JSJ-decomposition. Let $f : M \to M$ is a homeomorphism. Then*

1. $\mathrm{ind}(\mathbf{F}) \leq 1, \forall \mathbf{F} \in \mathrm{Fpc}(f)$, *hence* $L(f) \leq N(f)$;

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$$\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}, \quad \forall\ \mathbf{F} \in \mathrm{Fpc}(f),$$

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$$\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}, \quad \forall\ \mathbf{F} \in \mathrm{Fpc}(f),$$
hence $|L(f)| \leq N(f)$.
3. $\forall\ n > 3, \exists f$ *on a closed aspherical n-manifold such that*
$$L(f) > N(f).$$

Bounds for Seifert manifolds

Theorem (Z., 2012)

Suppose M is a compact connected orientable **Seifert manifold** *(closed or with boundary) with hyperbolic orbifold $X(M)$, and $f : M \to M$ is a homeomorphism. Then*

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2. $$\sum_{\mathrm{ind}(\mathbf{F})<-1} \{\mathrm{ind}(\mathbf{F})+1\} \geq \mathcal{B},$$
   where
   $$\mathcal{B} = \begin{cases} 4\big(3-\mathrm{rk}\pi_1(M)\big) & M \text{ is a closed surface } F\times S^1 \\ 4\big(2-\mathrm{rk}\pi_1(M)\big) & \text{others} \end{cases}.$$

The bound above is analogous to the one on graphs and surfaces.
For f **orient.-preserving**, [Jiang-Wang, 1992]: $\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}$.

Hyperbolic n-manifolds

- By a hyperbolic n-manifold ($n \geq 2$) we mean a quotient space
$$M = \mathbb{H}^n/\Gamma,$$
where $\mathbb{H}^n$ is the hyperbolic n-space, that is, the connected, simply connected Riemanian manifold of constant curvature -1, and Γ is a cocompact torsion-free discrete subgroup of the group $\mathrm{Isom}(\mathbb{H}^n)$ of all the isometries of $\mathbb{H}^n$.

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- A hyperbolic manifold (in this talk) is **compact** and has **empty boundary**.
- The isometry group $\mathrm{Isom}(M)$ of a hyperbolic n-manifold M of $n \geq 2$ is finite.
- [Belolipetsky-Lubotzky, 2005]: $\forall\ n \geq 2$ and every finite group G, $\exists$ infinitely many n-dimensional hyperbolic manifolds M with
$$\mathrm{Isom}(M) \cong G.$$

Bounded Index Property: hyperbolic n-manifolds

Theorem (Z., 2015)

For any hyperbolic n-manifold M^n $(n \geq 2)$, $\exists$ *a bound* $\mathcal{B}$, *such that for any self-homeomorphism* $f : M \to M$ *and any* $\mathbf{F} \in \mathrm{Fpc}(f)$,

$$|\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}.$$

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  $\mathrm{Isom}(M)$ finite $\Longrightarrow \mathcal{B} := \max\{\mathcal{B}_f | f \in \mathrm{Isom}(M)\} < \infty \Longrightarrow$

$$|\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B} < \infty.$$

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For any compact hyperbolic 3-manifold

Theorem (Z., 2013)

Let M^3 be a compact hyperbolic 3-manifold (orientable or nonorientable). Then for any homeomorphism $f : M \to M$,

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2. $$\sum_{\mathrm{ind}(f,\mathbf{F})<0} \mathrm{ind}(f, \mathbf{F}) > 1 - 2\mathrm{rk}\pi_1(M),$$
   where the sum is taken over all fixed point classes $\mathbf{F}$ *with* $\mathrm{ind}(f, \mathbf{F}) < 0$.

Bounds for hyperbolic 4-manifolds

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Theorem (Z., 2015)

Let M^4 be a hyperbolic 4-manifold. Then for any homeomorphism $f: M \to M$, we have

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}(M),$$

where $\mathcal{B}(M) = \max\{\dim H_*(M; \mathbb{Z}_p) | p \text{ is a prime}\}$. In particular, if f is not homotopic to the identity, then

$$\mathrm{ind}(f, \mathbf{F}) \leq 1, \qquad L(f) \leq N(f).$$

Theorem (Z., 2015)

Let M^n be a hyperbolic n-manifold ($n \geq 5$). If the isometry group $\mathrm{Isom}(M)$ is a **p-group** *($|\mathrm{Isom}(M)|$ is a power of some prime p), then for any homeomorphism $f : M \to M$, we have*

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \dim H_*(M; \mathbb{Z}_p),$$

where $\dim H_*(M; \mathbb{Z}_p)$ *denotes the dimension of the* $\mathbb{Z}_p$*-linear space*

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Question

Is there an analogous explicit bound for any compact hyperbolic n-manifold with $n \geq 5$?

Key points of Proofs of the three Theorems above

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- $n \geq 3$, Mostow Rigidity Thm $\Longrightarrow$ f can be homotopied to a unique **isometry** g of finite order.
- **F**: a compact hyperbolic submanifold, $|\mathrm{ind}(\mathbf{F})| = |\chi(\mathbf{F})| < \infty$.
- P.A. Smith Theory: Let X be a compact topological space and $t : X \to X$ a transformation of order a prime p. Suppose X has a triangulation in which t is simplicial. Let F denote the set of fixed points of t, and X' be the quotient space $X/(x = tx)$. The projection $X \to X'$ maps F homeomorphically onto a subset of X', which we again denote by F. Then for any q,

$$\dim H_q(X', F; \mathbb{Z}_p) + \sum_{r=q}^{\infty} \dim H_r(F; \mathbb{Z}_p) \leq \sum_{r=q}^{\infty} \dim H_r(X; \mathbb{Z}_p).$$

  In particular,

$$\dim H_*(F; \mathbb{Z}_p) \leq \dim H_*(X; \mathbb{Z}_p).$$

Thanks ! 谢 谢!