Fixed subgroups in direct products of free and surface groups

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Hanna Neumann Conjecture

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For any **free group**, Mineyev and Friedman proved the following theorem conjectured by [Neumann, 1957] independently. Dicks gave two versions of simplified proofs. A. Jaikin gave another new proof recently.

Let $\overline{\mathrm{rk}} := \max\{0, \mathrm{rk}(G) - 1\}$.

Theorem (Mineyev, Friedman, 2011)

Let F_n be a f.g. **free group***, and H, K any two f.g. subgroups of F_n. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

Intersection of subgroups: surface groups

Let G be a surface group, namely, $G \cong \pi_1(S)$ for a closed (possibly non-orientable) surface S with $\chi(S) < 0$.

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Theorem (Soma, 1991)

Let G be a f.g. **surface group**, *and H, K any two f.g. subgroups of G. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq 1161 \cdot \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

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Question 1

For any subgroups H, K of a **surface group**, does

$$\overline{\mathrm{rk}}(H \cap K) \leq \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K) \ ?$$

Mineyev claimed that the answer of the question above is affirmative.

Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Fixed subgroups: free groups

Theorem (Dyer-Scott, 1975)

Let $\phi \in \mathrm{Aut}(F_n)$ *be an automorphism with* **finite order** *of* F_n. *Then*

$$\mathrm{rkFix}\phi \leq \mathrm{rk}F_n.$$

Fixed subgroups: free groups

Theorem (Dyer-Scott, 1975)

Let $\phi \in \mathrm{Aut}(F_n)$ be an automorphism with **finite order** *of F_n. Then*

$$\mathrm{rkFix}\phi \leq \mathrm{rk}F_n.$$

Theorem (Bestvina-Handel, 1992)

Let ϕ be an **automorphism** *of F_n. Then* $\mathrm{rkFix}\phi \leq \mathrm{rk}F_n$.

Other alternative proofs (Sela, Paulin, Gaboriau-Jaeger-Levitt-Lustig,...)

Fixed subgroups: free groups

Theorem (Dicks-Ventura, 1996)

Let $\mathcal{B}$ be a family of **injective** *endomorphisms of F_n, then*

$$\mathrm{rkFix}\mathcal{B} \leq \mathrm{rk}F_n.$$

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Theorem (Dicks-Ventura, 1996)

Let $\mathcal{B}$ be a family of **injective** *endomorphisms of F_n, then*

$$\mathrm{rk}\mathrm{Fix}\mathcal{B} \leq \mathrm{rk}F_n.$$

They also showed that $\mathrm{Fix}\mathcal{B}$ is **inert** in F_n.

Definition

A subgroup A is inert in G if for every subgroup $B \leqslant G$,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk}B.$$

Fixed subgroups & fixed points on graphs & surfaces

Let $\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ (or $2 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ for some cases).

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite graph or a connected compact hyperbolic surface, and $f : X \to X$ is a **selfmap**. *Then*

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f*;*
2. *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X).$$

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Corollary

Bestvina-Handel results for free groups (Scott Conjecture).

Fixed subgroups: surface groups

Let G be a f.g. **surface group**.

Theorem (Jiang-Wang-Z., 2011)

Let ϕ be an **endomorphism** *of G. Then*

1. $\mathrm{rkFix}\phi \leq \mathrm{rk}G$, *with equality if and only if $\phi = id$;*
2. $\mathrm{rkFix}\phi \leq \frac{1}{2}\mathrm{rk}G$ *if ϕ is not epimorphic.*

[Nielsen,1929]: For any closed **orientable** surface S and **automorphism** ϕ of $\pi_1(S)$, $\mathrm{rkFix}\phi \leq \mathrm{rk}G$.

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Theorem (Wu-Z.,2014)

Let $\mathcal{B}$ be a family of **endomorphisms** *of G. Then*

1. $\mathrm{rkFix}\mathcal{B} \leq \mathrm{rk}G$, *with equality if and only if $\mathcal{B} = \{id\}$;*
2. $\mathrm{rkFix}\mathcal{B} \leq \frac{1}{2}\mathrm{rk}G$, *if $\mathcal{B}$ contains a non-epimorphic endomorphism*

Geometric subgroups of surface groups

A connected subsurface F of a connected surface S is called incompressible if the natural homomorphism $\pi_1(F) \to \pi_1(S)$ induced by the inclusion $F \hookrightarrow S$ is injective. We can think of $\pi_1(F)$ as a subgroup of $\pi_1(S)$. Subgroups which arise in this way are called geometric.

Geometric subgroups: inertia

Theorem (Nielsen, Jaco-Shalen)

The fixed subgroup of an **automorphism** *of a surface group is either cyclic or geometric.*

Geometric subgroups: inertia

Theorem (Nielsen, Jaco-Shalen)

The fixed subgroup of an **automorphism** *of a surface group is either cyclic or geometric.*

For geometric subgroups of a surface group, we prove that

Theorem (Wu-Z., 2014)

Any geometric subgroup A of a surface group G is inert in G, i.e.,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk} B \quad \textit{for} \quad \forall B \leqslant G.$$

Corollary (Wu-Z., 2014)

The fixed subgroup of any family of epimorphisms of a surface group G is inert in G.

Equalizers and Retracts: definitions

- Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A section of ϕ is a homomorphism $\sigma : H \to G$ such that
$$\phi\sigma = id : H \to H.$$
For any family $\mathcal{B}$ of sections of ϕ, the equalizer of $\mathcal{B}$ is
$$\mathrm{Eq}(\mathcal{B}) := \{h \in H | \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\} \leq H.$$

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- $id \in \mathcal{B} \Longrightarrow \mathrm{Eq}(\mathcal{B}) = \mathrm{Fix}(\mathcal{B})$.
- Suppose H is a subgroup of a group G. If there is a homomorphism $\pi : G \to G$ such that $\pi(G) \leq H$ and

$$\pi|_H = id : H \to H,$$

we say that H is a retract of G. If $H \neq G$, it is called a proper retract.

Equalizers and Retracts: results

We have the following relation between equalizers and retracts:

Lemma

Let G, H be two groups, and $\phi : G \to H$ an epimorphism. If $\mathcal{B}$ is a family of sections of ϕ, then for any section $\sigma \in \mathcal{B}$, $\sigma(H)$ is a retract of G, and

$$\sigma|_{\mathrm{Eq}(\mathcal{B})} : \mathrm{Eq}(\mathcal{B}) \to \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$$

is an isomorphism.

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is an isomorphism.

For **free groups**, Bergman showed

Proposition (Bergman, 1999)

1. *Any intersection of retracts of a f.g. free group is also a retract;*
2. *If $\phi : G \to H$ is an epimorphism of free groups with H f.g., then the equalizer of any family of sections of ϕ is a free factor in H.*

Retracts on surface groups

For a **surface group** G, we have

Proposition (Wu-Z., 2014)

1. *Any proper retract of* G *is free of rank* $\leq \frac{1}{2}\mathrm{rk}G$.

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1. *Any proper retract of* G *is free of rank* $\leq \frac{1}{2}\mathrm{rk}G$.
2. *If* H_1, H_2 *are two proper retracts of* G, *and* $H = \langle H_1, H_2\rangle \leq G$, *the subgroup generated by* H_1 *and* H_2, *then*
   (1) *If* $H < G$, *then* $H_1 \cap H_2$ *is a retract of both* H_1 *and* H_2, *and*
   $$\mathrm{rk}(H_1 \cap H_2) \leq \min\{\mathrm{rk}H_1, \mathrm{rk}H_2\}.$$
   (2) *If* $H = G$, *then* $H_1 \cap H_2$ *is cyclic (possibly trivial).*

For a **surface group** G, we have

Proposition (Wu-Z., 2014)

1. *Any proper retract of G is free of rank $\leq \frac{1}{2}\mathrm{rk}G$.*
2. *If H_1, H_2 are two proper retracts of G, and $H = \langle H_1, H_2\rangle \leq G$, the subgroup generated by H_1 and H_2, then*
   (1) *If $H < G$, then $H_1 \cap H_2$ is a retract of both H_1 and H_2, and*
   $$\mathrm{rk}(H_1 \cap H_2) \leq \min\{\mathrm{rk}H_1, \mathrm{rk}H_2\}.$$
   (2) *If $H = G$, then $H_1 \cap H_2$ is cyclic (possibly trivial).*
3. *If $\mathcal{R}$ is a family retracts of G, then*
   $$\mathrm{rk}(\bigcap_{H\in\mathcal{R}} H) \leq \min\{\mathrm{rk}H | H \in \mathcal{R}\} \leq \begin{cases} \mathrm{rk}G, & \mathcal{R} = \{G\} \\ \frac{1}{2}\mathrm{rk}G, & \mathcal{R} \neq \{G\} \end{cases}.$$

Question 2

Is every retract H of a surface/free group G inert in G? Namely, is

$$\mathrm{rk}(H \cap K) \leq \mathrm{rk}(K)$$

for any subgroup $K \leq G$?

Fixed subgroups & fixed points on Seifert manifolds

M: a comp. orient. **Seifert 3-manifold** with hyperbolic orbifold,

Theorem (Z., 2012)

Suppose $f : M \to M$ is a homeomorphism. Then

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f*;*
2. $\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0}\{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq \mathcal{B}$,
   where $\mathcal{B} = 4\big(2 - \mathrm{rk}\pi_1(M)\big)$.

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   where $\mathcal{B} = 4\big(2 - \mathrm{rk}\pi_1(M)\big)$.

Corollary (Z., 2012)

Let $f_\pi : \pi_1(M,x) \to \pi_1(M,x)$ be the induced automorphism and

$$\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(M,x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(M,x),$$

where x is in an **essential fixed point class***. Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

Fixed subgroups: Seifert manifold groups

Theorem (Z., 2013)

Suppose M is a compact orientable **Seifert** *3-manifold, and $f_\pi : \pi_1(M) \to \pi_1(M)$ is an automorphism induced by an* **orientation-reversing** *homeomorphism $f : M \to M$. Then*

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Remark. Analogue as above does NOT hold for orient.-preserving automorphism of Seifert manifold groups.

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Theorem (Lin-Wang, 2012)

Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a compact orientable **hyperbolic** *3-manifold with finite volume. Then*

$$\mathrm{rkFix}\phi < 2\mathrm{rk}G.$$

Inert, compressed and bounded

Definition

For a subgroup $A \leqslant G$,

- A is called inert in G, if for every subgroup $B \leqslant G$,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk}B.$$

- A is called compressed in G, if for every subgroup $A \leqslant B \leqslant G$,

$$\mathrm{rk}A \leq \mathrm{rk}B.$$

- A is called c-bounded in G, if

$$\mathrm{rk}A \leq c \cdot \mathrm{rk}G.$$

A is called bounded in G, if it is 1-bounded in G.

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  A is called bounded in G, if it is 1-bounded in G.

Remark: Inert $\Longrightarrow$ Compressed $\Longrightarrow$ Bounded.

Fixed subgroups of endomorphisms

For any family $B \subseteq \mathrm{End}(G)$,

Theorem (Bergman, 1999)

$\mathrm{Fix}\mathcal{B}$ *is* **bounded** *in* F_n.

Question (Bergman, 1999)

Is $\mathrm{Fix}\mathcal{B}$ **inert** in F_n?

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$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in* F_n.

Theorem (Z.-Ventura-Wu, 2015)

$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in any* **surface group**.

Fixed subgroups in product groups: most are bounded

Let $G = G_1 \times G_2 \times \cdots \times G_n$, each G_i is a f.g. free group or $\pi_1(S)$ for a closed surface S (maybe $\mathbb{R}P^2, 2\mathbb{R}P^2$ or a torus). We call it a product group.

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Theorem A (Z.-Ventura-Wu, 2015)

$$\mathrm{rkFix}\phi \leq \mathrm{rk}G \text{ for every } \phi \in \mathrm{Aut}(G)$$

$\Longleftrightarrow$ All G_i are of the same type (Euclidean or hyperbolic).

Euclidean type: $\mathbb{Z}$, $\pi_1(S)$ for $\chi(S) \geq 0$.
Hyperbolic type: F_n $(n > 1)$, $\pi_1(S)$ for $\chi(S) < 0$.

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Hyperbolic type: F_n $(n > 1)$, $\pi_1(S)$ for $\chi(S) < 0$.

Example (NOT satisfying the conditions of Theorem A)

Let $G = F_2 \times \mathbb{Z} = \langle a, b \rangle \times \langle t \rangle$ and

$$\phi \in \mathrm{Aut}(G) : a \mapsto at, b \mapsto b, t \mapsto t.$$

Then $\mathrm{Fix}\phi = \langle t, a^{-m}ba^m | m \in \mathbb{Z} \rangle$.

Fixed subgroups in product groups: few are compressed

Theorem B (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **compressed** in G for every $\phi \in \mathrm{Aut}(G)$, then G must be of one of the following forms:

(*euc1*) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or
(*euc2*) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or
(*euc3*) $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for some $p \geqslant 1$; or
(*euc4*) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$; or

(*hyp1*) $G = F_r \times NS_3^\ell$ for some $r \geqslant 2$, $\ell \geqslant 0$; or
(*hyp2*) $G = S_g \times NS_3^\ell$ for some $g \geqslant 2$, $\ell \geqslant 0$; or
(*hyp3*) $G = NS_k \times NS_3^\ell$ for some $k \geqslant 3$, $\ell \geqslant 0$.

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($hyp1$) $G = F_r \times NS_3^\ell$ for some $r \geqslant 2$, $\ell \geqslant 0$; or
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Question (Z.-Ventura-Wu, 2015)

Is the implication in Theorem B an equivalence?

Fixed subgroups in product groups: few are compressed

Proposition

If G is of form (euc3), i.e. $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for $p \geqslant 1$, then $\exists\ \phi \in \mathrm{Aut}(G)$, s.t. $\mathrm{Fix}\phi$ is NOT compressed, hence NOT inert.

Fixed subgroups in product groups: few are compressed

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Proof: Let $G = \langle a, b | bab^{-1}a\rangle \times \prod_{i=1}^{p} \langle c_i \rangle \times \langle d | d^2 \rangle$ and $\phi \in \mathrm{Aut}(G)$:

$$a \mapsto ad,\ \ b \mapsto ba,\ \ c_1 \mapsto c_1 d,\ \ c_i \mapsto c_i^{-1}, (i = 2, ..., p),\ \ d \mapsto d.$$

$\Longrightarrow \mathrm{Fix}\phi = \langle a^2, b^2, ac_1, d\rangle \cong \mathbb{Z}^3 \times (\mathbb{Z}/2\mathbb{Z})$, while $\mathrm{Fix}\phi \leqslant \langle ac_1, b, d\rangle$.

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Theorem C

Let G be a product group of **Euclidean** type. Then, $\mathrm{Fix}\phi$ is compressed in G for every $\phi \in \mathrm{Aut}(G) \Longleftrightarrow G$ is of one of the following forms:

(*euc1*) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or
(*euc2*) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or
(*euc4*) $G = NS_2^{\ell} \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$.

Examples: fixed subgroups NOT compressed

Proposition (5 factors)

If $G = G_1 \times \cdots \times G_5$*, each* G_i *is* $F_r(r \geq 2)$ *,* $S_g(g \geq 2)$ *or* $NS_k(k \geq 3)$*, then* $\exists\ \phi \in \mathrm{Aut}(G)$ *s.t.* $\mathrm{Fix}\phi$ *is NOT compressed in* G*.*

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Proof: Let $1 \neq h_i = [s_i, t_i] = s_i t_i s_i^{-1} t_i^{-1} \in G_i$, and

$$\phi_i \in \mathrm{Aut}(G_i): \ g \mapsto h_i g h_i^{-1}.$$

Then $\mathrm{Fix}\phi_i = \langle h_i \rangle$. Let $\phi = \phi_1 \times \cdots \times \phi_5 \in \mathrm{Aut}(G)$. Then

$$\mathrm{Fix}\phi = \langle s_1 t_1 s_1^{-1} t_1^{-1} \rangle \times \cdots \times \langle s_5 t_5 s_5^{-1} t_5^{-1} \rangle \cong \mathbb{Z}^5$$

while

$$\mathrm{Fix}\phi \leqslant \langle s_1 s_2 s_4, t_1 t_3 t_5, t_2 s_3, s_5 t_4 \rangle,$$

because

$$[s_1 s_2 s_4, t_1 t_3 t_5] = [s_1, t_1], \ \ [s_1 s_2 s_4, t_2 s_3] = [s_2, t_2],$$

$$[t_2 s_3, t_1 t_3 t_5] = [s_3, t_3], \ [s_1 s_2 s_4, s_5 t_4] = [s_4, t_4], \ [s_5 t_4, t_1 t_3 t_5] = [s_5, t_5].$$

Examples: fixed subgroups NOT compressed

Proposition (4 factors)

Let $G = G_0 \times NS_3^\ell (\ell \geq 3)$, G_0 is $F_r (r \geq 2)$, $S_g (g \geq 2)$ or $NS_k (k \geq 4)$. Then $\exists\ \phi \in \mathrm{Aut}(G)$ s.t. $\mathrm{Fix}\phi$ is NOT compressed in G.

Examples: fixed subgroups NOT compressed

Proposition (4 factors)

Let $G = G_0 \times NS_3^{\ell}(\ell \geq 3)$*,* G_0 *is* $F_r(r \geq 2)$ *,* $S_g(g \geq 2)$ *or* $NS_k(k \geq 4)$*. Then* $\exists\ \phi \in \mathrm{Aut}(G)$ *s.t.* $\mathrm{Fix}\phi$ *is NOT compressed in* G*.*

Proof: For $i = 1, 2, 3$, $\exists \phi_i \in \mathrm{Aut}(G_i)$, s.t. $\mathrm{Fix}\phi_i = \langle s_i t_i s_i^{-1} t_i^{-1} \rangle$.

- $G_0 = F_r = \langle a_1, \ldots, a_r \rangle$, $\phi_0 \in \mathrm{Aut}(G_0) : a_1 \mapsto a_1 a_2,\ a_i \mapsto a_i$, $i \geq 2$, $\mathrm{Fix}\phi_0 = \langle a_2, a_1 a_2 a_1^{-1}, a_3, \ldots, a_r \rangle \cong F_r$. We have $\mathrm{Fix}\phi_0 \times \cdots \times \mathrm{Fix}\phi_3 \leqslant H = \langle a_2 a_1 s_1 s_2, s_3 t_3 t_1 t_2, s_3 s_1 t_2, a_1 t_3 t_1 s_2, a_3, \ldots,$
- $G_0 = S_g$ or NS_k, we can construct an analog $\phi_0 \in \mathrm{Aut}(G)$.

Let $\phi = \phi_0 \times \phi_1 \times \phi_2 \times \phi_3 \times Id \times \cdots \times Id \in \mathrm{Aut}(G)$. Then

$$\mathrm{Fix}\phi = \mathrm{Fix}\phi_0 \times \cdots \times \mathrm{Fix}\phi_3 \times G_4 \times \cdots \times G_\ell \leqslant H \times G_4 \times \cdots \times G_\ell.$$

But $$\mathrm{rk}\mathrm{Fix}\phi > \mathrm{rk}(H \times G_4 \times \cdots \times G_\ell).$$

Fixed subgroups in product groups: less are inert

Theorem D

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **inert** in G for every $\phi \in \mathrm{Aut}(G)$, then G is of one of the forms:

(*euc1*) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or
(*euc2*) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or
(*euc4*) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$; or

(*hyp1'*) $G = F_n$ for some $n \geq 2$; or
(*hyp2'*) $G = \pi_1(S)$ for some closed surface $\chi(S) < 0$.

Open problems

Inertia Conjecture

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, the following are equivalent:

1. Every $\phi \in \mathrm{End}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
2. Every $\phi \in \mathrm{Aut}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
3. G is one of the forms (euc1),(euc2),(euc4), (hyp1') or (hyp2').

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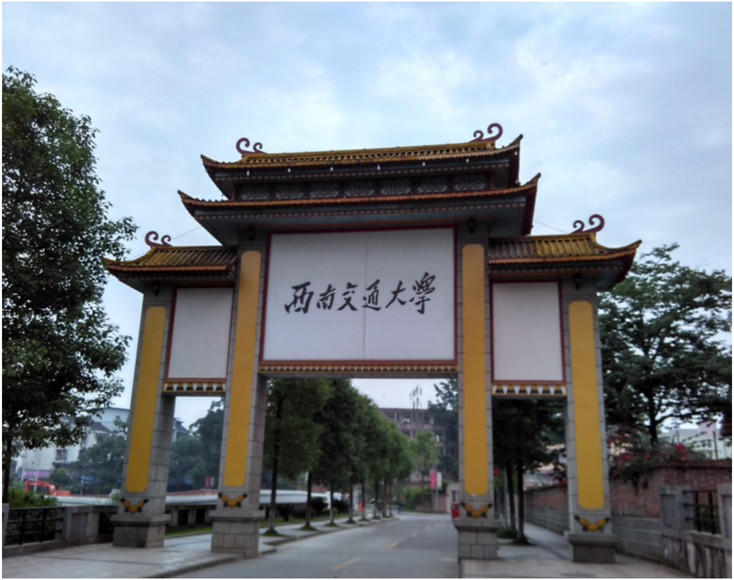
Remark

$$(1) \xRightarrow{trivial} (2) \xRightarrow{Theorem\ D} (3) \xRightarrow{???} (1)$$

$$\{(3) - (euc4)\} \xRightarrow[Wu-Z.,2014]{Dicks-Ventura,1996} (2)$$



Thanks! 谢 谢!