Bounds for fixed points on some manifolds

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Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

$$\mathrm{Fix} f := \{x \in X | f(x) = x\} = \bigsqcup_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathbf{F}$$

Definition

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

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Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Rank of a fixed point class

Definition

For a fixed point $x \in \mathbf{F}$, let

$$\mathrm{Stab}(f, x) := \{\gamma \in \pi_1(X, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(X, x),$$

where $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ is the induced endomorphism. It is independent of the choice of $x \in F$, up to isomorphism. For a fixed point class $\mathbf{F}$ of f, define the rank to be

$$\mathrm{rk}(\mathbf{F}) := \mathrm{rk}(f, x) := \mathrm{rk}\mathrm{Stab}(f, x), \quad \forall x \in \mathbf{F}.$$

For an empty fixed point class $\mathbf{F}$, we can make it nonempty by deforming f.

Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\operatorname{ind}(f_0, \mathbf{F}_0) = \operatorname{ind}(f_1, \mathbf{F}_1), \quad \operatorname{rk}(f_0, \mathbf{F}_0) = \operatorname{rk}(f_1, \mathbf{F}_1).$$

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Commutation invariance

Suppose $\phi : X \to Y$ and $\psi : Y \to X$ are maps. Then $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are said to differ by a commutation. The map ϕ sets up a natural one-one correspondence

$$\mathbf{F}_X \to \mathbf{F}_Y$$

from the fixed point classes of $\psi \circ \phi$ to the fixed point classes of $\phi \circ \psi$.

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Under the correspondence via commutation,

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Mutation invariance

Among selfmaps of compact polyhedra, homotopy and commutation generates an equivalence relation:

Definition

A sequence $\{f_i : X_1 \to X_i | i = 0, \cdots, k\}$ of self-maps is a mutation if for each i, either

1. $X_{i+1} = X_i$ and $f_{i+1} \simeq f_i$, or
2. f_{i+1} is obtained from f_i by commutation.

A mutation sets up a one-one correspondence between fixed point classes of the end maps.

Theorem (Mutation invariance)

The index $\mathrm{ind}(\mathbf{F})$ *and the rank* $\mathrm{rk}(\mathbf{F})$ *are mutation invariants.*

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Characteristic of a fixed point class

From now on, unless otherwise stated, we always assume X to be a graph, a surface or a Seifert manifold, and $f : X \to X$ is a selfmap. For convenience, we define another term.

Definition

The characteristic of a fixed point class $\mathbf{F}$ is defined as

$$\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rk}(\mathbf{F}).$$

with the exception is when $\mathrm{Stab}(f, \mathbf{F}) = \pi_1(S)$ for some closed hyperbolic surface $S \subset X$, in this case

$$\mathrm{chr}(\mathbf{F}) := \chi(S) = 2 - \mathrm{rk}(\mathbf{F}).$$

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Nielsen number & Lefschetz number

Definition

- A fixed point class **F** of f is essential if $\mathrm{ind}(f, \mathbf{F}) \neq 0$.
- Nielsen number $N(f) := \#\{\textit{essential fixed point classes of } f\}$.
- Lefschetz number

$$L(f) := \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

Lefschetz Fixed Point Theorem

$$\sum_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathrm{ind}(f, \mathbf{F}) = \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

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Bounds for graphs and surfaces

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite **graph** *or a connected compact hyperbolic* **surface**, *and $f : X \to X$ is a* **selfmap**. *Then*

(A) $\mathrm{ind}(\mathbf{F}) \le \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f*;*

(B) *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \ge 2\chi(X),$$

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B. Jiang, S.D. Wang, Q. Zhang, *Bounds for fixed points and fixed subgroups on surfaces and graphs*, Alg. Geom. Topology, 11(2011), 2297–2318.

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Fixed subgroups: free groups

As corollaries, we have

Theorem (Bestvina-Handel, 1992)

Let ϕ be an **automorphism** *of F_n. Then* $\mathrm{rkFix}\phi \leq \mathrm{rk}F_n$.

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Corollary (Jiang, 1998)

Let X be either a connected finite **graph** *(not a tree) or a connected compact hyperbolic* **surface**, *and $f : X \to X$ a* **selfmap**. *Then*

1. $\mathrm{ind}(\mathbf{F}) \leq 1$, $\forall \mathbf{F} \in \mathrm{Fpc}(f)$;
2. *Almost all fixed point classes have index ≥ -1, in the sense*
$$\sum_{\mathrm{ind}(\mathbf{F})<-1} \{\mathrm{ind}(\mathbf{F})+1\} \geq 2\chi(X).$$
3. $|L(f) - \chi(X)| \leq N(f) - \chi(X)$.

Theorem (Jiang-Wang-Z., 2011)

Let ϕ be an **endomorphism** *of a surface group G. Then*

1. $\mathrm{rkFix}\phi \leq \mathrm{rk}G$, *with equality if and only if $\phi = id$;*
2. $\mathrm{rkFix}\phi \leq \frac{1}{2}\mathrm{rk}G$ *if ϕ is not epimorphic.*

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Fixed subgroups: surface groups

Theorem (Wu-Z.,2014)

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Bounds for 3-manifolds

Theorem (Jiang-Wang, 1992)

Suppose a closed **aspherical 3-manifold** *M is finitely covered by an orientable 3-manifold which is either a Seifert manifold, or a hyperbolic 3-manifold, or admits a non-trivial JSJ-decomposition. Let $f : M \to M$ is a homeomorphism. Then*

1. $\mathrm{ind}(\mathbf{F}) \leq 1$, $\forall \mathbf{F} \in \mathrm{Fpc}(f)$, *hence* $L(f) \leq N(f)$;
2. *If M is orientable and f is* **orientation-preserving**, *then*
$$\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}, \quad \forall\ \mathbf{F} \in \mathrm{Fpc}(f),$$
hence $|L(f)| \leq N(f)$.
3. $\forall\ n > 3$, $\exists f$ *on a closed aspherical n-manifold such that*
$$L(f) > N(f).$$

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where the sum is taken over all **essential** *fixed point classes* $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$, *and*

$$\mathcal{B} = \begin{cases} 4(3 - \mathrm{rk}\pi_1(M)) & M \text{ is a closed surface } F \times S^1 \\ 4(2 - \mathrm{rk}\pi_1(M)) & \text{others} \end{cases}.$$

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As a corollary, we have

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Suppose M is a compact orientable **Seifert** *3-manifold with hyperbolic orbifold $X(M)$, and $f : M \to M$ is a homeomorphism. Then*

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3. $|L(f) - \mathcal{B}/2| \leq N(f) - \mathcal{B}/2$.

The bound above is analogous to the one on graphs and surfaces. For f **orient.-preserving**, [Jiang-Wang, 1992]: $\mathrm{ind}(\mathbf{F}) \in \{-1, 0, 1\}$.

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Suppose M is a compact orientable **Seifert** *3-manifold with hyperbolic orbifold $X(M)$, and $f : M \to M$ is a homeomorphism. Then*

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The bound above is analogous to the one on graphs and surfaces.
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Bounds for Seifert 3-manifolds, II

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Proposition (Z., 2012)

Suppose $f : M \to M$ is a homeomorphism of a compact orientable **Seifert** *3-manifold with hyperbolic orbifold. Let $f_\pi : \pi_1(M, x) \to \pi_1(M, x)$ be the induced automorphism and $\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(M, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(M, x)$, where x is in an* **essential** *fixed point class. Then*

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Hyperbolic n-manifolds

- By a hyperbolic n-manifold ($n \geq 2$) we mean a quotient space

$$M = \mathbb{H}^n/\Gamma,$$

where $\mathbb{H}^n$ is the hyperbolic n-space, that is, the connected, simply connected Riemanian manifold of constant curvature -1, and Γ is a cocompact torsion-free discrete subgroup of the group $\mathrm{Isom}(\mathbb{H}^n)$ of all the isometries of $\mathbb{H}^n$.

- A hyperbolic manifold (in this talk) is **compact** and has **empty boundary**.
- The isometry group $\mathrm{Isom}(M)$ of a hyperbolic n-manifold M of $n \geq 2$ is finite.
- [Belolipetsky-Lubotzky, 2005]: $\forall$ $n \geq 2$ and every finite group G, $\exists$ infinitely many n-dimensional hyperbolic manifolds M with

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For any hyperbolic n-manifold M^n $(n \geq 2)$, $\exists$ a bound $\mathcal{B}$, such that for any self-homeomorphism $f : M \to M$ and any $\mathbf{F} \in \mathrm{Fpc}(f)$,

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Proof:

- $n = 2$, [Jiang, 1998];
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Theorem (Z., 2013)

Let M^3 be a compact hyperbolic 3-manifold (orientable or nonorientable). Then for any homeomorphism $f: M \to M$,

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Let M^4 be a hyperbolic 4-manifold. Then for any homeomorphism $f : M \to M$, we have

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}(M),$$

where $\mathcal{B}(M) = \max\{\dim H_*(M; \mathbb{Z}_p) | p$ is a prime$\}$. In particular, if f is not homotopic to the identity, then

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Let M^n be a hyperbolic n-manifold ($n \geq 5$). If the isometry group $\mathrm{Isom}(M)$ is a **p-group** *($|\mathrm{Isom}(M)|$ is a power of some prime p), then for any homeomorphism $f: M \to M$, we have*

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- $n \geq 3$, Mostow Rigidity Thm $\Longrightarrow$ f can be homotopied to a unique **isometry** g of finite order.
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- P.A. Smith Theory: Let X be a compact topological space and $t : X \to X$ a transformation of order a prime p. Suppose X has a triangulation in which t is simplicial. Let F denote the set of fixed points of t, and X' be the quotient space $X/(x = tx)$. The projection $X \to X'$ maps F homeomorphically onto a subset of X', which we again denote by F. Then for any q,

$$\dim H_q(X', F; \mathbb{Z}_p) + \sum_{r=q}^{\infty} \dim H_r(F; \mathbb{Z}_p) \leq \sum_{r=q}^{\infty} \dim H_r(X; \mathbb{Z}_p).$$

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Open problems

A compact polyhedron X is said to have the *Bounded Index Property (BIP)* if there is an integer $B > 0$ such that for any map $f : X \to X$ and any fixed point class $\mathbf{F}$ of f, the index $|\mathrm{ind}(f, \mathbf{F})| \leq B$. X has the *Bounded Index Property for Homeomorphisms (BIPH)* if there is such a bound for all homeomorphisms $f : X \to X$.

Question (Jiang, 1998)

Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

Positive Examples:

- Graphs & surfaces with negative Euler characteristic have BIP;
- Closed aspherical 3-manifolds have BIPH for orientation preserving self-homeomorphisms;
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- Compact hyperbolic n-manifolds with $n \geq 2$ have BIPH.
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Thanks ！ 谢 谢！