Fixed subgroups in low-dimensional manifold groups

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Contents

Hanna Neumann Conjecture

For a **f.g.** (finitely generated) group G, let $\mathrm{rk}(G)$ denote the rank (i.e., the minimal number of generators) of G. There are lots of research on the intersection of subgroups in the literature.

For any **free group**, Mineyev and Friedman proved the following theorem conjectured by [Neumann, 1957] independently. Dicks gave two versions of simplified proofs. A. Jaikin gave another new proof recently.

Let $\overline{\mathrm{rk}} := \max\{0, \mathrm{rk}(G) - 1\}$.

Theorem (Mineyev, Friedman, 2011)

Let F_n be a f.g. **free group**, *and H, K any two f.g. subgroups of F_n. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

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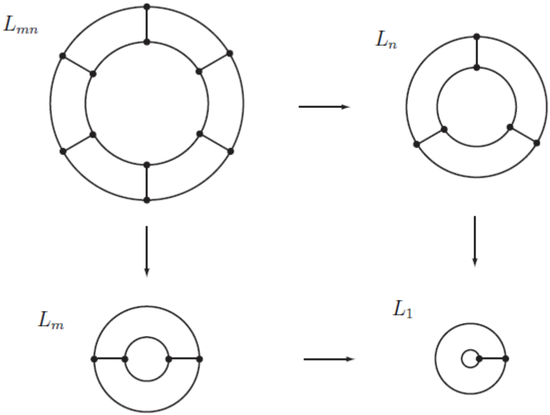
H. N. Conjecture: a special case



m and *n* are relatively prime

Intersection of subgroups: surface groups

Let G be a surface group, namely, $G \cong \pi_1(S)$ for a closed (possibly non-orientable) surface S with $\chi(S) < 0$.

Theorem (Soma, 1991)

Let G be a f.g. **surface group**, *and H, K any two f.g. subgroups of G. Then*

$$\overline{\mathrm{rk}}(H \cap K) \leq 1161 \cdot \overline{\mathrm{rk}}(H) \cdot \overline{\mathrm{rk}}(K).$$

Question 1

For any subgroups H, K of a **surface group**, does

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Mineyev claimed that the answer of the question above is affirmative.

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Intersection of subgroups: one-relator groups

Let $G = \langle X | r \rangle$ be a **one-relator group** where r is a cyclically reduced word in the free group on the generating set X.

A subset $Y \subset X$ is called a *Magnus subset* if Y omits a generator which appears in the relator r. A subgroup H of G is called a *Magnus subgroup* if $H = \langle Y \rangle$ for some Magnus subset Y of X, and hence by the Magnus Freiheitssatz, H is free of rank $|Y|$.

Theorem (Collins, 2004)

The intersection $\langle Y \rangle \cap \langle Z \rangle$ of two Magnus subgroups of the one-relator group G is either $\langle Y \cap Z \rangle$ or the free product of $\langle Y \cap Z \rangle$ with an infinite cyclic group and thus of rank $|Y \cap Z| + 1$.

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Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Fixed subgroups: free groups

Theorem (Dyer-Scott, 1975)

Let $\phi \in \mathrm{Aut}(F_n)$ be an automorphism with **finite order** *of F_n. Then*

$$\mathrm{rkFix}\phi \leq \mathrm{rk}F_n.$$

Theorem (Bestvina-Handel, 1992)

Let ϕ be an **automorphism** *of F_n. Then $\mathrm{rkFix}\phi \leq \mathrm{rk}F_n$.*

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Theorem (Dicks-Ventura, 1996)

Let $\mathcal{B}$ be a family of **injective** *endomorphisms of F_n, then*

$$\mathrm{rk}\mathrm{Fix}\mathcal{B} \leq \mathrm{rk}F_n.$$

They also showed that $\mathrm{Fix}\mathcal{B}$ is **inert** in F_n.

Definition

A subgroup A is inert in G if for every subgroup $B \leqslant G$,

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Fixed subgroups & fixed points on graphs & surfaces

Let $\mathrm{chr}(\mathbf{F}) := 1 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ (or $2 - \mathrm{rkFix}(f_{\pi,\mathbf{F}})$ for some cases).

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite graph or a connected compact hyperbolic surface, and $f : X \to X$ is a **selfmap**. *Then*

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f*;*
2. *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X).$$

Corollary

Bestvina-Handel results for free groups (Scott Conjecture).

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2. $\mathrm{rkFix}\phi \leq \frac{1}{2}\mathrm{rk}G$ *if ϕ is not epimorphic.*

[Nielsen,1929]: For any closed **orientable** surface S and **automorphism** ϕ of $\pi_1(S)$, $\mathrm{rkFix}\phi \leq \mathrm{rk}G$.

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Geometric subgroups of surface groups

A connected subsurface F of a connected surface S is called incompressible if the natural homomorphism $\pi_1(F) \to \pi_1(S)$ induced by the inclusion $F \hookrightarrow S$ is injective. We can think of $\pi_1(F)$ as a subgroup of $\pi_1(S)$. Subgroups which arise in this way are called geometric.

Geometric subgroups: inertia

Theorem (Nielsen, Jaco-Shalen)

The fixed subgroup of an **automorphism** *of a surface group is either cyclic or geometric.*

For geometric subgroups of a surface group, we prove that

Theorem* (Wu-Z., 2014)

If A is a geometric subgroup of a surface group G, then A is inert in G, i.e., for any subgroup B of G, we have $\mathrm{rk}(A \cap B) \leq \mathrm{rk} B$.

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The fixed subgroup of any family of epimorphisms of a surface group G is inert in G.

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Sketch of the proof of Theorem*

Using covering theory:

- A is a geometric subgroup of $G \iff \exists$ incomp. subsurface F of a closed surface S, s.t. $A = \pi_1(F, *) \leq \pi_1(S, *) = G$.
- We have two maps: the inclusion $i : F \hookrightarrow S$, and the covering $k : K \to S$ associated to B (i,e., $k_*(\pi_1(K, \tilde{*})) = B$).
- Consider the commutative diagram

$$\begin{array}{ccc} F_0 \subset \tilde{F} & \xrightarrow{i'} & K \\ p \downarrow & & k \downarrow \\ F & \xrightarrow{i} & S \end{array}$$

  where $p : \tilde{F} \to F$ is the **pull back** map of k via i, and F_0 is the component of $\tilde{F}$ containing the base point.
- $i_* p_*(\pi_1(F_0)) = A \cap B \quad \Longrightarrow \quad \mathrm{rk}(A \cap B) = \mathrm{rk}\pi_1(F_0) \leq \mathrm{rk}B.$

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Equalizers and Retracts: definitions

- Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A section of ϕ is a homomorphism $\sigma : H \to G$ such that
$$\phi\sigma = id : H \to H.$$
For any family $\mathcal{B}$ of sections of ϕ, the equalizer of $\mathcal{B}$ is
$$\mathrm{Eq}(\mathcal{B}) := \{h \in H | \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\} \leq H.$$

- $id \in \mathcal{B} \Longrightarrow \mathrm{Eq}(\mathcal{B}) = \mathrm{Fix}(\mathcal{B})$.
- Suppose H is a subgroup of a group G. If there is a homomorphism $\pi : G \to G$ such that $\pi(G) \leq H$ and
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we say that H is a retract of G. If $H \neq G$, it is called a proper retract.

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Equalizers and Retracts: results

We have the following relation between equalizers and retracts:

Lemma

Let G, H be two groups, and $\phi : G \to H$ an epimorphism. If $\mathcal{B}$ is a family of sections of ϕ, then for any section $\sigma \in \mathcal{B}$, $\sigma(H)$ is a retract of G, and

$$\sigma|_{\mathrm{Eq}(\mathcal{B})} : \mathrm{Eq}(\mathcal{B}) \to \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$$

is an isomorphism.

For **free groups**, Bergman showed

Proposition (Bergman, 1999)

1. *Any intersection of retracts of a f.g. free group is also a retract;*
2. *If $\phi : G \to H$ is an epimorphism of free groups with H f.g., then the equalizer of any family of sections of ϕ is a free factor in H.*

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Retracts on surface groups

For a **surface group** G, we have

Proposition (Wu-Z., 2014)

1. *Any proper retract of G is free of rank $\leq \frac{1}{2}\mathrm{rk}G$.*
2. *If H_1, H_2 are two proper retracts of G, and $H = \langle H_1, H_2 \rangle \leq G$, the subgroup generated by H_1 and H_2, then*
   (1) *If $H < G$, then $H_1 \cap H_2$ is a retract of both H_1 and H_2, and*
   $$\mathrm{rk}(H_1 \cap H_2) \leq \min\{\mathrm{rk}H_1, \mathrm{rk}H_2\}.$$
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3. *If $\mathcal{R}$ is a family retracts of G, then*
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Retracts: further questions

Question 2

Is every retract H of a surface/free group G inert in G? Namely, is

$$\mathrm{rk}(H \cap K) \leq \mathrm{rk}(K)$$

for any subgroup $K \leq G$?

Fixed subgroups & fixed points on Seifert manifolds

M: a comp. orient. **Seifert 3-manifold** with hyperbolic orbifold,

Theorem (Z., 2012)

Suppose $f : M \to M$ is a homeomorphism. Then

1. $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f*;*
2. $\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0}\{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq \mathcal{B}$,
   where $\mathcal{B} = 4\big(2 - \mathrm{rk}\pi_1(M)\big)$.

Corollary (Z., 2012)

Let $f_\pi : \pi_1(M, x) \to \pi_1(M, x)$ is the induced automorphism and

$$\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(M, x) | \gamma = f_\pi(\gamma)\} \subset \pi_1(M, x),$$

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Fixed subgroups: Seifert manifold groups

Theorem (Z., 2013)

Suppose $f_\pi : \pi_1(M) \to \pi_1(M)$ *is an automorphism induced by an* **orientation-reversing** *homeomorphism* $f : M \to M$*. Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

However, an analogue as above does **not hold** for a generic automorphism of Seifert manifold groups.

Example

Let $G = F_2 \times \mathbb{Z} = \langle a, b \rangle \times \langle t \rangle$ and

$$\phi \in \mathrm{Aut}(G) : a \mapsto at, b \mapsto b, t \mapsto t.$$

Then $\mathrm{Fix}\phi = \langle t, a^{-m}ba^m | m \in \mathbb{Z} \rangle$.

Fixed subgroups: Seifert manifold groups

Theorem (Z., 2013)

Suppose $f_\pi : \pi_1(M) \to \pi_1(M)$ *is an automorphism induced by an* **orientation-reversing** *homeomorphism* $f : M \to M$*. Then*

$$\mathrm{rkFix}(f_\pi) < 2\mathrm{rk}\pi_1(M).$$

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Fixed subgroups: hyperbolic 3-manifold groups

Theorem (Lin-Wang, 2012)

Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a compact orientable **hyperbolic 3-manifold** *with finite volume. Then*

$$\mathrm{rkFix}\phi < 2\mathrm{rk}G.$$

"$<$" is sharp. They also proved

Theorem (Lin-Wang, 2012)

There exists a sequence of automorphism $\phi_n : \pi_1(M_n) \to \pi_1(M_n)$ of closed hyperbolic 3-manifolds M_n such that $\mathrm{Fix}\phi_n$ is the group of a closed surface and

$$\frac{\mathrm{rkFix}\phi_n}{\mathrm{rk}\pi_1(M_n)} > 2 - \varepsilon \quad \text{as } n \to +\infty$$

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Inert, compressed and bounded

Definition

For a subgroup $A \leqslant G$,

- A is called inert in G, if for every subgroup $B \leqslant G$,

$$\mathrm{rk}(A \cap B) \leq \mathrm{rk} B.$$

- A is called compressed in G, if for every subgroup $A \leqslant B \leqslant G$,

$$\mathrm{rk} A \leq \mathrm{rk} B.$$

- A is called c-bounded in G, if

$$\mathrm{rk} A \leq c \cdot \mathrm{rk} G.$$

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Remark: Inert $\Longrightarrow$ Compressed $\Longrightarrow$ Bounded.

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Fixed subgroups of endomorphisms

For any family $B \subseteq \mathrm{End}(G)$,

Theorem (Bergman, 1999)

$\mathrm{Fix}\mathcal{B}$ *is* **bounded** *in* F_n.

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Is $\mathrm{Fix}\mathcal{B}$ **inert** in F_n?

Theorem (Martino-Ventura, 2004)

$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in* F_n.

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$\mathrm{Fix}\mathcal{B}$ *is* **compressed** *in any* **surface group**.

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Let $G = G_1 \times G_2 \times \cdots \times G_n$, each G_i is a f.g. free group or $\pi_1(S)$ for a closed surface S (maybe $\mathbb{R}P^2, 2\mathbb{R}P^2$ or a torus). We call it a product group.

Theorem A (Z.-Ventura-Wu, 2015)

$\mathrm{rkFix}\phi \leq \mathrm{rk}G$ for every $\phi \in \mathrm{Aut}(G)$
$\iff$ All G_i are of the same type (Euclidean or hyperbolic).

Euclidean type: $\mathbb{Z}$, $\pi_1(S)$ for $\chi(S) \geq 0$.
Hyperbolic type: F_n $(n > 1)$, $\pi_1(S)$ for $\chi(S) < 0$.

Example (NOT satisfying the conditions of Theorem A)

Let $G = F_2 \times \mathbb{Z} = \langle a, b \rangle \times \langle t \rangle$ and

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Fixed subgroups in product groups: most are bounded

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Fixed subgroups in product groups: few are compressed

Theorem B (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **compressed** in G for every $\phi \in \mathrm{Aut}(G)$, then G must be of one of the following forms:

(*euc1*) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or
(*euc2*) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or
(*euc3*) $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for some $p \geqslant 1$; or
(*euc4*) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$; or

(*hyp1*) $G = F_r \times NS_3^\ell$ for some $r \geqslant 2$, $\ell \geqslant 0$; or
(*hyp2*) $G = S_g \times NS_3^\ell$ for some $g \geqslant 2$, $\ell \geqslant 0$; or
(*hyp3*) $G = NS_k \times NS_3^\ell$ for some $k \geqslant 3$, $\ell \geqslant 0$.

Question 4

Is the implication in Theorem B an equivalence?

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Question 4

Is the implication in Theorem B an equivalence?

Example 1:

Let $G = F_2 \times F_2 = \langle t, u \rangle \times \langle a, b \rangle$, and

$$\phi \in \mathrm{Aut}(G) : t \mapsto t, u \mapsto tu, a \mapsto a, b \mapsto ab.$$

Then

$$\mathrm{Fix}\phi = \langle t, u^{-1}tu, a, b^{-1}ab \rangle \leqslant \langle t, bu, a \rangle.$$

Example 2:

Let $G = F_2 \times NS_4 = \langle t, u \rangle \times \langle a, b, c, d | aba^{-1}bcdc^{-1}d \rangle$, and

$$\phi \in \mathrm{Aut}(G) : t \mapsto t, u \mapsto tu, a \mapsto ab, b \mapsto b, c \mapsto cd, d \mapsto d.$$

Then

$$\mathrm{Fix}\phi = \langle t, u^{-1}tu, b, aba^{-1}, d, cdc^{-1} \rangle = \langle t, u^{-1}tu \rangle \times \langle b, aba^{-1}, d \rangle$$

but $\quad \mathrm{Fix}\phi \leqslant \langle t, au, b, d \rangle \cong F_4.$

Examples: fixed subgroups NOT compressed

Example 1:

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Examples: fixed subgroups NOT compressed

Example 3:

Let $G = NS_2 \times NS_2 \times \mathbb{Z}_2 = \langle a, b | aba^{-1}b \rangle \times \langle c, d | cdc^{-1}d \rangle \times \langle e | e^2 \rangle$,

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Then

$$\mathrm{Fix}\phi = \langle a, b^2, c^2, d, e \rangle \leqslant \langle a, bc, d, e \rangle$$

since $c^2 = a \cdot bc \cdot a^{-1} \cdot bc$ and $b^2 = bc \cdot bc \cdot c^{-2}$.

Examples: fixed subgroups NOT inert

Example 4:

Let $G = F_2 \times NS_3 = \langle t, u\rangle \times \langle a, b, c | aba^{-1}bc^2\rangle$, and

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Then

$$\mathrm{Fix}\phi = \langle t, u, aba^{-1}, b, c\rangle = \langle t, u\rangle \times \langle b, c\rangle.$$

Let $K = \langle at, u\rangle$. Then

$$\mathrm{Fix}\phi \cap K = \langle t^{-m}ut^{m} | m \in \mathbb{Z}\rangle$$

is infinite generated.

Theorem C (Z.-Ventura-Wu, 2015)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\phi$ is **inert** in G for every $\phi \in \mathrm{Aut}(G)$, then G is of one of the forms

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New progress: Compressedness in Euclid type

Proposition (Ventura-Wu-Z.)

If G is of form (euc3), i.e. $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for $p \geqslant 1$, then $\exists\ \phi \in \mathrm{Aut}(G)$, s.t. $\mathrm{Fix}\phi$ is NOT compressed, hence NOT inert.

Proof: Let $G = \langle a, b | bab^{-1}a\rangle \times \prod_{i=1}^{p}\langle c_i\rangle \times \langle d | d^2\rangle$ and $\phi \in \mathrm{Aut}(G)$:

$$a \mapsto ad,\ \ b \mapsto ba,\ \ c_1 \mapsto c_1 d,\ \ c_i \mapsto c_i^{-1}, (i = 2, ..., p),\ \ d \mapsto d.$$

$\Longrightarrow \mathrm{Fix}\phi = \langle a^2, b^2, ac_1, d\rangle \cong \mathbb{Z}^3 \times (\mathbb{Z}/2\mathbb{Z})$, while $\mathrm{Fix}\phi \leqslant \langle ac_1, b, d\rangle$.

Theorem B' (Ventura-Wu-Z.)

Let G be a product group of **Euclidean** type. Then, $\mathrm{Fix}\phi$ is **compressed** in G for every $\phi \in \mathrm{End}(G)$ $\Longleftrightarrow$ $\mathrm{Fix}\phi$ is **compressed** in G for every $\phi \in \mathrm{Aut}(G)$ $\Longleftrightarrow$ G is of one of the following forms:

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New progress: Compressedness in hyp. type

Proposition (5 factors)

If $G = G_1 \times \cdots \times G_5$, each G_i is $F_r (r \geq 2)$, $S_g (g \geq 2)$ or $NS_k (k \geq 3)$, then $\exists\ \phi \in \mathrm{Aut}(G)$ s.t. $\mathrm{Fix}\phi$ is NOT compressed in G.

Proof: Let $1 \neq h_i = [s_i, t_i] = s_i t_i s_i^{-1} t_i^{-1} \in G_i$, and

$$\phi_i \in \mathrm{Aut}(G_i): \ \ g \mapsto h_i g h_i^{-1}.$$

Then $\mathrm{Fix}\phi_i = \langle h_i \rangle$. Let $\phi = \phi_1 \times \cdots \times \phi_5 \in \mathrm{Aut}(G)$. Then

$$\mathrm{Fix}\phi = \langle s_1 t_1 s_1^{-1} t_1^{-1} \rangle \times \cdots \times \langle s_5 t_5 s_5^{-1} t_5^{-1} \rangle \cong \mathbb{Z}^5$$

while

$$\mathrm{Fix}\phi \leqslant \langle s_1 s_2 s_4, t_1 t_3 t_5, t_2 s_3, s_5 t_4 \rangle,$$

because

$$[s_1 s_2 s_4, t_1 t_3 t_5] = [s_1, t_1], \quad [s_1 s_2 s_4, t_2 s_3] = [s_2, t_2],$$

$$[t_2 s_3, t_1 t_3 t_5] = [s_3, t_3], \ [s_1 s_2 s_4, s_5 t_4] = [s_4, t_4], \ [s_5 t_4, t_1 t_3 t_5] = [s_5, t_5].$$

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New progress: Compressedness in hyp. type

Proposition (4 factors)

Let $G = G_0 \times NS_3^{\ell}(\ell \geq 3)$, G_0 is $F_r(r \geq 2)$, $S_g(g \geq 2)$ or $NS_k(k \geq 4)$. Then $\exists\ \phi \in \mathrm{Aut}(G)$ s.t. $\mathrm{Fix}\phi$ is NOT compressed in G.

Proof: For $i = 1, 2, 3$, $\exists \phi_i \in \mathrm{Aut}(G_i)$, s.t. $\mathrm{Fix}\phi_i = \langle s_i t_i s_i^{-1} t_i^{-1} \rangle$.

- $G_0 = F_r = \langle a_1, \ldots, a_r \rangle$, $\phi_0 \in \mathrm{Aut}(G_0) : a_1 \mapsto a_1 a_2$, $a_i \mapsto a_i$, $i \geq 2$, $\mathrm{Fix}\phi_0 = \langle a_2, a_1 a_2 a_1^{-1}, a_3, \ldots, a_r \rangle \cong F_r$. We have $\mathrm{Fix}\phi_0 \times \cdots \times \mathrm{Fix}\phi_3 \leqslant H = \langle a_2 a_1 s_1 s_2, s_3 t_3 t_1 t_2, s_3 s_1 t_2, a_1 t_3 t_1 s_2, a_3, \ldots,$
- $G_0 = S_g$ or NS_k, we can construct an analog $\phi_0 \in \mathrm{Aut}(G)$.

Let $\phi = \phi_0 \times \phi_1 \times \phi_2 \times \phi_3 \times Id \times \cdots \times Id \in \mathrm{Aut}(G)$. Then

$$\mathrm{Fix}\phi = \mathrm{Fix}\phi_0 \times \cdots \times \mathrm{Fix}\phi_3 \times G_4 \times \cdots \times G_\ell \leqslant H \times G_4 \times \cdots \times G_\ell.$$

But $$\mathrm{rk}\mathrm{Fix}\phi > \mathrm{rk}(H \times G_4 \times \cdots \times G_\ell).$$

New progress: Compressedness in hyp. type

Proposition (4 factors)

Let $G = G_0 \times NS_3^\ell (\ell \geq 3)$, G_0 *is* $F_r(r \geq 2)$, $S_g(g \geq 2)$ *or* $NS_k(k \geq 4)$. *Then* $\exists\ \phi \in \mathrm{Aut}(G)$ *s.t.* $\mathrm{Fix}\phi$ *is NOT compressed in* G.

Proof: For $i = 1, 2, 3$, $\exists \phi_i \in \mathrm{Aut}(G_i)$, s.t. $\mathrm{Fix}\phi_i = \langle s_i t_i s_i^{-1} t_i^{-1} \rangle$.

- $G_0 = F_r = \langle a_1, \ldots, a_r \rangle$, $\phi_0 \in \mathrm{Aut}(G_0) : a_1 \mapsto a_1 a_2$, $a_i \mapsto a_i$, $i \geq 2$, $\mathrm{Fix}\phi_0 = \langle a_2, a_1 a_2 a_1^{-1}, a_3, \ldots, a_r \rangle \cong F_r$. We have $\mathrm{Fix}\phi_0 \times \cdots \times \mathrm{Fix}\phi_3 \leqslant H = \langle a_2 a_1 s_1 s_2, s_3 t_3 t_1 t_2, s_3 s_1 t_2, a_1 t_3 t_1 s_2, a_3, \ldots,$
- $G_0 = S_g$ or NS_k, we can construct an analog $\phi_0 \in \mathrm{Aut}(G)$.

Let $\phi = \phi_0 \times \phi_1 \times \phi_2 \times \phi_3 \times Id \times \cdots \times Id \in \mathrm{Aut}(G)$. Then

$$\mathrm{Fix}\phi = \mathrm{Fix}\phi_0 \times \cdots \times \mathrm{Fix}\phi_3 \times G_4 \times \cdots \times G_\ell \leqslant H \times G_4 \times \cdots \times G_\ell.$$

But $$\mathrm{rkFix}\phi > \mathrm{rk}(H \times G_4 \times \cdots \times G_\ell).$$

New progress: Inertia in Euclid type

Proposition (Ventura-Wu-Z.)

Let $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$. Then $\exists \phi \in \mathrm{Aut}(G)$, s.t $\mathrm{Fix}\phi$ NOT inert in G.

Proof. For simple, let $G = \langle a, b | bab^{-1}a \rangle \times \langle c \rangle$ and $\phi \in \mathrm{Aut}(G)$: $a \mapsto a$, $b \mapsto ba$, $c \mapsto c$. $\Longrightarrow \mathrm{Fix}\phi = \langle a, b^2, c \rangle \cong \mathbb{Z}^3$, while

$$\mathrm{Fix}\phi \cap \langle ac, b \rangle = \langle ac, a^2, b^2 \rangle \cong \mathbb{Z}^3.$$

Theorem C' (Ventura-Wu-Z.)

Let G be a product group of **Euclidean** type. Then, $\mathrm{Fix}\phi$ is **inert** in G for every $\phi \in \mathrm{End}(G) \Longleftrightarrow \mathrm{Fix}\phi$ is **inert** in G for every $\phi \in \mathrm{Aut}(G) \Longleftrightarrow G$ is one of the following forms:

(*euc1*) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or

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Theorem D (Ventura-Wu-Z.)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, $\mathrm{Fix}\phi$ is **inert** in G for every $\phi \in \mathrm{Aut}(G) \iff G$ is one of the following forms:

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(*hyp1'*) $G = F_n$ for some $n \geq 2$; or

(*hyp2'*) $G = \pi_1(S)$ for some closed surface $\chi(S) < 0$.

Inertia Conjecture (Ventura-Wu-Z.)

Let G be a product group. Then, the following are equivalent:

1. Every $\phi \in \mathrm{End}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
2. Every $\phi \in \mathrm{Aut}(G)$ satisfies that $\mathrm{Fix}\phi$ is inert in G,
3. G is one of the forms (euc1),(euc2), (hyp1') or (hyp2').

Remark

$$(1) \xrightarrow{trivial} (2) \xleftrightarrow{Thm\ D} (3) \xrightarrow{???} (1),$$

$$\{(3) - (hyp1') - (hyp2')\} \xleftrightarrow{Thm\ C'} (1).$$

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谢　谢！　Thanks！