Bounds for fixed points on products of hyperbolic surfaces

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Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

$$\mathrm{Fix} f := \{x \in X | f(x) = x\} = \bigsqcup_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathbf{F}$$

Definition

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

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Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1).$$

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Commutation invariance

Suppose $\phi : X \to Y$ and $\psi : Y \to X$ are maps. Then $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are said to differ by a commutation. The map ϕ sets up a natural one-one correspondence

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from the fixed point classes of $\psi \circ \phi$ to the fixed point classes of $\phi \circ \psi$.

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Mutation invariance

Among selfmaps of compact polyhedra, homotopy and commutation generates an equivalence relation:

Definition

A sequence $\{f_i : X_i \to X_i | i = 0, \cdots, k\}$ of self-maps is a mutation if for each i, either

1. $X_{i+1} = X_i$ and $f_{i+1} \simeq f_i$, or
2. f_{i+1} is obtained from f_i by commutation.

A mutation sets up a one-one correspondence between fixed point classes of the end maps.

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The index $\mathrm{ind}(\mathbf{F})$ *is a mutation invariant.*

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Nielsen number & Lefschetz number

Definition

- A fixed point class **F** of f is essential if $\operatorname{ind}(f, \mathbf{F}) \neq 0$.
- Nielsen number $N(f) := \#\{\textit{essential fixed point classes of } f\}$.
- Lefschetz number

$$L(f) := \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

Lefschetz Fixed Point Theorem

$$\sum_{\mathbf{F} \in \mathrm{Fpc}(f)} \operatorname{ind}(f, \mathbf{F}) = \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

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Bounds for graphs and surfaces

Theorem (Jiang, 1998)

Let X be either a connected finite **graph***(not a tree) or a connected compact hyperbolic* **surface***, and $f : X \to X$ a* **selfmap***. Then*

1. $\mathrm{ind}(\mathbf{F}) \leq 1, \forall \mathbf{F} \in \mathrm{Fpc}(f)$;
2. *Almost all fixed point classes have index ≥ -1, in the sense*

$$\sum_{\mathrm{ind}(\mathbf{F})<-1} \{\mathrm{ind}(\mathbf{F}) + 1\} \geq 2\chi(X).$$

3. $|L(f) - \chi(X)| \leq N(f) - \chi(X)$.

Open problems

A compact polyhedron X is said to have the *Bounded Index Property (BIP)* if there is an integer $B > 0$ such that for any map $f : X \to X$ and any fixed point class **F** of f, the index $|\mathrm{ind}(f, \mathbf{F})| \leq B$. X has the *Bounded Index Property for Homeomorphisms (BIPH)* if there is such a bound for all homeomorphisms $f : X \to X$.

Question (Jiang, 1998)

Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

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Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

Positive examples

- [McCord, 1992]: Infra-solvmanifolds have BIP;
- [Jiang-Wang, 1992]: Closed aspherical 3-manifolds have BIPH for orientation preserving self-homeomorphisms;
- [Jiang, 1998; Jiang-Wang-Z., 2011]: Graphs & surfaces with $\chi < 0$ have BIP;
- [Z., 2012]: Orientable Seifert 3-manifolds with hyp. orbifold have BIPH;
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Bounds for products of hyperbolic surfaces

Suppose S_1 and S_2 are two connected compact surfaces with Euler characteristics $\chi_1 := \chi(S_1) \leq \chi_2 := \chi(S_2) < 0$, then $S_1 \times S_2$ has BIPH. More precisely,

Theorem (Z.-Zhao)

Let $f : S_1 \times S_2 \to S_1 \times S_2$ be a homeomorphism. Then the indices of the Nielsen fixed point classes of f are bounded:

1. *For every fixed point class* $\mathbf{F}$ *of f, we have*

$$2\chi_1 - 1 \leq \mathrm{ind}(f, \mathbf{F}) \leq (2\chi_1 - 1)(2\chi_2 - 1);$$

2. $|L(f) - 2\chi_1\chi_2| \leq (1 - 2\chi_1)N(f) + 2(\chi_1\chi_2 - \chi_1)$.

To prove the above Theorem, we first consider two good forms of selfmaps called **fiber-preserving maps** and **alternating homeomorphisms**, and then show that any homeomorphism f can be homotoped to one of the two good forms.

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Case 1: Fiber-preserving maps

Definition

A selfmap $f : S_1 \times S_2 \to S_1 \times S_2$ is called a fiber-preserving map, if

$$f = f_1 \times f_2 : S_1 \times S_2 \to S_1 \times S_2, \quad (a, b) \mapsto (f_1(a), f_2(b)),$$

where f_i is a selfmap of $S_i (i = 1, 2)$.

For any fiber-preserving map f, we have $\mathrm{Fix} f = \mathrm{Fix} f_1 \times \mathrm{Fix} f_2$, and each fixed point class $\mathbf{F}$ of f splits into a product of some fixed point classes of f_i, i.e.,

Lemma

1. $\mathbf{F} = \mathbf{F}_1 \times \mathbf{F}_2, \qquad \mathrm{ind}(f, \mathbf{F}) = \mathrm{ind}(f_1, \mathbf{F}_1) \cdot \mathrm{ind}(f_2, \mathbf{F}_2),$
where $\mathbf{F}_i$ is a fixed point class of f_i for $i = 1, 2$.
2. $L(f) = L(f_1) \cdot L(f_2), \quad N(f) = N(f_1) \cdot N(f_2).$

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By the Lemma above, we can show that the product $S_1 \times S_2$ has BIP for fiber-preserving maps.

Proposition (BIP for fiber-preserving maps)

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Let $S_1 = S_2$ be two copies of a connected compact hyperbolic surface S, and hence, their Euler characteristics $\chi_1 = \chi_2 = \chi(S) < 0$.

Definition

A self-homeomorphism $f : S_1 \times S_2 \to S_1 \times S_2$ is called an alternating homeomorphism, if

$$f = \tau \circ (f_1 \times f_2) : S_1 \times S_2 \to S_1 \times S_2$$

$$(a, b) \mapsto (f_2(b), f_1(a)),$$

where f_1, f_2 are two self-homeomorphisms of S, and τ is a transposition.

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induces an index-preserving one-to-one corresponding between the set $\mathrm{Fpc}(f_2 \circ f_1)$ of fixed point classes of $f_2 \circ f_1$ and the set $\mathrm{Fpc}(f)$ of fixed point classes of f.

Proof: Let $M = Df_1(a)$ and $N = Df_2(b)$ for $b = f_1(a)$. Then the differential $Df(a,b)$ of f at (a,b) is $\begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}$. Hence

$$\begin{aligned} \mathrm{ind}(f,(a,b)) &= sgn\det(I_4 - \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}) \\ &= sgn\det(I_2 - NM) \\ &= \mathrm{ind}(f_2 \circ f_1, a). \end{aligned}$$

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$$\begin{aligned} \mathrm{ind}(f,(a,b)) &= sgn\det(I_4 - \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix}) \\ &= sgn\det(I_2 - NM) \\ &= \mathrm{ind}(f_2 \circ f_1, a). \end{aligned}$$

Case 2: Alternating homeomorphisms

If $f = \tau \circ (f_1 \times f_2) : S_1 \times S_2 \to S_1 \times S_2$ is an alternating homeomorphism, then by the above lemma, we have

Lemma

$$N(f) = N(f_2 \circ f_1) = N(f_1 \circ f_2), \quad L(f) = L(f_2 \circ f_1) = L(f_1 \circ f_2).$$

Proposition (BIPH for alternating homeomorphisms)

1. $2\chi_1 - 1 \leq \mathrm{ind}(f, \mathbf{F}) \leq 1, \forall \mathbf{F} \in Fpc(f)$;
2. *Almost all fixed point classes have index* ≥ -1, *in the sense*
$$\sum_{\mathrm{ind}(f,\mathbf{F})<-1} \{\mathrm{ind}(f, \mathbf{F}) + 1\} \geq 2\chi_1;$$
3. $|L(f) - \chi_1| \leq N(f) - \chi_1$.

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Good forms of self-homeomorphisms of $S_1 \times S_2$

Recall that S_1 and S_2 be two compact hyperbolic surfaces.

Lemma (Z.-Ventura-Wu, 2015)

Let $G = \pi_1(S_1) \times \pi_1(S_2)$ and $\phi \in \mathrm{Aut}(G)$ be an automorphism. Then there exist automorphisms $\phi_i \in \mathrm{Aut}(\pi_1(S_i))$ such that ϕ must have one of the following forms:

1. *if $S_1 \ncong S_2$, then $\phi = \phi_1 \times \phi_2$;*
2. *if $S_1 \cong S_2$, then $\phi = \begin{cases} \phi_1 \times \phi_2 \\ \tau \circ (\phi_1 \times \phi_2) \end{cases}$,*
   where τ is a transposition.

Proposition

Let $f : S_1 \times S_2 \to S_1 \times S_2$ be a homeomorphism, where S_1, S_2 are two compact hyperbolic surfaces. Then

1. *if $S_1 \ncong S_2$, then f can be homotoped to a fiber-preserving homeomorphism $f_1 \times f_2$;*
2. *if $S_1 \cong S_2$, then f can be homotoped to either a fiber-preserving homeomorphism or an alternating homeomorphism.*

Proof: f homeomorphism $\Longrightarrow f_\pi = \phi_1 \times \phi_2$ or $f_\pi = \tau \circ (\phi_1 \times \phi_2)$, where $\phi_i \in \mathrm{Aut}(\pi_1 S_i)$. By Dehn-Nielsen-Bar Thm for hyperbolic surfaces, ϕ_i can be induced by a self-homeomorphism f_i of S_i. Hence

$$f_\pi = (f_1 \times f_2)_\pi \quad or \quad f_\pi = \big(\tau \circ (f_1 \times f_2)\big)_\pi .$$

S_i hyperbolic $\Longrightarrow S_1 \times S_2$ aspherical $\Longrightarrow f \simeq \begin{cases} f_1 \times f_2 \\ \tau \circ (f_1 \times f_2) \end{cases}.$ $\square$

Good forms of self-homeomorphisms of $S_1 \times S_2$

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Good forms of self-homeomorphisms of $S_1 \times S_2$

Proposition

Let $f : S_1 \times S_2 \to S_1 \times S_2$ be a homeomorphism, where S_1, S_2 are two compact hyperbolic surfaces. Then

1. *if $S_1 \not\cong S_2$, then f can be homotoped to a fiber-preserving homeomorphism $f_1 \times f_2$;*
2. *if $S_1 \cong S_2$, then f can be homotoped to either a fiber-preserving homeomorphism or an alternating homeomorphism.*

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Thanks ！ 谢 谢！