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Advances in Unified Strength Theory and its Generalization

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Abstract

It has been two decades since the first presentation of "A new model and theory on yield and failure of materials under complex stress state" at ICM-6 held at Kyoto, Japan in 1991. The twin-shear element model and a new strength theory were proposed at ICM-6. However, only the two equations were introduced, the characteristics and its applications of this strength theory have not been studied in details. A great deal of researches on this new strength theory and its applications are developed since then by Yu and other scholars at other Universities and Institutes in some countries. Some behaviour of the unified strength theory are described here. The advances in the unified strength theory and its applications are summarized briefly in the framework of continuum mechanics and engineering application.

Keywords: Unified Strength Theory; Unified Yield Criterion; twin shear element

1. Mechanical Model of the Unified Strength Theory

Mechanical model and mathematical modelling are powerful means for establishing and understanding the development of a new theory. Mechanical modelling is an abstraction, a formation of an idea or ideas that may involve the subject with special configurations. Mathematical modelling may involve relationships between continuous functions of space, time and other variations (Tayler 1986; Meyer 1985; Besseling and van der Liesen 1994; Wu, Lahav and Rees 1999).

To express the general nature of the strength theory, the cubic element is often used. It is clear that there are three principal stresses σ_1, σ_2 and σ_3 acting on the cubic element. The regular octahedral element

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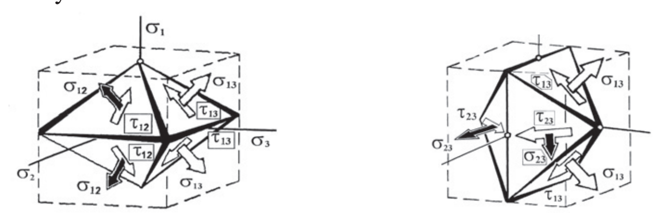
was proposed to introduce the Huber-von Mises yield criterion by Ros-Richinger-Nadai (1926-1931, see: Zyczkowski 1981). The Tresca-Mohr–Coulomb strength theory is used widely, however, the effect of the intermediate principal stress σ_2 and the effect of intermediate principal shear stress (τ_{12} or τ_{23}) are not taken into account in the Tresca-Mohr–Coulomb strength theory.

The twin-shear stress element and multi-shear element are proposed. The stress state (σ_1, σ_2, σ_3) was converted into the principal shear-stress state (τ_{13}, τ_{12}, τ_{23}) by Yu (Yu 1983, 1985, 1991) as shown in Fig.1.

Based on the orthogonal octahedral element and pentahedron element, a new unified strtength theory (yield criterion) can be developed (Yu 1991, 2004). The twin-shear orthogonal octahedral model is different from the regular octahedral model. The orthogonal octahedral model consists of two groups of four sections that are perpendicular to each other and are acted on by the maximum shear stress τ_{13} and the intermediate principal shear stress τ_{12} or τ_{23}.

It is worth noticing that the orthogonal octahedral model can be subjected to an affinity deformation but remain a parallehedron, which fills the space without gaps or overlapping. The orthogonal octahedral model, like the cubic element, is also a spatial equipartition, which consists of completely filling a volume with polyhedra of the same kind. The combination of many orthogonal octahedral models can be used as a continuous body. Obviously, the twin-shear pentahedron element is also a spatial equipatition. The effect of intermediate principal shear-stress (τ_{12} or τ_{23}) can be taken into account naturally in the mathematical modelling of strength theory.



(a) twin-shear stress (τ_{13}, τ_{12}, σ_{13}, σ_{12}) (b) twin-shear stress (τ_{13}, τ_{23}, σ_{13}, σ_{23})

Fig. 1. Principal stress state (σ_1, σ_2, σ_3) convert into the principal shear-stress state.

2. Mathematical Modelling and the Determination of the Material Parameters of the Unified Strength Theory

It is clear that there are three principal shear stresses τ_{13}, τ_{12} and τ_{23} in the three-dimensional principal stress state σ_1, σ_2 and σ_3. However, only two principal shear stresses are independent variables among τ_{13}, τ_{12} , τ_{23} because the maximum principal shear stress equals the sum of the other two, that is, $\tau_{13}=\tau_{12}+\tau_{23}$.

Since there are only two independent principal shear stresses, the shear stress state can also be converted into the twin-shear stress state (τ_{13}, τ_{12}, σ_{13}, σ_{12}) or (τ_{13}, τ_{23}, σ_{13}, σ_{23}). Considering all the stress components acting on the twin-shear element and the different effects of various stresses on the yield of materials, the mathematical modelling of the unified strength theory was proposed by Yu in 1991. It can be expressed as follows (Yu and He 1991; Yu 2004):

$$F=\tau_{13}+b\tau_{12}+\beta(\sigma_{13}+b\sigma_{12})=C, \quad \text{when} \quad \tau_{12}+\beta\sigma_{12}\geq\tau_{23}+\beta\sigma_{23} \tag{1a}$$

$$F'=\tau_{13}+b\tau_{23}+\beta(\sigma_{13}+b\sigma_{23})=C, \quad \text{when} \quad \tau_{12}+\beta\sigma_{12}\leq\tau_{23}+\beta\sigma_{23} \tag{1b}$$

The unified strength theory assumes that the yielding of materials begins when the sum of the two larger principal shear stresses and the corresponding normal stress function reaches a magnitude C. Where b is a parameter that reflects the influence of the intermediate principal shear stress τ_{12} or τ_{23} on the yield; β is the coefficient that represents the effect of the normal stress on the yield; C is a strength parameter.

The magnitude of parameters β and C can be determined by uniaxial tension strength σ_t and uniaxial compression strength σ_c , So the material constants β and C can be determined as follows:

$$\beta=\frac{\sigma_c-\sigma_t}{\sigma_c+\sigma_t}=\frac{1-\alpha}{1+\alpha},\quad C=\frac{2\sigma_c\sigma_t}{\sigma_c+\sigma_t}=\frac{2}{1+\alpha}\sigma_t \tag{2}$$

3. Mathematical Expression of the Unified Strength Theory

Substituting β and C into the Eq. (1a) and (1b), the unified strength theory is now obtained. It can be expressed in terms of the three principal stresses as follows:

$$F=\sigma_1-\frac{\alpha}{1+b}(b\sigma_2+\sigma_3)=\sigma_t,\ \text{when}\quad \sigma_2\le\frac{\sigma_1+\alpha\sigma_3}{1+\alpha} \tag{3a}$$

$$F'=\frac{1}{1+b}(\sigma_1+b\sigma_2)-\alpha\sigma_3=\sigma_t,\ \text{when}\quad \sigma_2\ge\frac{\sigma_1+\alpha\sigma_3}{1+\alpha} \tag{3b}$$

Equations (3a) and (3b) are the mathematical formula of the unified strength theory. *b* is a choice parameter of the yield criteria, it may be referred as the unified strength theory parameter. *b* is also a parameter reflecting the effect of the intermediate principal stress σ_2. The relationship among shear strength τ_0, the uniaxial tensile strength σ_t, uniaxial compressive strength σ_c and unified strength theory parameter *b* can be determined as follows:

$$b=\frac{(1+\alpha)\tau_0-\sigma_t}{\sigma_t-\tau_0},\quad \alpha=\frac{\sigma_t}{\sigma_c} \tag{4}$$

The ratio of shear strength to tensile strength of materials can be introduced as follows:

$$\alpha_\tau=\frac{\tau_0}{\sigma_t}=\frac{1+b}{1+b+\alpha} \tag{5}$$

It is shown that:

1. The ratio of shear strength to tensile strength $\alpha_\tau=\tau_0/\sigma_t$ of brittle materials ($\alpha_\tau<1$) is lower than that of ductile materials ($\alpha_\tau=1$). This agrees with the experimental data.
2. The yield surface may be non-convex when the ratio of shear strength to tensile strength $\alpha_\tau<1/(1+\alpha)$ or $\alpha_\tau>2/(2+\alpha)$.
3. The shear strength of the material is lower than the tensile strength of material. This is true for non-SD materials. It needs, however, further study for SD materials.

A great number of new failure criteria can be generated from the unified strength theory by changing α and *b*. A unified yield criteria can be deduced from the unified strength theory when α=1, as follows.

$$f=\sigma_1-\frac{1}{1+b}(b\sigma_2+\sigma_3)=\sigma_y \tag{6a}$$

$$f'=\frac{1}{1+b}(\sigma_1+b\sigma_2)-\sigma_3=\sigma_y \tag{6b}$$

4. Yield Surfaces of the UST

The yield surfaces in stress space of the unified strength theory are usually a semi-infinite hexagonal cone with unequal sides and a dodecahedron cone with unequal sides. The shape and size of the yield hexagonal cone depends on the parameter *b* and on the tension-compression strength ratio α. The 3D computer images of yield surface for the unified strength theory in the stress space is shown in Fig. 2.

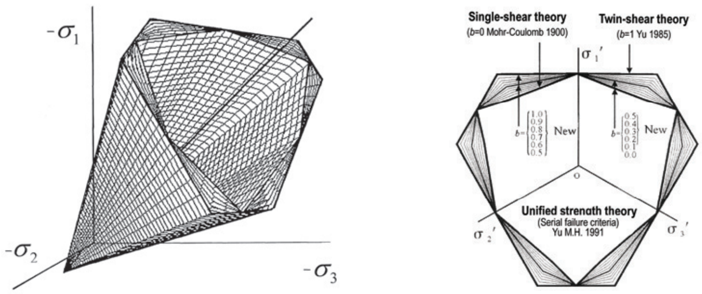


Fig. 2. Yield surfaces of the unified strength theory in stress space

The SD effect, hydrostatic stress effect, normal stress effect, effect of the Intermediate principal stress, the effect of intermediate principal shear stress and the effect of the twin-shear stresses are all taken into account in the unified strength theory.

The variation of yield surfaces of the unified strength theory (UST) in deviatoric plane can be also expressed as Fig.3 with α and $k=\sqrt{3}\dfrac{1+b}{1+b+\alpha}$. It is given by Kolupaev and Altenbach (2009, 2010).

The lower bound is provided by the single-shear strength theory (the Mohr-Coulomb strength theory, or the unified strength theory with b=0). The upper bound is given by the generalized twin-shear strength theory (Yu et al. 1985), or the unified strength theory with b=1. The median is a new series of yield criteria deduced from UST with b=1/2. Other series of new yield criteria can also be deduced from UST with b=1/4 or b=3/4, as shown in Fig.3. All the convex yield surfaces are situated between two bounds of the twin-shear theory and the single-shear theory, as shown in Fig.3 (Altenbach H and Kolupaev VA 2008; Kolupaev and Altenbach 2009, 2010; Yu, Xia and Kolupaev 2009).

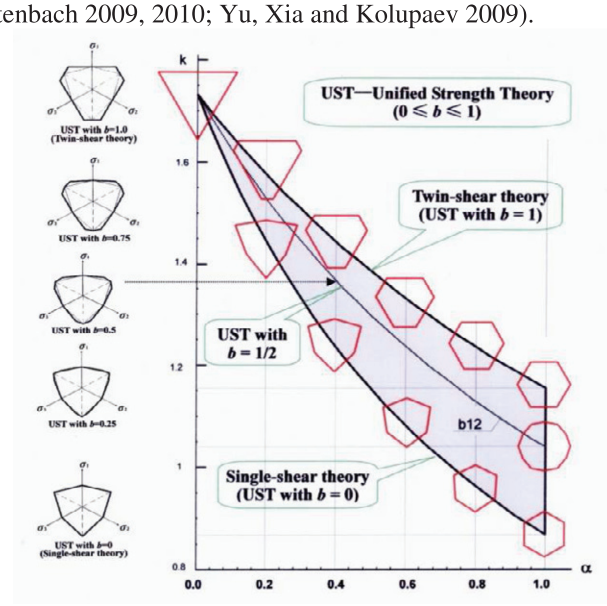


Fig.3 Variation of the shape of yield loci of the unified strength theory with parameters b(k) and α

5. Generalization and Applications of the Unified Strength Theory

The unified strength theory can be extended from convex loci to non-convex loci. The yield loci of UST are convex when yield criterion parameter b ($0 \le b \le 1$), and the yield loci of UST are non-convex when yield criterion parameter $b< 0$ or $b>1$. The behavior, however, has not studied so far. It may be a mathematical signification. Some comments and reviews are made by Shen (2004), Teodorescu (2006), Qian and Qi (2008), Altenbach and Kolupaev (2008), Kolupaev and Altenbach (2009, 2010) et al.

The advances in strength theories are briefly illustrated in Fig.4. It shows the development from Single-Shear theory to Three-Shear theory, then from Twin-Shear theory to the unified strength theory (a set of the serial criteria).

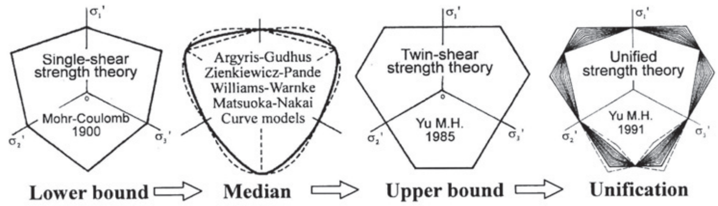


Fig. 4. Advances in strength theories

The unified strength theory is a completely new theory system. A series of new yield surfaces and yield loci can be deduced from the unified strength theory. The unified strength theory forms an entire spectrum of convex and nonconvex criteria, which can be used to describe many kinds of engineering materials. The significance of the Yu's unified strength theory is summarized as follows:

1. It provides a system of yield and failure criteria adapted for most materials, from metallic materials to rocks, concretes, soils, polymers, etc. It is suitable for more kinds of isotropic materials.
2. It gives a relation among the single-shear criterion, the twin-shear criterion, and a series of new criteria. It gives good agreement with experimental results for various materials.
3. The unified strength theory has been applied successfully to analyze the elastic limit, plastic limit capacities, and the dynamic response for some structures under static and moderate impulsive load. A series of new results can be obtained by using the unified strength theory (Yu 2004).
4. The unified strength theory is generalized to unified slip field for plane strain problems and unified characteristics field for plane stress and axisymmetric plastic problems. These new results are summarized in the book published by Springer in 2006 (Yu 2006).
5. The unified strength theory is easy to use for analytical solutions of plastic problems. Limit, Shakedown and Dynamic Plastic Analyses of Structures by using the unified strength theory are given in the book published by Springer and ZJU Pressin 2009 (Yu et al. 2009).
6. It may be implemented in some finite element codes, The unified strength theory is also easy to use for numerical solutions of plastic problems. Computational Plasticity: with emphasis on the Application of the Unified Strength Theory is published by Springer and ZJU Press in 2011 (Yu et al. 2011).

Acknowledgements

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