



# 自然对流控制方程及其 无量纲化方法

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# 自然对流

自然对流:

由于流体内部存在温度差, 使得各部分流体的密度不同, 温度高的流体密度小, 必然上升; 反之, 从而引起流体内部的流动, 为自然对流, 这种没有外部抵抗力的作用, 仅靠流体内部的温度差, 而使流体流动, 从而产生的换热现象, 称为自然对流传热。

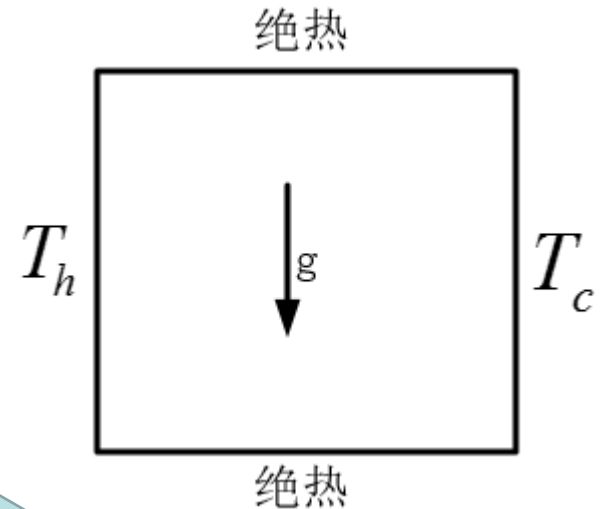
控制方程:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g_y$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\lambda}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$



浮力项

浮力项是密度的函数:  $g_y = f(\rho)$





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## Boussinesq假设:

- 1) 除密度外，其他物性为常数。（适用温差约30-50℃）
- 2) 对密度，仅考虑动量方程中与体积力有关的项，其余各项中密度为常数。

$$p = p_0 + p'$$

$p_0 \rightarrow$  温度为  $T_0$  时，静止压力  
 $p' \rightarrow$  流动发生时的压力变动。

静止时:  $u = v = 0; p = p_0, p' = 0.$

将其代入动量方程:

$$0 = \frac{1}{\rho_0} \frac{\partial p_0}{\partial x} \quad \rho_0 \rightarrow T_0 \text{ 时流体的密度}$$

$$0 = \frac{1}{\rho_0} \frac{\partial p_0}{\partial y} + g_y$$

流动时:  $u \neq 0, v \neq 0$

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{1}{\rho} \left( \frac{\partial p_0}{\partial x} + \frac{\partial p'}{\partial x} \right) = -\frac{1}{\rho} \frac{\partial p'}{\partial x} \quad \textcircled{1}$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} + g_y = g_y - \frac{1}{\rho} \left( \frac{\partial p_0}{\partial y} + \frac{\partial p'}{\partial y} \right) = g_y - \frac{\rho_0}{\rho} g_y - \frac{1}{\rho} \frac{\partial p'}{\partial y} \quad \textcircled{2}$$





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此外，对于流体的密度，当  $T \approx T_0$  时，有如下关系式成立：

$$\frac{1}{\rho} = \frac{1 + \beta_0(T - T_0)}{\rho_0} \quad \textcircled{3} \quad \beta_0 \rightarrow T = T_0 \text{ 时，流体的膨胀系数。}$$

$$\text{即：} \beta/T_0 = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p \Big|_{T_0} = \frac{1}{1/\rho} \frac{\partial}{\partial T} \left( \frac{1}{\rho} \right) \Big|_{T_0} \approx \rho_0 \frac{\beta_0}{\rho_0} = \beta_0$$

将③代入②得：

$$g_y - \frac{1}{\rho} \frac{\partial p}{\partial y} = g_y - [1 + \beta(T - T_0)] g_y - \frac{1}{\rho} \frac{\partial p'}{\partial y} = -g_y \beta(T - T_0) - \frac{1}{\rho} \frac{\partial p'}{\partial y} \quad \textcircled{4}$$

将④代入原动量方程，得：

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p'}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g_y \beta_0 (T - T_0)$$



将  $p'$  记作  $p$ ， $T_0$ 、 $\beta_0$  为参考点之值，在本例中可分别为  $\beta_a, T_a = \frac{T_c + T_h}{2}$





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原方程组化为：

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g_y \beta_a (T - T_a)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\lambda}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

无量纲化：

$$X = \frac{x}{x^*}, Y = \frac{y}{y^*}, x^* = y^* = L^*$$

$$U = \frac{u}{u^*}, V = \frac{v}{u^*}, P = \frac{p}{p^*}$$

$$\tau = \frac{t}{t^*}, T = \frac{T - T^{**}}{T^*}$$

构造  $T^{**}$  : 当  $T = T_h$  时,  $T = 0.5$

当  $T = T_c$  时,  $T = -0.5$

或构造  $T^{**}$  : 当  $T = T_h$  时,  $T = 1$

当  $T = T_c$  时,  $T = 0$

其中:  $T^* = T_h - T_c, T^{**} = \frac{T_h + T_c}{2}$





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代入上述方程组，得：

$$\frac{u^*}{L^*} \frac{\partial U}{\partial X} + \frac{u^*}{L^*} \frac{\partial V}{\partial Y} = 0 \Rightarrow \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\frac{u^*}{t^*} \frac{\partial U}{\partial \tau} + \frac{u^{*2}}{L^*} U \frac{\partial U}{\partial X} + \frac{u^{*2}}{L^*} V \frac{\partial U}{\partial Y} = -\frac{1}{\rho} \frac{p^*}{L^*} \frac{\partial P}{\partial X} + \frac{\nu u^*}{L^{*2}} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right)$$

两边同乘  $\frac{L^*}{U^{*2}}$ ，得：

$$\boxed{\frac{L^*}{t^* u^*}} \frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\boxed{\frac{1}{\rho} \frac{p^*}{u^{*2}}} \frac{\partial P}{\partial X} + \boxed{\frac{\nu}{L^* u^*}} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right)$$

Y方向上：

$$-g_y \beta_a (T - T_a) = -g_y \beta_a [(TT^* + T^{**}) - (T_a T^* + T^{**})]$$

$$\because g_y = -g \Rightarrow \text{上式} = g \beta_a T^* (T - T_a)$$

$$T_a = (T_h + T_c) / 2 = (0.5 - 0.5) / 2 = 0, \text{ 记 } \beta_a = \beta$$





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Y方向上变为:

$$\boxed{\frac{L^*}{t^* u^*}} \frac{\partial V}{\partial \tau} + U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = - \boxed{\frac{1}{\rho u^{*2}} p^*} \frac{\partial P}{\partial X} + \boxed{\frac{\nu}{L^* u^*}} \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \boxed{\frac{L^*}{u^{*2}} g \beta T^*} T$$

能量方程:

$$\frac{\partial(TT^* + T^{**})}{t^* \partial \tau} + \frac{u^* T^*}{L^*} U \frac{\partial T}{\partial X} + \frac{u^* T^*}{L^*} V \frac{\partial T}{\partial Y} = \frac{\alpha T^*}{L^{*2}} \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right)$$

两边同乘  $\frac{L^*}{U^* T^*}$ , 得:

$$\boxed{\frac{L^*}{t^* u^*}} \frac{\partial T}{\partial \tau} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \boxed{\frac{\alpha}{L^* u^*}} \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right)$$

$$\text{Pr} = \frac{\text{粘性扩散}}{\text{热扩散}}$$

$$\textcircled{1} = 1 \Rightarrow L^* = t^* u^*$$

$$\textcircled{5} = 1 \Rightarrow u^* = \frac{\alpha}{L^*} = \frac{\alpha}{H}$$

$$\textcircled{2} = 1 \Rightarrow p^* = \rho u^{*2}$$

$$\textcircled{3} = \frac{\nu}{L^* u^*} = \frac{\nu}{H \frac{\alpha}{H}} = \frac{\nu}{\alpha} = \text{Pr}$$

$$t^* = \frac{L^*}{u^*} = \frac{H^2}{\alpha}$$

$$p^* = \rho \left( \frac{\alpha}{H} \right)^2$$

表明温度边界层与流动边界层的关系，反应流体物理性质对流动传热过程的影响。





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$$\textcircled{4} \frac{L^*}{u^{*2}} g \beta T^* = \frac{H}{\alpha^2 / H^2} g \beta (T_h - T_c) = \frac{g \beta (T_h - T_c) H^3 \nu}{\alpha \cdot \alpha} = \frac{g \beta (T_h - T_c) H^3 \nu}{\alpha \cdot \nu} = Ra \cdot Pr = Gr \cdot Pr^2$$

$Ra$ ：瑞利数，自然对流传热中，传热系数关联的无量纲参数。

$$Ra = \frac{g \beta (T_h - T_c) H^3}{\alpha \nu}$$

$Ra = Gr \cdot Pr \rightarrow Gr$  为格拉晓夫数

$$Gr = \frac{g \beta (T_h - T_c) H^3}{\nu^2}$$

，浮力与粘性力的比值，自然对流重要参数。

无量纲化结果：

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + Pr \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right)$$

$$\frac{\partial V}{\partial \tau} + U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial Y} + Pr \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + Gr Pr^2 T$$

$$\frac{\partial T}{\partial \tau} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right)$$







# 自然对流

边界条件:

$$\begin{aligned}
 X = 0, \quad T = 0.5, \quad U = V = 0 \\
 X = 1, \quad T = -0.5, \quad U = V = 0 \\
 Y = 0, 1 \quad \frac{\partial T}{\partial Y} = 0, \quad U = V = 0
 \end{aligned}$$

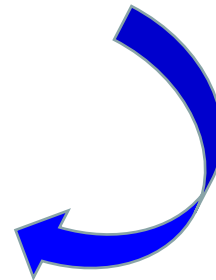


两点说明:

(1) 若  $T_c = 0, T_h = 1$ , 则  $Gr Pr^2 (T - 0.5)$

(2) 若令  $\textcircled{3} = 1, u^* = \frac{\nu}{H}$

$$\textcircled{5} \text{式} = \frac{\alpha}{u^* L^*} = \frac{\alpha}{\frac{\nu}{H} H} = \frac{\alpha}{\nu} = \frac{1}{Pr}$$



$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right)$$

$$\frac{\partial V}{\partial \tau} + U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial Y} + \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + GrT$$

$$\frac{\partial T}{\partial \tau} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \frac{1}{Pr} \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right)$$

