An efficient finite difference/Hermite–Galerkin spectral method for time-fractional coupled sine–Gordon equations on multidimensional unbounded domains and its application in numerical simulations of vector solitons

Shimin Guo, Liquan Mei, Yanren Hou, Zhengqiang Zhang

School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, 710049, China

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

ARTICLE INFO

Article history:
Received 9 April 2018
Received in revised form 16 September 2018
Accepted 16 November 2018
Available online 29 November 2018

Keywords:
Coupled sine–Gordon equations
Caputo fractional derivative
Hermite polynomial/function
Unbounded domain

ABSTRACT

This study is devoted to the numerical simulation of vector solitons described by the time-fractional coupled sine–Gordon equations in the sense of Caputo fractional derivative, where the problem is defined on the multidimensional unbounded domains \( \mathbb{R}^d (d = 2, 3) \). For this purpose, we employ the Hermite–Galerkin spectral method with scaling factor for the spatial approximation to avoid the errors introduced by the domain truncation, and we apply the finite difference method based on the Crank–Nicolson method for the temporal discretization. Comprehensive numerical studies are carried out to verify the accuracy and the stability of our method, which shows that the method is convergent with \( (3 - \max\{\alpha_1, \alpha_2\}) \)-order accuracy in time and spectral accuracy in space. Here, \( \alpha_i (1 < \alpha_i < 2, i = 1, 2) \) are the orders of the Caputo fractional derivative. In addition, the effect of the Caputo fractional derivative on the evolutions of the vector solitons is numerically studied. Finally, several numerical simulations for both two- and three-dimensional cases of the problem are performed to illustrate the robustness of the method as well as to investigate the collisions of circular and elliptical ring vector solitons.

© 2018 Elsevier B.V. All rights reserved.

1. Introduction

In this article, we focus on the following time-fractional coupled sine–Gordon equations on multidimensional unbounded domains

\[
\begin{align*}
\frac{\mathcal{C}_t^{\alpha_1} u_1(x, t)}{\partial t} - \Delta u_1(x, t) &= -\beta^2 \sin(u_1(x, t) - u_2(x, t)), \quad x \in \mathbb{R}^d, \quad 0 < t \leq T, \quad (a) \\
\frac{\mathcal{C}_t^{\alpha_2} u_2(x, t)}{\partial t} - c^2 \Delta u_2(x, t) &= \sin(u_1(x, t) - u_2(x, t)), \quad x \in \mathbb{R}^d, \quad 0 < t \leq T, \quad (b)
\end{align*}
\]

subject to the following initial and boundary conditions

\[
\begin{align*}
u_1(x, 0) &= \phi_1(x), \quad \frac{\partial u_1(x, 0)}{\partial t} = \psi_1(x), \quad x \in \mathbb{R}^d, \quad (a) \\
u_2(x, 0) &= \phi_2(x), \quad \frac{\partial u_2(x, 0)}{\partial t} = \psi_2(x), \quad x \in \mathbb{R}^d, \quad (b) \\
\lim_{|x| \to \infty} u_1(x, t) &= \lim_{|x| \to \infty} u_2(x, t) = 0, \quad 0 \leq t \leq T, \quad (c)
\end{align*}
\]
where $\Delta$ is the Laplacian, $x = (x_1, \ldots, x_d)$ stands for the multivariables, $d (= 2, 3)$ denotes the dimension of the space, and $\delta D_\alpha^\beta u(x, t)$ ($i = 1, 2$) are the Caputo fractional derivatives of order $\alpha_i$ expressed by

$$\delta D_\alpha^\beta u(x, t) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{\partial^\beta u(x, s)}{\partial s^{\beta}} \frac{ds}{(t - s)^{\alpha - 1}}, 1 < \alpha_i < 2. \tag{3}$$

As the generalization of the Frenkel–Kontorova dislocation model where the shear of one part of a crystal is considered with respect to the rigid base [1], the coupled sine–Gordon equations are derived under the assumption that the parts of a crystal are deformable [2–4]. Nowadays, this model becomes one of the basic equations in the modern nonlinear wave theory. In (1)–(1)(b), the constant $c$ is the ratio of the acoustic velocities of the components $u_1$ and $u_2$, and the parameter $\beta$ stands for the ratio of masses of particles in the “lower” and the “upper” parts of the crystal [2]. In the case of $\alpha_1 = \alpha_2 = 2$, Eqs. (1)(a)–(1)(b) reduce to the classical integer-order coupled sine–Gordon equations as in Refs. [2,5]. For this special case, the system (1)(a)–(1)(b) has some interesting physical features: (i) The system (1)(a)–(1)(b) is Lagrangian with the density [2]

$$L = \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial t} \right)^2 + \left( \frac{\partial u_2}{\partial t} \right)^2 - |\nabla u_1|^2 - c^2 |\nabla u_2|^2 \right] + \cos(\beta u_1 - u_2) - 1. \tag{ii}$$

(ii) The system (1)(a)–(1)(b) has the following conservation law for energy [5]

$$\frac{\partial}{\partial t} \left[ \frac{1}{2} \left( \frac{\partial u_1}{\partial t} \right)^2 + \left( \frac{\partial u_2}{\partial t} \right)^2 + |\nabla u_1|^2 + c^2 |\nabla u_2|^2 \right] - \nabla \cdot \left[ \frac{\partial u_1}{\partial t} \nabla u_1 + c^2 \frac{\partial u_2}{\partial t} \nabla u_2 \right] = 0.$$
numerical simulations of the collisions of vector solitons for the two-dimensional (respectively three-dimensional) case of the problem (1)(a)–(2)(c) are carried out. Concluding remarks are given in Section 5.

2. Numerical method

2.1. Temporal discretization

In this subsection, we show the temporal discretization of problem (1)(a)–(2)(c) by using the finite difference method based on the Crank–Nicholson method. For a given positive integer \( n_f \), let \( \tau = T/n_f \), \( t_m = m\tau (0 \leq m \leq n_f) \). Clearly, the time domain \([0, T]\) is covered by \([t_m | 0 \leq m \leq n_f]\). For a given grid function \([u^n | 0 \leq n \leq n_f]\), we define

\[
\begin{align*}
\frac{u_{n}^{n+1} - u_{n}^{n}}{\tau} &= \frac{u_{n+1}^{n} - u_{n}^{n}}{\tau} \quad \text{and} \quad \frac{u_{n}^{n+1} - u_{n}^{n-1}}{\tau} = \frac{u_{n+1}^{n} - u_{n}^{n}}{\tau}.
\end{align*}
\]

Now, we present the following two lemmas which will be used for the temporal discretization of problem (1)(a)–(2)(c).

**Lemma 2.1** ([39]). Suppose \( 1 < \alpha < 2 \), \( g(t) \in C^2[0, t_0] \). It holds that

\[
\begin{align*}
\frac{1}{\Gamma(2 - \alpha)\alpha} &\int_0^\infty \frac{g^{(s)}(t)}{(t_0 - s)^{\alpha-1}} ds - \frac{t^{1-\alpha}}{\Gamma(3 - \alpha)} \left[ a_0 u_{n-1}^{n-1} - \sum_{k=1}^{n-1} (a_{n-1-k} - a_n) u_{n-1-k}^{n-1} g(0) \right] \\
&\leq \frac{1}{\Gamma(3 - \alpha)} \left[ \frac{2 - \alpha}{12} + \frac{2^{3-\alpha}}{3 - \alpha} \right] \max_{0 \leq t \leq t_0} |g''(t)| t^{3-\alpha},
\end{align*}
\]

where \( a_k = (k + 1)^{2-\alpha} - k^{2-\alpha}, k \geq 0 \).

**Lemma 2.2** ([40]). Let \( g(t) \in C^3[t_{k-1}, t_k], t_{k-1/2} = t_k - \frac{\tau}{2} \). It holds that

\[
\frac{1}{2} \left| g(t_k) + g(t_{k-1}) - \frac{1}{\tau} [g(t_k) - g(t_{k-1})] + \frac{\tau^2}{16} \int_0^1 \left[ g^{(s)} \left( t_{k-1/2} + \frac{s\tau}{2} \right) + g^{(s)} \left( t_{k-1/2} - \frac{s\tau}{2} \right) \right] (1 - s^2) ds \right|
\]

Let \( \rho_i(x, t) = \frac{\partial u_i(x, t)}{\partial t} \) and \( \theta_i(x, t) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{\partial \rho_i(x, s)}{\partial s} \frac{ds}{(t - s)^{\alpha-1}} \) \((i = 1, 2)\), we transform (1)(a)–(1)(b) into

\[
\begin{align*}
\frac{\partial \theta_1(x, t)}{\partial t} - \Delta u_1(x, t) &= -\beta^2 \sin(u_1(x, t) - u_2(x, t)), \quad x \in \mathbb{R}^d, \quad 0 < t < T, \quad (a) \\
\frac{\partial \theta_2(x, t)}{\partial t} + c^2 \Delta u_2(x, t) &= \sin(u_1(x, t) - u_2(x, t)), \quad x \in \mathbb{R}^d, \quad 0 < t < T. \quad (b)
\end{align*}
\]

Define

\[
\rho_i^n = \rho_i(x, t_n), \quad \theta_i^n = \theta_i(x, t_n), \quad u_i^n = u_i^n(x, t_n), \quad i = 1, 2.
\]

Using Taylor formula, it follows from (4)(a) and (4)(b) that

\[
\begin{align*}
\theta_1^{n+1/2} - \Delta u_1^{n+1/2} &= -\beta^2 G^{n+1/2} + r_1^{n+1/2}, \quad (a) \\
\theta_2^{n+1/2} + c^2 \Delta u_2^{n+1/2} &= G^{n+1/2} + r_2^{n+1/2}, \quad (b)
\end{align*}
\]

where

\[
G^{n+1/2} = \frac{1}{2} \left( \sin(u_1^n - u_2^n) + \sin(u_1^{n+1} - u_2^{n+1}) \right), \quad r_1^{n+1/2} = O(\tau^2), \quad r_2^{n+1/2} = O(\tau^2).
\]

Considering Lemma 2.1, we have

\[
\theta_i^n = \frac{1}{\Gamma(2 - \alpha)} \int_0^n \frac{\partial \rho_i(x, s)}{\partial s} \frac{ds}{(t - s)^{\alpha-1}} = \frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} \left[ a_0 \rho_i^n - \sum_{k=1}^{n-1} (a_{n-k} - a_n) \rho_i^k - a_{n-1} \rho_i^0 \right] + O(\tau^{3-\alpha}), \quad i = 1, 2.
\]

Thus, we obtain

\[
\begin{align*}
\theta_1^{n+1/2} &= \frac{1}{2} (\theta_1^n + \theta_1^{n+1}) = \frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} \left[ a_0 \rho_1^{n+1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_n) \rho_1^{k+1/2} - a_{n-1} \rho_1^0 \right] + r_3^{n+1/2}, \quad (a) \\
\theta_2^{n+1/2} &= \frac{1}{2} (\theta_2^n + \theta_2^{n+1}) = \frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} \left[ a_0 \rho_2^{n+1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_n) \rho_2^{k+1/2} - a_{n-1} \rho_2^0 \right] + r_4^{n+1/2}, \quad (b)
\end{align*}
\]

and there exists positive constant \( C \) such that

\[
|r_3^{n+1/2}| \leq C \tau^{3-\alpha}, \quad |r_4^{n+1/2}| \leq C \tau^{3-\alpha}.
\]
Substituting (6)(a) (respectively (6)(b)) into (5)(a) (respectively (5)(b)), we get the following equations

\[
\begin{align*}
\tau^{1-a_1} \frac{1}{\Gamma(3 - \alpha_1)} \left[ a_0 \rho_1 n^{-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) \rho_1^{k-1/2} - a_{n-1} \rho_1 \right] - \Delta u_1^{n-1/2} &= -\beta^2 G^{n-1/2} + r_1^{n-1/2} - r_3^{n-1/2}, \\
\tau^{1-a_2} \frac{1}{\Gamma(3 - \alpha_2)} \left[ a_0 \rho_2 n^{-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) \rho_2^{k-1/2} - a_{n-1} \rho_2 \right] - c^2 \Delta u_2^{n-1/2} &= G^{n-1/2} + r_2^{n-1/2} - r_4^{n-1/2}.
\end{align*}
\]

(7)

Considering Lemma 2.2, we obtain

\[
\begin{align*}
\rho_1^{n-1/2} &= \delta_1 u_1^{n-1/2} + r_5^{n-1/2}, \\
\rho_2^{n-1/2} &= \delta_2 u_2^{n-1/2} + r_6^{n-1/2},
\end{align*}
\]

(8)

where \(r_5^{n-1/2} = O(\tau^2), \ r_6^{n-1/2} = O(\tau^2).

Substituting (8)(a) (respectively (8)(b)) into (7)(a) (respectively (7)(b)), and noticing that \(\rho_1^0 = \psi_1\) and \(\rho_2^0 = \psi_2\), we transform (1)(a)-(1)(b) into the following equivalent forms

\[
\begin{align*}
\tau^{1-a_1} \frac{1}{\Gamma(3 - \alpha_1)} \left[ a_0 \delta_1 u_1^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) \delta_1 u_1^{k-1/2} - a_{n-1} \delta_1 \right] - \Delta u_1^{n-1/2} &= -\beta^2 G^{n-1/2} + R_1, \\
\tau^{1-a_2} \frac{1}{\Gamma(3 - \alpha_2)} \left[ a_0 \delta_2 u_2^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) \delta_2 u_2^{k-1/2} - a_{n-1} \delta_2 \right] - c^2 \Delta u_2^{n-1/2} &= G^{n-1/2} + R_2,
\end{align*}
\]

(9)

where

\[
\begin{align*}
R_1 &= r_1^{n-1/2} - r_3^{n-1/2} - \tau^{1-a_1} \frac{1}{\Gamma(3 - \alpha_1)} \left[ a_0 f_5^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) f_5^{k-1/2} \right], \\
R_2 &= r_2^{n-1/2} - r_4^{n-1/2} - \tau^{1-a_2} \frac{1}{\Gamma(3 - \alpha_2)} \left[ a_0 f_6^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) f_6^{k-1/2} \right].
\end{align*}
\]

(10)

Because \(r_5^{n-1/2} = O(\tau^2)\) and \(r_6^{n-1/2} = O(\tau^2)\), we have

\[
\begin{align*}
- \tau^{1-a_1} \frac{1}{\Gamma(3 - \alpha_1)} \left[ a_0 f_5^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) f_5^{k-1/2} \right] &= O(\tau^3 - \alpha_1), \\
- \tau^{1-a_2} \frac{1}{\Gamma(3 - \alpha_2)} \left[ a_0 f_6^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) f_6^{k-1/2} \right] &= O(\tau^3 - \alpha_2).
\end{align*}
\]

(11)

According (11)(a) and (11)(b), and recalling \(r_1^{n-1} = O(\tau^2), r_2^{n-1} = O(\tau^2), r_3^{n-1} = O(\tau^3 - \alpha_1), r_4^{n-1} = O(\tau^3 - \alpha_2)\), we conclude that

\[
R_1 = O(\tau^3 - \alpha_1), \quad R_2 = O(\tau^3 - \alpha_2).
\]

(12)

Dropping the terms \(R_1\) and \(R_2\) in (9)(a) and (9)(b), respectively, we obtain the following semi-discrete approximation for (1)(a)-(1)(b)

\[
\begin{align*}
\tau^{1-a_1} \frac{1}{\Gamma(3 - \alpha_1)} \left[ a_0 \delta_1 u_1^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) \delta_1 u_1^{k-1/2} - a_{n-1} \delta_1 \right] - \Delta u_1^{n-1/2} &= -\beta^2 G^{n-1/2}, \quad 1 \leq n \leq n_T, \\
\tau^{1-a_2} \frac{1}{\Gamma(3 - \alpha_2)} \left[ a_0 \delta_2 u_2^{n-1/2} - \sum_{k=1}^{n-1} (a_{n-k} - a_{n-k}) \delta_2 u_2^{k-1/2} - a_{n-1} \delta_2 \right] - c^2 \Delta u_2^{n-1/2} &= G^{n-1/2}, \quad 1 \leq n \leq n_T.
\end{align*}
\]

(13)

2.2. Spatial discretization

In this subsection, we present the spatial discretization for the semi-discrete approximation (13)(a)-(13)(b) with initial and boundary conditions (2)(a)-(2)(c) by using the Hermite–Galerkin method with scaling factor. Let \((\cdot, \cdot)\) be the inner product on the space \(L^2(\mathbb{R}^d)\) with the \(L^2\)-norm \(\| \cdot \|\) as usual.

Firstly, we introduce some properties of Hermite polynomials/functions. Let \(H_m(x_i)\) be the standard Hermite polynomial of degree \(m_i\) with univariate \(x_i (i = 1, \ldots, d)\), which satisfies the following three-term recurrence relation

\[
\begin{align*}
H_{m+1}(x_i) &= x_i \sqrt{\frac{2}{m_i + 1}} H_m(x_i) - \sqrt{\frac{m_i}{m_i + 1}} H_{m-1}(x_i), \quad m_i \geq 1, \\
H_0(x_i) &= \pi^{-1/4}, \quad H_1(x_i) = \sqrt{2} \pi^{-1/4} x_i.
\end{align*}
\]

(14)
By introducing the scaling factor \( \lambda_i > 0 \), we define a sequence of generalized Hermite function as follows

\[
\hat{H}_m(x_i; \lambda_i) = \sqrt{\lambda_i} \frac{1}{\sqrt{2^m m!}} e^{-i\lambda_i x_i^2 / 2} H_m(\lambda_i x_i), \quad i = 1, \ldots, d.
\]

Thanks to the three-term recurrence relation (14), we obtain

\[
\begin{aligned}
\hat{H}_{m+1}(x_i; \lambda_i) &= \lambda_i x_i \sqrt{\frac{m}{m+1}} \hat{H}_m(x_i; \lambda_i) - \sqrt{\frac{m}{m+1}} \hat{H}_{m-1}(x_i; \lambda_i), \quad m \geq 1, \\
\hat{H}_0(x_i; \lambda_i) &= \sqrt{\lambda_i} \pi^{-1/4} e^{-i\lambda_i x_i^2 / 2}, \quad \hat{H}_1(x_i; \lambda_i) = \lambda_i^{3/2} \sqrt{2\pi}^{-1/4} x_i e^{-i\lambda_i x_i^2 / 2}.
\end{aligned}
\] (15)

The orthogonality of the generalized Hermite functions is

\[
\int_{-\infty}^{\infty} \hat{H}_m(x_i; \lambda_i) \hat{H}_n(x_i; \lambda_i) dx_i = \begin{cases} 
1, & m = n_i, \quad 0, & m_i \neq n_i.
\end{cases}
\] (16)

Additionally, we have the following property for the generalized Hermite functions

\[
\frac{\partial}{\partial x_i} \hat{H}_m(x_i; \lambda_i) = -\lambda_i x_i \sqrt{\frac{m}{2}} \hat{H}_{m-1}(x_i; \lambda_i) + \lambda_i \sqrt{\frac{m}{2}} \hat{H}_m(x_i; \lambda_i). \tag{17}
\]

For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d \) and \( \mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d \), the d-dimensional generalized Hermite functions with scaling factor are defined as

\[
\hat{H}_m(\mathbf{x}; \mathbf{\lambda}) = \hat{H}_{m_1}(x_1; \lambda_1) \hat{H}_{m_2}(x_2; \lambda_2) \cdots \hat{H}_{m_d}(x_d; \lambda_d).
\] (18)

Now, we define some useful function spaces. For any integer \( N > 0 \), we define

\[
\mathbf{\hat{P}}_{N,1} = \{ \hat{H}_m(\mathbf{x}; \mathbf{\lambda}) \mathbf{m} = (m_1, m_2, \ldots, m_d), 0 \leq m_1, \ldots, m_d \leq N \}.
\]

Then we define one-dimensional tensor of \( \mathbf{\hat{P}}_{N,1} \) as follows

\[
\mathbf{\hat{P}}_N = \mathbf{\hat{P}}_{N,1} \otimes \cdots \otimes \mathbf{\hat{P}}_{N,d} = \{ \hat{H}_m(\mathbf{x}; \mathbf{\lambda}) \mathbf{m} = (m_1, m_2, \ldots, m_d), 0 \leq m_1, \ldots, m_d \leq N \}.
\]

For the original problem (1)\(\text{a)}\)–(2)\(\text{c)}\), the fully-discrete scheme is to find \( u^n_{1,N}, u^n_{2,N} \in \mathbb{P}_d^N \), such that for all \( v_N \in \mathbf{\hat{P}}_N^N \),

\[
\begin{aligned}
\frac{t^{\alpha_1}}{\Gamma(3-\alpha_1)} &\left[ a_0(\delta t u_{1, N}^{n-1/2}, v_N) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) (\delta t u_{1, k, N}^{n-1/2}, v_N) - a_{n-1} (H_N \psi_1, v_N) \right] \\
&\quad + \langle \nabla u_{1, N}^{n-1/2}, \nabla v_N \rangle = -\frac{\beta^2}{2} \left( I_N \sin(u_{1}^{n-1} - u_{1}^{n-1}) + I_N \sin(u_{2}^{n-1} - u_{2}^{n-1}), v_N \right), \quad 1 \leq n \leq n_f, \tag{a} \\
\frac{t^{\alpha_2}}{\Gamma(3-\alpha_2)} &\left[ a_0(\delta t u_{2, N}^{n-1/2}, v_N) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) (\delta t u_{2, k, N}^{n-1/2}, v_N) - a_{n-1} (H_N \psi_2, v_N) \right] \\
&\quad + c^2(\nabla u_{2, N}^{n-1/2}, \nabla v_N) = \frac{1}{2} (I_N \sin(u_{1}^{n} - u_{2}^{n}) + I_N \sin(u_{1}^{n-1} - u_{2}^{n-1}), v_N), \quad 1 \leq n \leq n_f. \tag{b}
\end{aligned}
\] (19)

Here, \( I_N : C(\mathbb{R}^d) \rightarrow \mathbb{P}_d^N \) is the Hermite–Gauss interpolation operator satisfying

\[
(I_N u)(\eta_1^{n_1}, \ldots, \eta_d^{n_d}) = u(\eta_1^{n_1}, \ldots, \eta_d^{n_d}), \quad 0 \leq j_1, \ldots, j_d \leq N.
\]

where \( \{\eta_i^n\}_{0 \leq i \leq N} \) are the Hermite–Gauss point in the \( x_i \)-direction with respect to the scaling factor \( \lambda_i \) (i = 1, 2, ... , d).

**Remark 1.** In this paper, an efficient finite difference/Hermite–Galerkin spectral method is constructed for problem (1)\(\text{a)}\)–(2)\(\text{c)}\). Here, we apply the Crank–Nicolson method for both the linear terms \(-\Delta u_1 \) and \(-c^2 \Delta u_2 \) in the original problem) and nonlinear terms \(-\beta^2 \sin(u_1 - u_2) \) and \( \sin(u_1 - u_2) \) in the original problem) in time. In other words, our numerical method is implicit in time. Generally, the Crank–Nicolson method is numerically stable [41]. Thus, we conclude that our numerical method is stable to solve problem (1)\(\text{a)}\)–(2)\(\text{c)}\).

**Remark 2.** By virtue of (12), we can see the small terms \( R_1 \) in (9)\(\text{a)} \) and \( R_2 \) in (9)\(\text{b)} \) are \( O(t^{3-\alpha_1}) \) and \( O(t^{3-\alpha_2}) \), respectively. Dropping these two terms, we obtain the semi-discrete approximations (13)\(\text{a)} \)–(13)\(\text{b)} \) in time. Therefore, we conclude that the temporal convergence order of our numerical scheme is \( \min(3 - \alpha_1, 3 - \alpha_2) = 3 - \max(\alpha_1, \alpha_2) \). In addition, we apply the Hermite–Galerkin spectral method with scaling factor for the spatial approximation. The typical convergence rate of the spectral method is \( O(N^{-m}) \) [36], where \( N \) and \( m \) are the polynomial degree and the regularity index of the underlying function, respectively. Thus, we expect that our numerical scheme has the exponential convergence rate (or the spectral accuracy) in space if the solutions of problem (1)\(\text{a)}\)–(2)\(\text{c)} \) are sufficiently smooth with respect to the spatial variables.

The above discussions on the stability and convergence of the numerical method are checked by numerical tests in Sections 3.1 and 4.1 for 2D and 3D cases of the problem (1)\(\text{a)}\)–(2)\(\text{c)} \), respectively.
2.3. Implementation of the numerical scheme

In this subsection, we present the detailed implementation of the finite difference/Hermite–Galerkin spectral method \((19)\)–\((19)\). Introduce the parameter \(\mu_i\):

\[
\mu_i = r^{\nu - \alpha} \Gamma(3 - \alpha), \quad i = 1, 2,
\]

and define the matrices

\[
P^{\psi_i} = (p_n^{\psi_i})_{k,l=0} = \left( \begin{array}{c} \widetilde{H}_i(x_n; \lambda_i, \lambda_i) \\ \widetilde{H}_i(x_n; \lambda_i, \lambda_i) \\ \vdots \\ \widetilde{H}_i(x_n; \lambda_i, \lambda_i) \end{array} \right)_{k,l=0}^N, \quad Q^{\psi_i} = (q_n^{\psi_i})_{k,l=0} = \left( \begin{array}{c} \partial_3 \widetilde{H}_i(x_n; \lambda_i, \lambda_i) \\ \partial_3 \widetilde{H}_i(x_n; \lambda_i, \lambda_i) \\ \vdots \\ \partial_3 \widetilde{H}_i(x_n; \lambda_i, \lambda_i) \end{array} \right)_{k,l=0}^N, \quad i = 1, 2, \ldots, d.
\]

Using the properties of Hermite function expressed in \((16)\) and \((17)\), we find the elements of the matrices \(P^{\psi_i}\) and \(Q^{\psi_i}\) can be computed as

\[
\begin{align*}
(P^{\psi_i})_{k,l} &= \begin{cases} 
\frac{\lambda_i^2}{2}(l+1), & k = l, \\
\frac{\lambda_i^2}{2}(l+1), & k = l+1, \\
0, & \text{otherwise},
\end{cases} \\
(Q^{\psi_i})_{k,l} &= \begin{cases} 
\frac{\lambda_i^2}{2}(l+1)(l+2), & k = l+2, \\
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

Consider the following generalized eigenvalue problems

\[
P^{\psi_i} \bf{w}^\psi_i = \alpha^{\psi_i} Q^{\psi_i} \bf{w}^\psi_i, \quad i = 1, 2, \ldots, d.
\]

Denote by \(\Omega^{\psi_i}\) the diagonal matrix whose diagonal elements \(\alpha^{\psi_i}\) are the eigenvalues of \((20)\), and let \(E^{\psi_i}\) be the matrix formed by the corresponding eigenvector of \((20)\). Clearly, we have

\[
P^{\psi_i} E^{\psi_i} = Q^{\psi_i} E^{\psi_i} \Omega^{\psi_i}, \quad i = 1, 2, \ldots, d.
\]

Because the matrices \(P^{\psi_i}\) and \(Q^{\psi_i}\) are symmetric and positive definite, the diagonal elements of \(\Omega^{\psi_i}\) are all real and positive. In addition, we have \((E^{\psi_i})^{-1} = (E^{\psi_i})^T\).

Next, we give the matrix representation of the finite difference/Hermite–Galerkin spectral method \((19)\)–\((19)\) for both 2D and 3D problems and present the algorithms for the corresponding nonlinear algebraic system.

Two-dimensional case \((d = 2)\). For this case, we set the function space \(\tilde{P}^2_N\) as

\[
\tilde{P}^2_N = \tilde{P}_{N,1} \otimes \tilde{P}_{N,2} = \text{span}\{ \widetilde{H}_m(x_1; \lambda_1, \lambda_2) | m_1 = 0, 1, \ldots, N \}.
\]

Denote

\[
\Psi_1 = \left( \begin{array}{c} \Psi_{1,m}^{U_1} \\ \Psi_{1,m}^{U_2} \end{array} \right), \quad \Psi_2 = \left( \begin{array}{c} \Psi_{2,m}^{U_1} \\ \Psi_{2,m}^{U_2} \end{array} \right),
\]

\[
H^n = \left( \begin{array}{c} H_1 \sin(u_1^n - u_2^n) \\ H_2 \sin(u_1^n - u_2^n) \end{array} \right), \quad \Psi_{1,m}^{U_1} = \left( \begin{array}{c} \mathbf{u}_{m,1}^{U_1} \\ \mathbf{u}_{m,2}^{U_1} \end{array} \right), \quad \Psi_{2,m}^{U_1} = \left( \begin{array}{c} \mathbf{u}_{m,1}^{U_2} \\ \mathbf{u}_{m,2}^{U_2} \end{array} \right),
\]

Taking \(v_{n} = \widetilde{H}_i(x_n; \lambda_1, \lambda_2)\) in \((19)\) and \((19)\) with \(d = 2\) for \(k_1, k_2 = 0, 1, \ldots, N\), and recalling that \(a_0 = 1\), we can see \((19)\) and \((19)\) are equivalent to the following algebraic system

\[
\begin{align}
(P^{\psi_1} U_1^{P^2_2} + \frac{\mu_1 \tau}{2} (Q^{\psi_1} U_1^{P^2_2} + P^{\psi_1} U_1^{Q^2_2})) &= G_1^{n-1} - \frac{\mu_1 \tau \beta^2}{2} H^n, \quad (a) \\
(P^{\psi_2} U_1^{P^2_2} + \frac{\mu_2 \tau \beta^2}{2} (Q^{\psi_2} U_1^{P^2_2} + P^{\psi_2} U_1^{Q^2_2})) &= G_2^{n-1} - \frac{\mu_2 \tau \beta^2}{2} H^n, \quad (b)
\end{align}
\]

where

\[
\begin{align}
G_1^{n-1} &= P^{\psi_1} U_1^{n-1} P^{\psi_2} + \sum_{k=1}^{n} (a_{n-k} - a_{n-k}) P^{\psi_1}(U_1^{n-1} - U_1^{k-1}) P^{\psi_2} + \tau a_{n-k} \Psi_1, \\
&\quad - \frac{\mu_1 \tau}{2} (Q^{\psi_1} U_1^{n-1} P^{\psi_2} + P^{\psi_1} U_1^{k+1} P^{\psi_2}), \quad (a) \\
G_2^{n-1} &= P^{\psi_2} U_1^{n-1} P^{\psi_2} + \sum_{k=1}^{n} (a_{n-k} - a_{n-k}) P^{\psi_2}(U_2^{n-1} - U_2^{k+1}) P^{\psi_2} + \tau a_{n-k} \Psi_2, \\
&\quad - \frac{\mu_2 \tau \beta^2}{2} (Q^{\psi_2} U_2^{n-1} P^{\psi_2} + P^{\psi_2} U_2^{k+1} P^{\psi_2}), \quad (b)
\end{align}
\]
Noticing that $H^o = H^o(u_{3,N}^o, u_{2,N}^o)$, we can solve the algebraic system (22)(a)–(22)(b) at a fixed time level $n$ by combining the matrix decomposition method [36,42] and iteration algorithm:

\begin{enumerate}
\item Pre-computing: compute the matrices $E^i$ and $\tau^i$ ($i = 1, 2$) from the generalized eigenvalue problems (21), and compute $(Q^i E^i)^{-1}$ and $(Q^3 E^3)^{-T}$.
\item Compute $G_1^{-1}$ and $G_2^{-1}$ from (23)(a) and (23)(b), respectively.
\item Set $U_1^0 = U_2^{-1}$ and $U_2^0 = U_2^{-1}$.
\item Set $u_{1,0}^n = \sum_{m_1, m_2 = 0}^N \hat{w}_{1,m_1m_2}^n \hat{h}_{m_1}(x_1; \lambda_1) \hat{h}_{m_2}(x_2; \lambda_2)$ and $u_{2,0}^n = \sum_{m_1, m_2 = 0}^N \hat{w}_{2,m_1m_2}^n \hat{h}_{m_1}(x_1; \lambda_1) \hat{h}_{m_2}(x_2; \lambda_2)$.
\item Let $K$ be a suitable positive integer, $\varepsilon$ be a suitably small positive constant.
\begin{align*}
\text{for } k = 0 : K \\
&\text{Compute } \tilde{H} = (I_n \sin (u_{1,N}^n - u_{2,N}^n) - \hat{h}_{m_1}(x_1; \lambda_1) \hat{h}_{m_2}(x_2; \lambda_2))^N_{m_1, m_2 = 0} ; \\
&\text{Compute } S_1 = (s_{1,i,j} \big| \sum_{j=0}^N (Q^1 E^1)^{-1}(G_1^{-1} - \frac{\mu_1 T^2}{2} H) (Q^3 E^3)^{-T} ; \\
&\text{Compute } S_2 = (s_{2,i,j} \big| \sum_{j=0}^N (Q^2 E^1)^{-1}(G_2^{-1} + \frac{\mu_1 T^2}{2} H) (Q^3 E^3)^{-T} ; \\
&\text{Solve } (\alpha_i^1 \alpha_j^2 + \frac{\mu_2 T^2 \gamma^2}{2} (\alpha_j^1 + \alpha_i^1)) b_{1,i,j} = s_{1,i,j} \text{ with } i, j = 0, 1, \ldots \text{, } N \text{ to obtain the matrix } B_1 = (b_{1,i,j})_{i,j=0}^N ; \\
&\text{Solve } (\alpha_i^1 \alpha_j^2 + \frac{\mu_2 T^2 \gamma^2}{2} (\alpha_j^1 + \alpha_i^1)) b_{2,i,j} = s_{2,i,j} \text{ with } i, j = 0, 1, \ldots \text{, } N \text{ to obtain the matrix } B_2 = (b_{2,i,j})_{i,j=0}^N ; \\
&\text{Compute } U_1^{n,k+1} = E^1 b_1(E^3)^T \text{ and } U_2^{n,k+1} = E^1 b_2(E^3)^T ; \\
&\text{Compute } u_{1,0}^{n,k+1} = \sum_{m_1, m_2 = 0}^N \hat{w}_{1,m_1m_2}^{n,k+1} \hat{h}_{m_1}(x_1; \lambda_1) \hat{h}_{m_2}(x_2; \lambda_2) \text{ and } u_{2,0}^{n,k+1} = \sum_{m_1, m_2 = 0}^N \hat{w}_{2,m_1m_2}^{n,k+1} \hat{h}_{m_1}(x_1; \lambda_1) \hat{h}_{m_2}(x_2; \lambda_2) ; \\
&\text{If } \|u_{1,0}^{n,k+1} - u_{1,0}^n\| \leq \varepsilon \text{ and } \|u_{2,0}^{n,k+1} - u_{2,0}^n\| \leq \varepsilon \\
&\text{Break;}
\end{align*}
\end{enumerate}

6) Set $U_1^n = U_1^{n,k+1}$ and $U_2^n = U_2^{n,k+1}$.

\textbf{Three-dimensional case} ($d = 3$). In this subsection, we employ the Einstein summation convention as a pair of repeated subscripted variable implies the summation of the subscripted variable from 0 to $N$. Let us define

$$\mathbb{P}_3^N = \hat{P}_{N,1} \otimes \hat{P}_{N,2} \otimes \hat{P}_{N,3} = \text{span} \left\{ \hat{h}_{m_1}(x_1; \lambda_1) \hat{h}_{m_2}(x_2; \lambda_2) \hat{h}_{m_3}(x_3; \lambda_3), \ m_1, m_2, m_3 = 0, 1, \ldots, N \right\},$$

and denote

\begin{align*}
\psi_{1,k_1k_2k_3} &= (I_N \psi_1 \hat{h}_{k_1}(x_1; \lambda_1) \hat{h}_{k_2}(x_2; \lambda_2) \hat{h}_{k_3}(x_3; \lambda_3)) \ , \ 
\psi_{2,k_1k_2k_3} &= (I_N \psi_2 \hat{h}_{k_1}(x_1; \lambda_1) \hat{h}_{k_2}(x_2; \lambda_2) \hat{h}_{k_3}(x_3; \lambda_3)) , \ 
\psi_{3,k_1k_2k_3} &= (I_N \psi_3 \hat{h}_{k_1}(x_1; \lambda_1) \hat{h}_{k_2}(x_2; \lambda_2) \hat{h}_{k_3}(x_3; \lambda_3)) .
\end{align*}

Setting $v_N = \hat{H}_k(x_1; \lambda_1)\hat{H}_k(x_2; \lambda_2)\hat{H}_k(x_3; \lambda_3)$ in (19)(a)–(19)(b) with $d = 3$ and $k_1, k_2, k_3 = 0, 1, \ldots, N$, we find that the equations are equivalent to the following algebraic system:

\[
\begin{align*}
&\left\{
\begin{array}{l}
p_{k_1,m_1}^n \tilde{u}_{m_1,m_2,m_3}^n + \frac{\mu_1 \tau}{2} \left( q_{k_1,m_1}^{k_1-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_1,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) = \frac{\mu_1 \tau}{2} \left( q_{k_1,m_1}^{k_1-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_1,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) + \frac{\mu_1 \tau \beta^2}{2} h_{k_1,k_2}^n, \\
p_{k_1,m_3}^n \tilde{u}_{m_1,m_2,m_3}^n + \frac{\mu_1 \tau}{2} \left( q_{k_1,m_1}^{k_1-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_1,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) = \frac{\mu_1 \tau}{2} \left( q_{k_1,m_1}^{k_1-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_1,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) + \frac{\mu_1 \tau \beta^2}{2} h_{k_1,k_3}^n,
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
&\left\{
\begin{array}{l}
p_{k_2,m_1}^n \tilde{u}_{m_1,m_2,m_3}^n + \frac{\mu_2 \tau}{2} \left( q_{k_2,m_1}^{k_2-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_2,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) = \frac{\mu_2 \tau}{2} \left( q_{k_2,m_1}^{k_2-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_2,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) + \frac{\mu_2 \tau \beta^2}{2} h_{k_2,k_3}^n,
\end{array}
\right.
\end{align*}
\]

where

\[
\begin{align*}
&g_{k_1,k_2}^{n-1} = \frac{\mu_1 \tau}{2} \left( q_{k_1,m_1}^{k_1-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_1,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) + \frac{\mu_1 \tau \beta^2}{2} h_{k_1,k_2}^n,
&g_{k_2,k_3}^{n-1} = \frac{\mu_2 \tau}{2} \left( q_{k_2,m_1}^{k_2-1} \tilde{u}_{m_1,m_2,m_3}^n + p_{k_2,m_2}^n \tilde{u}_{m_1,m_2,m_3}^n \right) + \frac{\mu_2 \tau \beta^2}{2} h_{k_2,k_3}^n.
\end{align*}
\]

Because $h_{k_1,k_2}^n = h_{k_1,k_2}^n(u_{1,n}^n, u_{2,n}^n)$, we shall solve the algebraic system (24)(a)–(24)(b) at a fixed time level $n$ by the following algorithm which is based on the matrix decomposition method and iteration algorithm:

**Algorithm II.**

1. Pre-computing: compute the matrices $E^3$ and $\Omega^3$ ($i = 1, 2, 3$) from the generalized eigenvalue problems (20), and compute $(Q^{i}E^3)^{-1}(Q^{i}E^{i})^{-T}$, $(E^3)^{-1} = \left( \begin{pmatrix} e_{1}^{i} \end{pmatrix} \right)^{-1}_{k_1,k_2}^N$, and $(Q^3)^{-1} = \left( \begin{pmatrix} q_{2}^{3} \end{pmatrix} \right)^{-1}_{k_1,k_2}^N$.

2. Compute $g_{k_1,k_2}^{n-1}$ and $g_{k_2,k_3}^{n-1}$ with $k_1, k_2, k_3 = 0, 1, \ldots, N$ from (25)(a) and (25)(b), respectively.

3. Set $\tilde{u}_{1,1,1,0}^n = \tilde{u}_{1,1,1,0}^n$ and $\tilde{u}_{2,2,2,0}^n = \tilde{u}_{2,2,2,0}^n$, with $m_1, m_2, m_3 = 0, 1, \ldots, N$.

4. Set $u_{1,1,1,0}^n = \sum_{m_1,m_2,m_3=0}^N \tilde{u}_{1,1,1,0}^n$, $\tilde{H}_m(x_1; \lambda_1)\tilde{H}_m(x_2; \lambda_2)\tilde{H}_m(x_3; \lambda_3)$ and $u_{2,2,2,0}^n = \sum_{m_1,m_2,m_3=0}^N \tilde{u}_{2,2,2,0}^n$, $\tilde{H}_m(x_1; \lambda_1)\tilde{H}_m(x_2; \lambda_2)\tilde{H}_m(x_3; \lambda_3)$.

5. Let $K$ be a suitable positive small constant, $\varepsilon$ be a suitably small small positive constant. For $k = 0 : K$

- Compute $\tilde{h}_{m_1,m_2} = \left( \begin{array}{c} \sin \left( \frac{\pi k_1}{2} \right) \tilde{u}_{1,m_1,m_2}^n \tilde{u}_{1,m_1,m_2}^n \end{array} \right)$.

- Compute $s_{1,m_2}^j = (e_{1}^{i})^{-1}(q_{1,m_2}^{i})^{-1}(g_{1,m_2,m_3}^{n-1} - \frac{\mu_1 \tau \beta^2}{2} h_{1,m_2,m_3}^n)$ to obtain the matrix $S_{1,j} = (s_{1,m_2}^j)_{m_1,m_2=0}^N$ with $j = 0, 1, \ldots, N$.

- Compute $s_{2,m_2}^j = (e_{2}^{i})^{-1}(q_{2,m_2}^{i})^{-1}(g_{2,m_2,m_3}^{n-1} + \frac{\mu_2 \tau \beta^2}{2} h_{2,m_2,m_3}^n)$ to obtain the matrix $S_{2,j} = (s_{2,m_2}^j)_{m_1,m_2=0}^N$ with $j = 0, 1, \ldots, N$.

- Compute $R_{1,j} = (r_{1,m_2}^j)_{m_1,m_2=0}^N = (Q^i E^3)^{-1}S_{1,j}(Q^i E^{i})^{-T}$ and $R_{2,j} = (r_{1,m_2}^j)_{m_1,m_2=0}^N = (Q^i E^3)^{-1}S_{2,j}(Q^i E^{i})^{-T}$ with $j = 0, 1, \ldots, N$.

- Solve $\left( (\omega_{1,j}^i + \frac{\mu_1 \tau}{2}) \omega_{1,j}^i + \frac{\mu_1 \tau}{2} \omega_{2,j}^i \omega_{2,j}^i \right) b_{1,m_1,j} = r_{1,m_1,j}$ to obtain the matrix $B_{1,j} = (b_{1,m_1,j})_{m_1,m_2=0}^N$ with $j = 0, 1, \ldots, N$. 

3. Numerical results for 2D problem

3.1. Test example

We now consider two-dimensional time-fractional coupled sine–Gordon equations with initial and boundary conditions in the following form:

\[
\begin{align*}
\frac{\partial^\alpha u_1(x_1, x_2, t)}{\partial t^\alpha} & + \frac{1}{2} \left( \sin(u_1(x_1, x_2, t)) - \sin(u_2(x_1, x_2, t)) \right) + f_1(x_1, x_2, t) = 0, \\
\frac{\partial^\alpha u_2(x_1, x_2, t)}{\partial t^\alpha} & + \frac{1}{2} \left( \sin(u_2(x_1, x_2, t)) - \sin(u_1(x_1, x_2, t)) \right) + f_2(x_1, x_2, t) = 0,
\end{align*}
\]

(26)

The exact solutions depending on \(\alpha_1\) and \(\alpha_2\) are expressed as

\[u_1(x_1, x_2, t) = t^{2+\alpha_1} \exp(5x_1^2 \pm 5x_2^2), \quad u_2(x_1, x_2, t) = t^{2+\alpha_2} \exp(-3x_1^2 - 3x_2^2).\]

And the forcing terms \(f_1(x_1, x_2, t)\) and \(f_2(x_1, x_2, t)\) can be numerically obtained.

In Fig. 1, we investigate the spatial errors in the sense of \(L^2\)-norm for different values of \(\alpha_1\) and \(\alpha_2\) at \(T = 1\) by letting \(N\) vary and fixing the time step \(\tau = 10^{-3}\). Here, a logarithmic scale is used for the error-axis. Clearly, we can see from plots (a) and (b) that the \(L^2\)-errors decay exponentially for all the values of \(\alpha_1\) and \(\alpha_2\), i.e., our numerical scheme arrives at the spectral accuracy in space.

In Table 1, we test the convergence rates in \(L^2\)-errors for temporal discretization by fixing \(N = 100\) and changing time step from 1/100 to 1/400. Obviously, we can see the temporal convergence order of our numerical scheme is close to \(3 - \max(\alpha_1, \alpha_2)\) for all the values of \(\alpha_1\) and \(\alpha_2\).

To demonstrate the stability of our numerical method for 2D problem (26), we consider the small perturbations of the initial conditions, i.e., \(u_1(x_1, x_2, 0) = 0 + \epsilon_1, \quad \frac{\partial u_1(x_1, x_2, 0)}{\partial t} = 0 + \epsilon_2, \quad u_2(x_1, x_2, 0) = 0 + \epsilon_3, \quad \frac{\partial u_2(x_1, x_2, 0)}{\partial t} = 0 + \epsilon_4\). In Fig. 2, comparing the exact solutions with small perturbations at \(t = 5\), we can conclude that our method is stable.

3.2. Circular ring vector solitons

Consider (1) (a)–(1) (b) with boundary condition (2) (c) and the following initial conditions

\[
\begin{align*}
&u_1(x_1, x_2, 0) = \arctan \left( \exp \left( 1 - \sqrt{5x_1^2 + 5x_2^2} \right) \right), \quad \frac{\partial u_1(x_1, x_2, 0)}{\partial t} = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \\
&u_2(x_1, x_2, 0) = - \arctan \left( \exp \left( 1 - \sqrt{5x_1^2 + 5x_2^2} \right) \right), \quad \frac{\partial u_2(x_1, x_2, 0)}{\partial t} = 0, \quad (x_1, x_2) \in \mathbb{R}^2.
\end{align*}
\]
Fig. 1. $L^2$-errors for $u_1$ (plot (a)) and $u_2$ (plot (b)) as a function of the polynomial degree $N$ for two-dimensional problem (26) at $T = 1$ with different values of $\alpha_1$ and $\alpha_2$. Other parameters are $\tau = 1e^{-3}$, $c = \beta = 1$, and $\lambda_1 = \lambda_2 = 2.5$.

Fig. 2. Exact solutions and numerical solutions with perturbations of initial-conditions for two-dimensional problem (26) at $T = 5$ with $N = 50$, $\tau = 1e^{-1}$, $c = \beta = 1$, $\alpha_1 = 1.6$, and $\alpha_2 = 1.3$. Plots (a) and (d): Exact solutions for $u_1$ and $u_2$; Plots (b) and (e): Numerical solutions for $u_1$ and $u_2$ with perturbations $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, $\epsilon_3 = 10^{-3}$, and $\epsilon_4 = 10^{-4}$; Plots (c) and (f): Numerical solutions for $u_1$ and $u_2$ with perturbations $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, $\epsilon_3 = 10^{-1}$, and $\epsilon_4 = 10^{-4}$.

The time evolutions of the circular ring vector solitons are presented in Fig. 3 with $N = 50$, $\tau = 1e^{-1}$, $c = \beta = 1$, $\alpha_1 = 1.5$, and $\alpha_2 = 1.7$ at $t = 0, 0.7, 1.5$, and 2.5 in terms of $\sin(u_1/2)$ and $\sin(u_2/2)$. Plots (a) and (e) show the initial stages of the components $u_1$ and $u_2$, respectively. As time goes on, it can be seen that the oscillations and radiations of the circular ring vector solitons begin to form and continue to form up to $t = 0.7$, which is depicted in plots (b) and (f) for components $u_1$ and $u_2$, respectively. At $t = 1.5$, plots (c) and (g) which respectively depict the components $u_1$ and $u_2$ show that the circular ring vector solitons are nearly formed again. As is shown in plots (d) and (h) for components $u_1$ and $u_2$, respectively, the circular ring vector solitons shrink again at $t = 2.5$.

In Fig. 4 (respectively Fig. 5), we investigate the effects of the orders of Caputo fractional derivative $\alpha_1$ and $\alpha_2$ on the forms of the circular ring vector soliton in terms of $\sin(u_1/2)$ (respectively $\sin(u_2/2)$). From Fig. 4, we can see increasing the values of $\alpha_1$ and $\alpha_2$ can enlarge the "holes" of the circular ring vector solitons of the component $u_1$. Moreover, the amplitudes of the circular ring vector solitons of the component $u_1$ also increase with the increase of $\alpha_1$ and $\alpha_2$. From Fig. 5, we can see the parameters $\alpha_1$ and $\alpha_2$ have the same qualitative
Table 1

$L^2$-errors for two-dimensional problem (26) at $T = 1$ with $N = 100$, $c = \beta = 1$, and $\lambda_1 = \lambda_2 = 2.5$.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$3\max(\alpha_1, \alpha_2)$</th>
<th>$\tau$</th>
<th>$L^2$-Error</th>
<th>Order</th>
<th>$L^2$-Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>1.8</td>
<td>1.2</td>
<td>1/100</td>
<td>1.89598e−3</td>
<td>—</td>
<td>1.28647e−2</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/200</td>
<td>8.02227e−4</td>
<td>1.2409</td>
<td>5.60095e−3</td>
<td>1.1997</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/300</td>
<td>4.86811e−4</td>
<td>1.2320</td>
<td>3.44323e−3</td>
<td>1.1999</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/400</td>
<td>3.42000e−4</td>
<td>1.2273</td>
<td>2.43802e−3</td>
<td>1.2000</td>
</tr>
<tr>
<td>1.9</td>
<td>1.2</td>
<td>1.1</td>
<td>1/100</td>
<td>3.09670e−2</td>
<td>—</td>
<td>4.46670e−3</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/200</td>
<td>1.44355e−2</td>
<td>1.1011</td>
<td>1.97224e−3</td>
<td>1.1794</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/300</td>
<td>9.23811e−3</td>
<td>1.1008</td>
<td>1.23453e−3</td>
<td>1.1554</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/400</td>
<td>6.73064e−3</td>
<td>1.1008</td>
<td>8.88316e−4</td>
<td>1.1441</td>
</tr>
<tr>
<td>1.1</td>
<td>1.7</td>
<td>1.3</td>
<td>1/100</td>
<td>1.21277e−3</td>
<td>—</td>
<td>6.95711e−3</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/200</td>
<td>4.87379e−4</td>
<td>1.3152</td>
<td>2.82740e−3</td>
<td>1.2990</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/300</td>
<td>2.86331e−4</td>
<td>1.3118</td>
<td>1.66938e−3</td>
<td>1.2995</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/400</td>
<td>1.96477e−4</td>
<td>1.1008</td>
<td>8.88316e−4</td>
<td>1.1441</td>
</tr>
</tbody>
</table>

**Fig. 3.** Time evolutions of two dimensional circular ring vector solitons at times $t = 0$, 0.7, 1.5, and 2.5 with $N = 50$, $\tau = 1e − 1$, $c = \beta = 1$, $\alpha_1 = 1.5$, and $\alpha_2 = 1.7$. The first row (resp. the second row): Surface plots for $\sin(u_1/2)$ (resp. $\sin(u_2/2)$).

effects on the component $u_2$ as that on the component $u_1$. Our numerical simulations reveal that: (i) the radiations of the circular ring vector solitons increase with the increase of $\alpha_1$ and $\alpha_2$, and (ii) increasing the values of $\alpha_1$ and $\alpha_2$ could concentrate the energy in the physical system and cause the enhancement of amplitudes of the circular ring vector solitons.

### 3.3. Collision of three circular ring vector solitons

In this example, we study the time evolutions of three circular ring vector solitons for problem (1)(a)–(1)(b) with boundary condition (2)(c) and the following initial conditions

\[
\begin{aligned}
    u_1(x_1, x_2, 0) &= 4 \arctan \left( \exp \left( 4 - \sqrt{(x_1 - 8)^2 + x_2^2} \right) \right) + \\
    &\sum_{i=1}^{2} 4 \arctan \left( \exp \left( 4 - \sqrt{(x_1 + 4)^2 + (x_2 + (-1)^i \cdot 4\sqrt{3})^2} \right) \right), (x_1, x_2) \in \mathbb{R}^2, \\
    u_2(x_1, x_2, 0) &= -u_1(x_1, x_2, 0), (x_1, x_2) \in \mathbb{R}^2, \\
    \frac{\partial u_1(x_1, x_2, 0)}{\partial t} &= \frac{\partial u_2(x_1, x_2, 0)}{\partial t} = \text{sech} \left( 4 - \sqrt{(x_1 - 8)^2 + x_2^2} \right) + \\
    &\sum_{i=1}^{2} 4 \text{sech} \left( 4 - \sqrt{(x_1 + 4)^2 + (x_2 + (-1)^i \cdot 4\sqrt{3})^2} \right). (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
\]

The problem is numerical solved with $\tau = 1e − 1$, $N = 50$, $c = \beta = 1$, $\alpha_1 = 1.9$, and $\alpha_2 = 1.8$. 


3.4. Collision of four circular ring vector solitons

In this case, we simulate the interactions of four circular ring vector solitons of the problem (1)(a)–(1)(b) with boundary condition (2)(c) and the following initial conditions

\[
\begin{align*}
    u_1(x_1, x_2, 0) &= \sum_{i=1}^{2} \sum_{j=1}^{2} 4 \arctan \left( \exp \left( 4 - \sqrt{(x_1 - (-1)^i \cdot 6)^2 + (x_2 + (-1)^j \cdot 6)^2} \right) \right), \ (x_1, x_2) \in \mathbb{R}^2, \\
    u_2(x_1, x_2, 0) &= -u_1(x_1, x_2, 0), \ (x_1, x_2) \in \mathbb{R}^2, \\
    \frac{\partial u_1(x_1, x_2, 0)}{\partial t} &= \frac{\partial u_2(x_1, x_2, 0)}{\partial t} = \sum_{i=1}^{2} \sum_{j=1}^{2} 4 \sech \left( 4 - \sqrt{(x_1 - (-1)^i \cdot 6)^2 + (x_2 + (-1)^j \cdot 6)^2} \right), \ (x_1, x_2) \in \mathbb{R}^2.
\end{align*}
\]

Here we solve the problem with \( \tau = 1 + 1, N = 50, \alpha = 1, \beta = 1.8, \) and \( \alpha_2 = 1.7. \)

We present the collisions of four circular ring vector solitons which are in terms of \( \sin(u_1/2) \) (respectively \( \sin(u_2/2) \)) in Fig. 8 (respectively Fig. 9). As we can see, the complex interactions of four circular ring vector solitons occur, which leads to the formation of large square ring vector solitons. Moreover, the values of the center of the new emerged square ring vector solitons vary rapidly.

4. Numerical results for 3D problem

4.1. Test example

Now, we consider three-dimensional time-fractional coupled sine–Gordon equations with initial and boundary conditions in the following forms:

\[
\begin{align*}
    \frac{\partial^\alpha}{\partial t^\alpha} u_1(x_1, x_2, x_3, t) - \Delta u_1(x_1, x_2, x_3, t) &= -\beta^2 \sin(u_1(x_1, x_2, x_3, t)) - u_2(x_1, x_2, x_3, t) + f_1(x_1, x_2, x_3, t), \ (x_1, x_2, x_3) \in \mathbb{R}^3, 0 < t \leq T, \\
    \frac{\partial^\alpha}{\partial t^\alpha} u_2(x_1, x_2, x_3, t) - \Delta u_2(x_1, x_2, x_3, t) &= \sin(u_1(x_1, x_2, x_3, t)) - u_2(x_1, x_2, x_3, t) + f_2(x_1, x_2, x_3, t), \ (x_1, x_2, x_3) \in \mathbb{R}^3, 0 < t \leq T, \\
    u_1(x_1, x_2, x_3, 0) &= \frac{\partial u_1(x_1, x_2, x_3, 0)}{\partial t} = u_2(x_1, x_2, x_3, 0) = 0, \ (x_1, x_2, x_3) \in \mathbb{R}^3, \ t = 0, \\
    \lim_{|x_1| + |x_2| + |x_3| \to \infty} u_1(x_1, x_2, x_3, t) &= 0, \ t > 0, \ (x_1, x_2, x_3) \in \mathbb{R}^3, \\
    \lim_{|x_1| + |x_2| + |x_3| \to \infty} u_2(x_1, x_2, x_3, t) &= 0, \ 0 \leq t \leq T.
\end{align*}
\]
Fig. 5. Surface plots (the first row) and density plots (the second row) of $\sin(u_i/2)$ for different values of $\alpha_1$ and $\alpha_2$ at $t = 2.3$. The parameters are the same as that in Fig. 4.

Fig. 6. Surface plots (the first row) and density plots (the second row) of $\sin(u_i/2)$ for the collisions of three circular vector solitons with $\alpha_1 = 1.9, \alpha_2 = 1.8, \epsilon = \beta = 1, N = 50,$ and $\tau = 1e - 1$.

We present the following exact solutions which depend on $\alpha_1$ and $\alpha_2$

$$u_1(x_1, x_2, x_3, t) = t^{2+\alpha_1} \text{sech}(x_1^2 + x_2^2 + x_3^2), \quad u_2(x_1, x_2, x_3, t) = t^{2+\alpha_2} \exp(-x_1^2 - x_2^2 - x_3^2),$$

where the forcing terms $f_1(x_1, x_2, x_3, t)$ and $f_2(x_1, x_2, x_3, t)$ can be numerically obtained.

To study the accuracy of our numerical method in space, we depict the $L^2$-errors in semilog scale for different values of $\alpha_1$ and $\alpha_2$ by letting $N$ vary in Fig. 10. Here, we choose $T = 1, \tau = 1e - 3, \epsilon = \beta = 1, \lambda_1 = \lambda_2 = 2.5$. Clearly, from both the plots (a) and (b), we observe an exponential convergence as expected.

To verify the convergence rates for temporal discretization, we present the $L^2$-errors by changing time step from $1e - 1$ to $5e - 3$. Here, we set $T = 1, N = 100, \epsilon = \beta = 1$, and $\lambda_1 = \lambda_2 = 2.5$. For all the values of $\alpha_1$ and $\alpha_2$, the Table 2 shows that the convergence order in time is close to $(3 - \max(\alpha_1, \alpha_2))$.

In Fig. 11, we investigate the stability of our numerical method for 3D problem (27) by considering the small perturbations of the initial conditions, i.e., $u_1(x_1, x_2, x_3, 0) = 0 + \epsilon_1, \frac{\partial u_1(x_1, x_2, x_3, 0)}{\partial t} = 0 + \epsilon_2, u_2(x_1, x_2, x_3, 0) = 0 + \epsilon_3$, and $\frac{\partial u_2(x_1, x_2, x_3, 0)}{\partial t} = 0 + \epsilon_4$. Comparing
Fig. 7. Surface plots (the first row) and density plots (the second row) of \( \sin(u_1/2) \) for the collisions of three circular vector solitons. The parameters are the same as that in Fig. 6.

Fig. 8. Surface plots (the first row) and density plots (the second row) of \( \sin(u_1/2) \) for the collision of four circular vector solitons with \( \alpha_1 = 1.8, \alpha_2 = 1.7, c = \beta = 1, N = 50, \) and \( \tau = 1e-1 \).

the isosurfaces of the exact solutions to that of the numerical solutions with perturbations at \( t = 6 \) in the figure, we conclude that our method is also stable for the 3D problem (see Table 2).

4.2. Collision of two elliptical ring vector solitons

In this subsection, we simulate the time evolutions of two elliptical ring vector solitons of the problem \((1)(a)\)–\((1)(b)\) with the boundary condition \((2)(c)\) and the following initial conditions

\[
\begin{align*}
  u_1(x_1, x_2, x_3, 0) &= \sum_{i=1}^{2} 5 \arctan \left( \exp \left( 1 - \sqrt{\frac{\left( x_1 + (-1)^i \cdot 7 \right)^2}{3} + x_2^2 + x_3^2} \right) \right), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \\
  u_2(x_1, x_2, x_3, 0) &= -u_1(x_1, x_2, x_3, 0), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \\
  \frac{\partial u_1(x_1, x_2, x_3, 0)}{\partial t} &= \sum_{i=1}^{2} 5 \text{sech} \left( 2 - \sqrt{\frac{\left( x_1 + (-1)^i \cdot 7 \right)^2}{3} + x_2^2 + x_3^2} \right), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \\
  \frac{\partial u_2(x_1, x_2, x_3, 0)}{\partial t} &= -\frac{\partial u_1(x_1, x_2, x_3, 0)}{\partial t}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{align*}
\]

where we solve the problem by our numerical method with \( N = 50, c = \beta = 1, \) and \( \tau = 1e-1 \).
Fig. 9. Surface plots (first row) and density plots (second row) of $\sin(u_2/2)$ for the collision of four circular vector solitons. The parameters are the same as that in Fig. 8.

Fig. 10. $L_2$-errors for $u_1$ (plot (a)) and $u_2$ (plot (b)) as a function of the polynomial degree $N$ for three-dimensional problem (27) at $T = 1$ with different values of $\alpha_1$ and $\alpha_2$. Other parameters are $\tau = 1e^{-3}$, $c = \beta = 1$, and $\lambda_1 = \lambda_2 = 2.5$.

**Table 2**

$L^2$-errors for three-dimensional problem (27) at $T = 1$ with $N = 100$, $c = \beta = 1$, and $\lambda_1 = \lambda_2 = 2.5$.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$3\text{max}(\alpha_1, \alpha_2)$</th>
<th>$\tau$</th>
<th>$L^2$-Error for $u_1$</th>
<th>Order</th>
<th>$L^2$-Error for $u_2$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>1.5</td>
<td></td>
<td>$1e^{-1}$</td>
<td>1.23286e–1</td>
<td>1.58001e–1</td>
<td>1.11801e–1</td>
<td>1.1062e–1</td>
</tr>
<tr>
<td>1.4</td>
<td>1.5</td>
<td></td>
<td>$5e^{-2}$</td>
<td>3.57904e–2</td>
<td>6.49259e–2</td>
<td>3.41667e–2</td>
<td>1.1018e–1</td>
</tr>
<tr>
<td>1.4</td>
<td>1.5</td>
<td></td>
<td>$5e^{-3}$</td>
<td>5.19088e–2</td>
<td>8.87821e–3</td>
<td>5.75999e–2</td>
<td>1.1062e–1</td>
</tr>
<tr>
<td>1.4</td>
<td>1.5</td>
<td></td>
<td>$1e^{-2}$</td>
<td>2.60006e–2</td>
<td>3.82528e–3</td>
<td>5.75999e–1</td>
<td>1.1062e–1</td>
</tr>
<tr>
<td>1.4</td>
<td>1.5</td>
<td></td>
<td>$5e^{-3}$</td>
<td>8.52147e–3</td>
<td>1.4956e–3</td>
<td>2.11987e–3</td>
<td>1.4958e–3</td>
</tr>
<tr>
<td>1.4</td>
<td>1.5</td>
<td></td>
<td>$3e^{-3}$</td>
<td>3.01554e–3</td>
<td>5.99032e–3</td>
<td>2.11987e–3</td>
<td>1.4958e–3</td>
</tr>
</tbody>
</table>

Figs. 12 and 13 show the collisions of two elliptical ring vector solitons in terms of $\sin(u_1/2)$ and $\sin(u_2/2)$, respectively. As we can see from the figures, a large dumbbell-shaped vector solitons which are resulted from the collisions of two elliptical ring vector solitons are emerged with time going on.
4.3. Collision of four elliptical–circular ring vector solitons

Consider the interactions of four elliptical–circular ring vector solitons of the problem (1)(a)–(1)(b) with the boundary condition (2)(c) and the following initial conditions

\[
\begin{align*}
    u_1(x_1, x_2, x_3, 0) &= \sum_{i=1}^{2} 5 \arctan \left( \exp \left( 1 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right) \right) + \\
        &+ \sum_{i=1}^{2} 5 \arctan \left( \exp \left( 1 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right) \right), (x_1, x_2, x_3) \in \mathbb{R}^3, \\
    u_2(x_1, x_2, x_3, 0) &= -\sum_{i=1}^{2} 5 \arctan \left( \exp \left( 1 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right) \right) - \\
        &- \sum_{i=1}^{2} 5 \arctan \left( \exp \left( 1 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right) \right), (x_1, x_2, x_3) \in \mathbb{R}^3, \\
    \frac{\partial u_1(x_1, x_2, x_3, 0)}{\partial t} &= \sum_{i=1}^{2} 5 \text{sech} \left( 1 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right) + \\
        &+ \sum_{i=1}^{2} 5 \text{sech} \left( 1 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right), (x_1, x_2, x_3) \in \mathbb{R}^3, \\
    \frac{\partial u_2(x_1, x_2, x_3, 0)}{\partial t} &= -\sum_{i=1}^{2} 5 \text{sech} \left( 2 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right) - \\
        &- \sum_{i=1}^{2} 5 \text{sech} \left( 2 - \sqrt{\frac{\left( x_1 + (\cdot)^2 \right)^2 + x_2^2 + x_3^2}{3}} \right), (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{align*}
\]

Here, the problem is numerically solved with \( N = 50, c = \beta = 1 \), and \( \tau = 1e-1 \).

We show the collisions of four elliptical–circular ring vector solitons which are in terms of \( \sin(u_1/2) \) and \( \sin(u_2/2) \) in Figs. 14 and 15, respectively. In particular, Fig. 14 depicts the time evolutions of four elliptical ring solitons for the component \( u_1 \), while the interactions
Fig. 12. Isosurfaces $u_1 = 0.8$ (the first row) and slices of $u_1$ at $x_2 = 0$ (the second row) for the collisions of two elliptical vector solitons with $\alpha_1 = 1.8, \alpha_2 = 1.6, N = 50, c = \beta = 1$, and $t = 1e - 1$.

Fig. 13. Isosurfaces $u_2 = -0.5$ (the first row) and slices of $u_2$ at $x_2 = 0$ (the second row) for the collisions of two elliptical vector solitons. The parameters are the same as that in Fig. 12.

of four circular ring solitons for the component $u_2$ are shown in Fig. 15. From these two figures, we can see the large square ring vector solitons are formed because of the interactions of four elliptical–circular ring vector solitons.

5. Conclusion

In this paper, the vector solitons of time-fractional coupled sine–Gordon equations which are defined on two- and three-dimensional unbounded domains are numerically investigated by using the finite difference/Hermite–Galerkin spectral method. The problem is directly solved on unbounded domains by considering Hermite–Galerkin spectral method with scaling factor, which makes it an advantage that the errors introduced by domain truncations can be avoided. In addition, the finite difference method based on the Crank–Nicolson method is used to discretize the temporal Caputo fractional derivative. The derived nonlinear algebraic system is solved by using the matrix decomposition method and iteration algorithm. Our numerical results show that: (i) the proposed method can arrive at the spectral accuracy in the sense that the $L^2$-errors decay exponentially in space; (ii) the numerical method is convergent of order $(3 - \max\{\alpha_1, \alpha_2\})$ in time where the parameters $\alpha_1$ and $\alpha_2$ are the orders of Caputo fractional derivative for the components $u_1$ and $u_2$, respectively; (iii) the method is stable against the small external perturbations of the initial conditions; (iv) increasing the values of $\alpha_1$ and $\alpha_2$ can enhance the radiations of the vector solitons and concentrate the energy in the physical system which makes the amplitudes of the vector solitons higher; (v) it is highlighted that the nonlinear dynamics of the interactions of vector solitons is quite complicated and may involve a sequence of a variety of different interaction scenarios, including collision, radiation, oscillation, and energy exchange. Therefore, by using the proposed computational method, we conclude that our successful simulation of such complicated nonlinear behavior of the vector solitons.
isostatic forces u₁ = 0.7 (the first row) and slices of u₁ at x₂ = 0 (the second row) for the collisions of four elliptical–circular ring vector solitons with c = β = 1, α₁ = 1.5, α₂ = 1.7, N = 50, and τ = 1e−1.

Fig. 14. Isosurfaces u₂ = −0.8 (the first row) and slices of u₂ at x₂ = 0 (the second row) for the collisions of four elliptical–circular ring vector solitons. The parameters are the same as that in Fig. 14.

Acknowledgments

The authors express their sincere thanks to the referees for their valuable comments which led to an improved version.

References