The fractional variational iteration method using He's polynomials

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A B S T R A C T

In this Letter, by introducing He’s polynomials in the correct functional, we propose a new fractional variational iteration method to solve nonlinear time-fractional partial differential equations involving Jumarie’s modified Riemann–Liouville derivative. Several examples have been solved to illustrate the proposed method is quite effective and convenient for solving kinds of nonlinear fractional order problems.

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1. Introduction

In recent years, fractional differential equations (FDEs) have gained much attention due to exact description of nonlinear phenomena in fluid flow, biology, physics, engineering and other areas of science. The aim of the present work is to construct solitary pattern solutions of nonlinear time-fractional dispersive equations given in the following models:

- The time-fractional Boussinesq-type equation:
  \[ D_t^{2\alpha} u - u_{xx} - (u^2)_{xx} + [u(x)]_{xx} = 0, \quad 0 < \alpha \leq 1. \]  

- The time-fractional Boussinesq-type equation:
  \[ D_t^{2\alpha} u + a(u^2)_{xx} - b[u(x)]_{xx} = 0, \quad 0 < \alpha \leq 1, \quad a, b > 0. \]  

- The time-fractional Klein–Gordon-type equation:
  \[ D_t^{2\alpha} u - a(u^2)_{xx} + b(u^2)_{xxxx} = 0, \quad 0 < \alpha \leq 1, \quad a, b > 0. \]  

Recently, various Boussinesq-type and Klein–Gordon-type equations have arisen in a large range of physical phenomena. For example, the Boussinesq-type equations can be used to describe small oscillations of nonlinear beams, long waves over an even slope, shallow-water waves, shallow fluid layers and nonlinear atomic chains, and so on. The Klein–Gordon-type equations can be applied to study complex group velocity and energy transport in absorbing media, short waves in nonlinear dispersive models, propagation of dislocations within crystals, etc. In reality, a physical phenomenon may depend not only on the time instant but also the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order. Therefore, it is clear that Eqs. (1)–(3) are very important in the field of mathematical physics. For better understanding the mechanisms of the complicated nonlinear physical phenomena, searching for explicit solutions of the aforementioned three nonlinear time-fractional dispersive equations is of great importance. In the past, many powerful methods have been established and developed to obtain numerical and analytical solutions of FDEs, such as finite difference method, Adomian decomposition method (ADM), differential transform method (DTM) and so on. Thanks to the efforts of many researchers, several FDEs have been investigated and solved, such as the nonlinear time-fractional advection partial differential equation, space- and time-fractional Fokker–Planck equation, time-fractional diffusion equation, fractional generalized Burgers’ fluid, fractional heat- and wave-like equations, etc.

Motivated and inspired by the on-going research in these areas, we modify the fractional variational iteration method (FVIM) by introducing He’s homotopy perturbation (He’s polynomials) in the correct functional, which is called the fractional...
variational iteration method using He’s polynomials (FVIMHP). This method is a combination of the variational iteration method (VIM) [26–28], the homotopy perturbation method (HPM) [29, 30] and the definition of the Jumarie’s modified Riemann–Liouville derivative [1–4]. The Jumarie’s modified Riemann–Liouville derivative has many interesting properties [4]. For example, it can be applied to functions which are differentiable or not. The FVIMHP provides solutions of the problem in a rapid convergent series which may lead to the solution in a closed form.

This Letter is organized as follows: In Section 2, the basic definition of the Jumarie’s modified Riemann–Liouville derivative and the main steps of the FVIMHP are given. To show the efficiency of the FVIMHP, in Section 3 we construct the analytical solutions of Eqs. (1)–(3). In Section 4, some conclusions are given.

2. Modified Riemann–Liouville derivative and FVIMHP

Assume \( f : R \rightarrow R, x \rightarrow f(x) \) denote a continuous (but not necessarily differentiable) function. The fractional Riemann–Liouville integral is defined as

\[
I^\alpha_x f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha - 1} f(\xi) \, d\xi, \quad \alpha > 0.
\]

(4)

The Jumarie’s modified Riemann–Liouville derivative of order \( \alpha \) is defined by the expression [2]

\[
f^{(\alpha)}(x) := \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha - 1} (f(\xi) - f(0)) \, d\xi, \quad \alpha < 0.
\]

(5)

For positive \( \alpha \), we will set

\[
f^{(\alpha)}(x) = (f^{(\alpha - 1)}(x))', \quad 0 < \alpha < 1,
\]

\[
f^{(\alpha)}(x) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} (f(\xi) - f(0)) \, d\xi,
\]

(6)

and

\[
f^{(\alpha)}(x) := \left( f^{(\alpha)}(x) \right)^{(\alpha - n)}, \quad n \leq \alpha < n + 1, \quad n \geq 1.
\]

(7)

G. Jumarie’s fractional derivative of order \( \alpha \) is defined by the limit

\[
f^{(\alpha)} = \lim_{\varepsilon \to 0} \frac{\Delta^\alpha f(x)}{\varepsilon^\alpha},
\]

(8)

where

\[
\Delta^\alpha f(x) = (f(x) - f(x + \varepsilon)).
\]

(9)

In Eq. (9), \( f \) is the function. The Jumarie’s modified Riemann–Liouville derivative (via integral) is strictly equivalent to Eq. (8) [3, 22].

The integral with respect to \((dx)\alpha\) is defined as the solution of the fractional differential equation

\[
dy = f(x)(dx)^{\alpha}, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha \leq 1,
\]

(10)

which is provided by the following result [3]:

Let \( f(x) \) denote a continuous function, then the solution of Eq. (10) is defined by the equality

\[
y = \int f(\xi)(d\xi)^{\alpha} = \alpha \int (x - \xi)^{\alpha - 1} f(\xi) \, d\xi, \quad 0 < \alpha \leq 1.
\]

(11)

We present the essential steps of the fractional variational iteration method using He’s polynomials as follows:

Step 1: Suppose that a nonlinear equation, say in two independent variables \( x \) and \( t \), is given by

\[
D^\gamma_t u(x, t) = L(u(x, t)) + N(u(x, t)) + g(x, t),
\]

(12)

where \( D^\gamma_t (\cdot) \) is the Jumarie’s modified Riemann–Liouville derivative, \( \gamma > 0 \), \( L \) is a linear operator, \( N \) is a nonlinear operator, \( u = u(x, t) \) is an unknown function, and \( g(x, t) \) is the forcing term.

Step 2: According to the FVIM, we can construct the following correct functional

\[
u_{k+1}(x, t) = u_k(x, t) + l^\gamma_k \left\{ \lambda(\tau) \left( D^\gamma_t u_k(x, \tau) - L(u_k(x, \tau)) \right) - N(\tilde{u}_k(x, \tau)) - g(x, \tau) \right\},
\]

(13)

where \( \lambda \) is the Lagrange multiplier, which can be identified optimally via the variational theory. The subscript \( k \geq 0 \) denotes the \( k \)-th approximation, the function \( \tilde{u}_k \) is considered as a restricted variation, that is \( \delta u_k = 0 \).

Step 3: We use HPM in the correction functional (13) as follows

\[
\sum_{k=0}^{\infty} q^k u_k(x, t) = u_0(x, t) + q \left\{ \sum_{k=1}^{\infty} q^k u_k(x, t) \right. \]

\[
+ l^\gamma_k \left. \left\{ \lambda(\tau) \left( \sum_{k=0}^{\infty} q^k D^\gamma_t u_k(x, \tau) \right) \right. \right.

\[
- \sum_{k=0}^{\infty} q^k L(u_k(x, \tau)) - \sum_{k=0}^{\infty} q^k N(\tilde{u}_k(x, \tau)) \right. \right.

\[
- g(x, \tau) \right\} \right\},
\]

(14)

which is the FVIMHP and is formulated by coupling of FVIM and He’s polynomials. \( \lambda \) is the general Lagrange multiplier obtained in Step 2, \( q \in [0, 1] \) is an imbedding parameter, and \( u_0 \) is an initial approximation of Eq. (12).

Step 4: Comparing with the coefficients of the same power of \( q \) in the both sides of the expression (14), \( u_i \) (\( i = 0, 1, 2, \ldots \)) can be obtained. According to the HPM, we have

\[
u = u_0 + u_1 + u_2 + \cdots
\]

(15)

which is the solution of Eq. (12).

3. Applications of the FVIMHP

In this section, we consider the nonlinear time-fractional partial differential equations (1)–(3) to demonstrate the performance and efficiency of the FVIMHP.

3.1. The time-fractional Boussinesq-type equation

Consider the nonlinear time-fractional Boussinesq-type equation

\[
D^\alpha_t u - u_{xx} - (u^2)_{xx} + [u(u)]_{xxx} = 0, \quad 0 < \alpha \leq 1,
\]

with the initial conditions
\( u(x, 0) = -2(c^2 - 1) \sinh^2 \left( \frac{1}{2} x \right) \).

\( D_t^\alpha u(x, 0) = (c^2 - 1) \sinh(x) \),

where \( c \) is an arbitrary constant.

Its iteration formula, according to the fractional calculus and FVIMHP given in Section 2, can be constructed as follows:

\[
\sum_{k=0}^{\infty} q^k u_k = u_0 + q \left\{ \sum_{k=1}^{\infty} q^k u_k + \frac{t^\alpha}{\Gamma(1 + \alpha)} \sum_{k=0}^{\infty} q^k D_{2t}^\alpha u_k \right. \\
- \sum_{k=0}^{\infty} q^k (u_k)_{xx} - \sum_{k=0}^{\infty} q^k (u_k^a)_{xx} \\
+ \left. \sum_{k=0}^{\infty} q^k [(u_k)_{xx}]_{xx} \right\} \right. \\
= u_0 + q \left\{ \sum_{k=1}^{\infty} q^k u_k \\
+ \frac{1}{\Gamma(1 + \alpha)} \int_0^t (t - \tau)^\alpha \left( \sum_{k=0}^{\infty} q^k D_{2t}^\alpha u_k \right) \\
- \sum_{k=0}^{\infty} q^k (u_k)_{xx} - \sum_{k=0}^{\infty} q^k (u_k^a)_{xx} \\
+ \sum_{k=0}^{\infty} q^k [(u_k)_{xx}]_{xx} \right\} \right\}.
\]

(16)

From the initial approximation \( u_0(x, t) = -(c^2 - 1) [\cosh(x) - \sinh(x)] \frac{t^\alpha}{\Gamma(1 + \alpha)} - 1 \), we can derive

\( q^0: \ u_0(x, t) = -(c^2 - 1) \left[ \cosh(x) - \sinh(x) \right] \frac{t^\alpha}{\Gamma(1 + \alpha)} - 1 \).

\( q^1: \ u_1(x, t) = -(c^2 - 1) \left[ c^2 \cosh(x) \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \\
- c^3 \sinh(x) \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} \right] \).

\( q^2: \ u_2(x, t) = -(c^2 - 1) \left[ c^4 \cosh(x) \frac{t^{4\alpha}}{\Gamma(1 + 4\alpha)} \\
- c^5 \sinh(x) \frac{t^{5\alpha}}{\Gamma(1 + 5\alpha)} \right] \).

\( q^3: \ u_3(x, t) = -(c^2 - 1) \left[ c^6 \cosh(x) \frac{t^{6\alpha}}{\Gamma(1 + 6\alpha)} \\
- c^7 \sinh(x) \frac{t^{7\alpha}}{\Gamma(1 + 7\alpha)} \right] \).

\vdots

Consequently, we have the following solution of Eq. (1) in a series form

\[
u(x, t) = -(c^2 - 1) \left[ \cosh(x) \left( 1 + c^2 \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \cdots \right) - 1 \right] \\
+ c^4 \frac{t^{4\alpha}}{\Gamma(1 + 4\alpha)} + \cdots - 1 \\
+ (c^2 - 1) \sinh(x) \left[ c^\alpha \frac{t^\alpha}{\Gamma(1 + \alpha)} + c^3 \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} + \cdots \right].
\]

The solitary pattern solution in a closed form of Eq. (1) is given by

\[
u(x, t) = -(c^2 - 1) \left[ \cosh(x) \cosh(ct^\alpha, \alpha) \\
- \sinh(x) \sinh(ct^\alpha, \alpha) - 1 \right],
\]

(17)

where the functions \( \cosh(z, \alpha) \) and \( \sinh(z, \alpha) \) are defined as

\[
\cosh(z, \alpha) = \sum_{n=0}^{\infty} \frac{2^n}{\Gamma(1 + 2n\alpha)}.
\]

\[
\sinh(z, \alpha) = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{\Gamma(1 + (2n + 1)\alpha)}.
\]

If we select the initial approximation \( u_0(x, t) = -(c^2 - 1) [\cosh(x) - \sinh(x)] \frac{t^\alpha}{\Gamma(1 + 2\alpha)} + 1 \), the solitary pattern solution of Eq. (1) will be of the form

\[
u(x, t) = -(c^2 - 1) \left[ \cosh(x) \cosh(ct^\alpha, \alpha) \\
- \sinh(x) \sinh(ct^\alpha, \alpha) + 1 \right].
\]

(18)

It is interesting to point out that when \( \alpha = 1 \), we have \( \cosh(z, 1) = \cosh(z) \) and \( \sinh(z, 1) = \sinh(z) \). Therefore, we get the solitary wave solutions

\[
u(x, t) = -2(c^2 - 1) \sinh^2 \left( \frac{1}{2} (x - ct) \right),
\]

and

\[
u(x, t) = 2(c^2 - 1) \cosh^2 \left( \frac{1}{2} (x - ct) \right)
\]

immediately upon replacing \( \alpha \) by 1 in (17) and (18) respectively, which are exactly the same as solutions obtained in [31,32].

3.2. The time-fractional \( B(2, 1, 1) \)-type equation

Consider the following time-fractional \( B(2, 1, 1) \)-type equation

\[
D_t^{2\alpha} u + a(u^2)_{xx} - b[u(u)_x]_x = 0, \quad 0 < \alpha \leq 1, \ a, b > 0,
\]

with the initial conditions

\[
u(x, 0) = \frac{2c^2}{a} \sinh^2 \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right).
\]

\[
D_t^\alpha u(x, 0) = -\sqrt{\frac{1}{ab}} \cosh \left( \sqrt{\frac{a}{b}} \right).
\]

where \( c \) is an arbitrary constant.

Its iteration formula, according to the fractional calculus and FVIMHP described in Section 2, can be constructed as follows:

\[
\sum_{k=0}^{\infty} q^k u_k = u_0 + q \left\{ \sum_{k=1}^{\infty} q^k u_k + \frac{t^\alpha}{\Gamma(1 + \alpha)} \sum_{k=0}^{\infty} q^k D_{2t}^\alpha u_k \right. \\
- \sum_{k=0}^{\infty} q^k (u_k)_{xx} - \sum_{k=0}^{\infty} q^k (u_k^a)_{xx} \\
+ \left. \sum_{k=0}^{\infty} q^k [(u_k)_{xx}]_{xx} \right\} \right. \\
= u_0 + q \left\{ \sum_{k=1}^{\infty} q^k u_k \\
+ \frac{1}{\Gamma(1 + \alpha)} \int_0^t (t - \tau)^\alpha \left( \sum_{k=0}^{\infty} q^k D_{2t}^\alpha u_k \right) \\
- \sum_{k=0}^{\infty} q^k (u_k)_{xx} - \sum_{k=0}^{\infty} q^k (u_k^a)_{xx} \\
+ \sum_{k=0}^{\infty} q^k [(u_k)_{xx}]_{xx} \right\} \right\}.
\]

(19)
From the initial approximation $u_0(x,t) = \frac{c^2}{a} \left[ \cosh \left( \sqrt{\frac{a}{b}} x \right) - \sqrt{\frac{a}{b}} \sinh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^\alpha}{\Gamma(1+\alpha)} - 1 \right]$, we can derive

$$q^0: \quad u_0(x,t) = \frac{c^2}{a} \left[ \cosh \left( \sqrt{\frac{a}{b}} x \right) - \sqrt{\frac{a}{b}} \sinh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^\alpha}{\Gamma(1+\alpha)} - 1 \right].$$

$$q^1: \quad u_1(x,t) = \frac{c^2}{a} \left[ \frac{a^2 c^2}{b^2} \cosh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \sqrt{\frac{a^2 c^2}{b^2}} \sinh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right].$$

$$q^2: \quad u_2(x,t) = \frac{c^2}{a} \left[ \frac{a^3 c^4}{b^5} \cosh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - \sqrt{\frac{a^3 c^4}{b^5}} \sinh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} \right].$$

$$q^3: \quad u_3(x,t) = \frac{c^2}{a} \left[ \frac{a^4 c^6}{b^7} \cosh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} - \sqrt{\frac{a^4 c^6}{b^7}} \sinh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^{7\alpha}}{\Gamma(1+7\alpha)} \right].$$

Consequently, we have the following solution of Eq. (2) in a series form

$$u(x,t) = \frac{c^2}{a} \left[ \cosh \left( \sqrt{\frac{a}{b}} x \right) \left( 1 + \frac{a^2 c^2}{b^2} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^3 c^4}{b^5} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots \right) - 1 \right].$$

The solitary pattern solution in a closed form of Eq. (2) is given by

$$u(x,t) = \frac{c^2}{a} \left[ \cosh \left( \sqrt{\frac{a}{b}} x \right) \cosh \left( \sqrt{\frac{a}{b}} c t^{\alpha}, \alpha \right) - \sinh \left( \sqrt{\frac{a}{b}} c t^{\alpha}, \alpha \right) - 1 \right].$$

If we select the initial approximation $u_0(x,t) = \frac{c^2}{a} \left[ \cosh \left( \sqrt{\frac{a}{b}} x \right) - \sqrt{\frac{a}{b}} \sinh \left( \sqrt{\frac{a}{b}} x \right) \frac{t^\alpha}{\Gamma(1+\alpha)} + 1 \right]$, the solitary pattern solution will be of the form

$$u(x,t) = -\frac{c^2}{a} \left[ \cosh \left( \sqrt{\frac{a}{b}} x \right) \cosh \left( \sqrt{\frac{a}{b}} c t^{\alpha}, \alpha \right) - \sinh \left( \sqrt{\frac{a}{b}} c t^{\alpha}, \alpha \right) + 1 \right].$$

Substituting $\alpha = 1$ in (20) and (21) respectively, we get the following solitary wave solutions of Eq. (2)

$$u(x,t) = \frac{-2c^2}{a} \sin^2 \left( \frac{1}{2} \sqrt{\frac{a}{b}} (x - ct) \right).$$

which are exactly the same as solutions obtained in [32,33].

### 3.3. The time-fractional Klein–Gordon-type equation

Consider the following time-fractional Klein–Gordon-type equation

$$D_t^{2\alpha} \left[ u - a(u^2)_{xx} + b(u^2)_t \right] = 0, \quad 0 < \alpha \leq 1, \quad a, b > 0,$$

with the initial conditions

$$u(x,0) = -\frac{4c^2}{3a} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right),$$

$$D_t^{2\alpha} \left[ u(x,0) \right] = \frac{c^3}{3\sqrt{ab}} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right),$$

where $c$ is an arbitrary constant.

Its iteration formula, according to the fractional calculus and FVIIMHP given in Section 2, can be constructed as follows:

$$\sum_{k=0}^{\infty} q^k u_k = u_0 + \left\{ \sum_{k=0}^{\infty} q^k u_k + c_0 \left( \frac{t^{\alpha}}{\Gamma(1+\alpha)} \left( \sum_{k=0}^{\infty} q^k D_t^{2\alpha} u_k \right) \right) - \sum_{k=0}^{\infty} q^k a(u_k^2)_{xx} + \sum_{k=0}^{\infty} q^k b(u_k^2)_t \right\}.$$
Consequently, we have the following solution of Eq. (3) in a series form

\[
u(x, t) = \frac{-2c^2}{3a} \left[ \cosh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) \left( 1 + \frac{ac^2}{2b} t^{2\alpha} \right) + \frac{a^2c^4}{24b^2} \frac{t^{4\alpha}}{\Gamma(1 + 4\alpha)} + \cdots \right] - 1 \right]
+ \frac{2c^2}{3a} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) \left[ \frac{c}{2} \sqrt{\frac{a}{b}} t^{\alpha} + \cdots \right]
+ \frac{c^3}{2^3} \frac{a^3}{b^3} t^{3\alpha} + \cdots .
\]

The solitary pattern solution in a closed form of Eq. (3) is given by

\[
u(x, t) = \frac{-2c^2}{3a} \left[ \cosh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) \cosh \left( \frac{c}{2} \sqrt{\frac{a}{b}} t^{\alpha}, \alpha \right) \right]
- \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) \sinh \left( \frac{c}{2} \sqrt{\frac{a}{b}} t^{\alpha}, \alpha \right) - 1 \right].
\]

If we select the initial approximation \( u_0(x, t) = \frac{2c^2}{3a} \left[ \cosh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) - \frac{1}{2} \sqrt{\frac{a}{b}} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) t^{\alpha} + 1 \right] \), the solitary pattern solution will be of the form

\[
u(x, t) = \frac{2c^2}{3a} \left[ \cosh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) \cosh \left( \frac{c}{2} \sqrt{\frac{a}{b}} t^{\alpha}, \alpha \right) \right]
- \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} x \right) \sinh \left( \frac{c}{2} \sqrt{\frac{a}{b}} t^{\alpha}, \alpha \right) + 1 \right] .
\]

Substituting \( \alpha = 1 \) in (23) and (24) respectively, we get the following solitary wave solutions of Eq. (3)

\[
u(x, t) = -\frac{4c^2}{3a} \sinh^2 \left( \frac{1}{4} \sqrt{\frac{a}{b}} (x - ct) \right),
\]
and

\[
u(x, t) = -\frac{4c^2}{3a} \cosh^2 \left( \frac{1}{4} \sqrt{\frac{a}{b}} (x - ct) \right) .
\]

which are exactly the same as solutions obtained in [31,32].

4. Conclusion

In this Letter, we applied the fractional variational iteration method using He’s polynomials for solving nonlinear time-fractional partial differential equations. As applications of the proposed method, the nonlinear time-fractional differential equations (1)–(3) with initial conditions have been successfully solved. Furthermore, as the FVIMHP does not require discretization of the variables, it is not affected by computational round off errors, and one is not faced with necessity of large computer memory and time.

Moreover, the basic idea of the FVIMHP proposed in this Letter can also be used to solve other fractional differential equations, such as time-fractional dispersive KdV-type equations [34,35], time-fractional Navier–Stokes equation [36], time-fractional Burgers equations [37] and so on.

Finally, some improved variational iteration methods [38,39] and improved homotopy perturbation methods [40–42] have emerged in recent research. By combining these two kinds of improved methods, it is possible to construct new methods to accelerate the convergence of the solution.

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