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Nonperturbative procedure for stable K-brane

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We propose a novel first-order formalism for a type of K-brane system. An example solution is presented and studied. We illustrate how the deviation parameter affects the properties of the model, such as the stability of the solutions and the localization of fermions and gravitons. We argue that our solution is stable against linear perturbations. The tensor zero mode of a graviton can be localized on the brane while the scalar zero mode cannot. The localization condition for fermion is also discussed.

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I. INTRODUCTION

Scalar fields with noncanonical kinetic terms, namely, the K-fields, were initially introduced to cosmology as a new mechanism of inflation [1–3]. Since then, K-fields have been extensively studied in many fields. One of its interesting applications is the modeling of thick K-branes. Unlike the standard thick brane models, where branes are usually generated by canonical scalar field(s) (see [4–14] for some of the original papers of the brane world, and Refs. [15–18] for reviews), thick K-branes are domain wall branes generated by scalar fields with noncanonical kinetic terms [19–24].

Like any kind of brane world model, the study of thick K-brane models also contains at least three nontrivial issues: solutions, stability, and properties (for example, the localization of bulk matter fields and gravitons). The stability of thick K-brane has been generally discussed in Ref. [25], where the stability conditions for solutions of a large class of thick K-brane models were derived.

As to the issue of finding solutions, it is worth mentioning that there are interesting dualities between some K-field models and the standard model. In other words, some K-field models support the same solution given by the model with the standard kinetic term of the scalar field [26]. Such K-field models are called the twin-like models of the corresponding standard model, and vice versa. Inspired by the original work [26], some authors studied twin-like models in the brane world [27–30]. The twin-like duality offers us an alternative way to find simple analytical K-brane solutions. However, some of the models can be distinguished from the standard model only when linear perturbations (especially the scalar perturbations) are considered, while some are indistinguishable even at the linear order [30,31], and hence might be phenomenologically trivial.

In this paper we follow another route to search for analytical solutions, i.e., the first-order formalism. In this formalism, the original second-order Einstein equations are rewritten as some first-order equations of the superpotential (an arbitary function of the background scalar field). The first application of this formalism in thick K-brane models was proposed by Bazeia *et al.* [22] to solve the following two types of models:

(i) type I: $F(X) = X - \alpha X^2$, and
(ii) type II: $F(X) = -X^2/2$.

Here, X and $F(X)$ represent the standard and generalized kinetic terms of the background scalar field, respectively. The parameter α in the type I model represents the deviation from the standard model, so let us call it *the deviation parameter.*

Assuming α to be small, the authors of Ref. [22] found some analytical (but not exact) solution for the type I model. The trapping of bulk fermions on the corresponding branes was discussed in Ref. [24].

In the present paper, we report a new first-order formalism that enables us to obtain exact analytical K-brane solutions of the type I model. The stability of our solution against liner perturbation as well as the localization of fermions and gravitons are studied. The type II model has been analytically solved in Ref. [22], so we will omit it here.

In the next section, we briefly review the K-brane model and the stability condition for an arbitrary solution. We revisit the type I model of Ref. [22] in Sec. III, where a new first-order formalism is established to solve the system. In particular, we study the Sine-Gordon superpotential as an example, and give the corresponding solution. With this solution, we study how the deviation parameter α affects the properties of the model, including the localization of fermions (Sec. IV), and the localization of gravitons of both tensor and scalar parts (see Sec. V). In the end, we give a brief summary of our results.

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II. A REVIEW ON K-BRANE AND STABILITY CONDITIONS

We study the simplest thick K-brane model, where a background K-field minimally couples with gravity:

$$S = \int d^5x\sqrt{-g}\left(\frac{1}{2\kappa_5^2}R + \mathcal{L}(\phi, X)\right). \tag{1}$$

Here, $\kappa_5^2 = 8\pi G_5$ is the gravitational coupling constant and G_5 is the five-dimensional Newtonian constant. $X \equiv -\frac{1}{2}g^{MN}\nabla_M\phi\nabla_N\phi$ represents the kinetic term of the background scalar field ϕ. In the standard model of a thick brane, $\mathcal{L} = X - V(\phi)$, where $V(\phi)$ is an arbitrary potential of the scalar field. The Einstein equations are

$$G_{MN} \equiv R_{MN} - \frac{1}{2}g_{MN}R = \kappa_5^2 T_{MN}, \tag{2}$$

where the energy-momentum tensor is defined as

$$T_{MN} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{MN}} = g_{MN}\mathcal{L} + \mathcal{L}_X\nabla_M\phi\nabla_N\phi. \tag{3}$$

In this paper, we always use latin letters, $M, N, \cdots$, as the indices of bulk coordinates, and greek letters, $\mu, \nu, \cdots$, as brane coordinate indices. We always use $\mathcal{L}_g$ to denote the derivative of $\mathcal{L}$ with respect to g, e.g., $\mathcal{L}_X \equiv \partial\mathcal{L}/\partial X$. For simplicity, the extra dimension is labeled as $y \equiv x^5$. Then the general metric that preserves four-dimensional Poincaré symmetry takes the following form:

$$ds^2 = \mathrm{e}^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{4}$$

where the four-dimensional Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, and $\mathrm{e}^{2A(y)}$, is called the warp factor. With this metric, we can explicitly write the Einstein equations as

$$-3\partial_y^2A = \kappa_5^2\mathcal{L}_X(\partial_y\phi)^2, \tag{5a}$$

$$6(\partial_yA)^2 = \kappa_5^2(\mathcal{L} + \mathcal{L}_X(\partial_y\phi)^2). \tag{5b}$$

The equation of motion for the scalar field is given by

$$(\partial_y^2\phi)(\mathcal{L}_X + 2X\mathcal{L}_{XX}) + \mathcal{L}_\phi - 2X\mathcal{L}_{X\phi} = -4\mathcal{L}_X(\partial_y\phi)(\partial_yA). \tag{6}$$

This equation can be derived from Eqs. (5). Therefore, only two of the dynamical equations are independent.

In principle, one can find uncountable domain wall solutions with different $\mathcal{L}$ (simply because we cannot fix solutions of A, ϕ, and $V(\phi)$ by using only two independent equations). However, not all the solutions are stable against small perturbations around them. The stability of a general class of K-brane models was studied in Ref. [25]; the conclusion is that models with

$$\mathcal{L}_X > 0, \qquad \gamma \equiv 1 + 2\frac{\mathcal{L}_{XX}X}{\mathcal{L}_X} > 0, \tag{7}$$

are always stable against linear perturbations.

III. THE MODEL AND FIRST-ORDER FORMALISM

Let us study the following model:

$$\mathcal{L} = X - \alpha X^2 - V(\phi). \tag{8}$$

The deviation parameter α can take any value provided that the stability conditions (7) are satisfied. Suppose the scalar field is a kink: $\phi(\pm\infty) = \pm v$ with v being a constant, and $\phi(0) = 0$. Then the stability conditions imply a lower bound on the parameter:

$$\alpha > -\frac{1}{3k^2v^2} \equiv \alpha_c. \tag{9}$$

The same model was studied in Refs. [22,32,33], where the authors assumed that the first-order derivative of the warp factor is an arbitrary function of ϕ, called the superpotential $W(\phi)$:

$$\partial_yA = -\frac{\kappa_5^2}{3}W(\phi). \tag{10}$$

Then the Einstein equations (5) can be rewritten as

$$\partial_y\phi + \alpha(\partial_y\phi)^3 = W_\phi, \tag{11}$$

$$V = \frac{1}{2}(\partial_y\phi)^2 + \frac{3}{4}\alpha(\partial_y\phi)^4 - \frac{2}{3}\kappa_5^2W^2. \tag{12}$$

In the standard model ($\alpha = 0$), Eq. (11) reduces to

$$\partial_y\phi = W_\phi. \tag{13}$$

Thus, by introducing the superpotential, one can rewrite the original second-order Einstein equations (5) into two first-order Eqs. (10) and (11). For this reason, this method is called the first-order formalism. After being given a suitable $W(\phi)$, the system can be analytically solved [34].

However, when $\alpha \neq 0$, the relation between $\partial_y\phi$ and W_ϕ becomes rather complex, so that analytical analysis of the system is achievable only for a small deviation parameter, $0 < \alpha \ll 1$; see Refs. [22,32,33]. When α became large, one has to come back to numerical calculation. Therefore, it is interesting to search for a new approach, from which exact analytical solutions can be obtained even when α is very large.

Let us first observe Eq. (5a) and notice that when $\alpha \neq 0$, the right-hand side of Eq. (5a) contains α while the left-hand side is a function of W. This means W must contains α. So, instead of Eq. (10), we assume

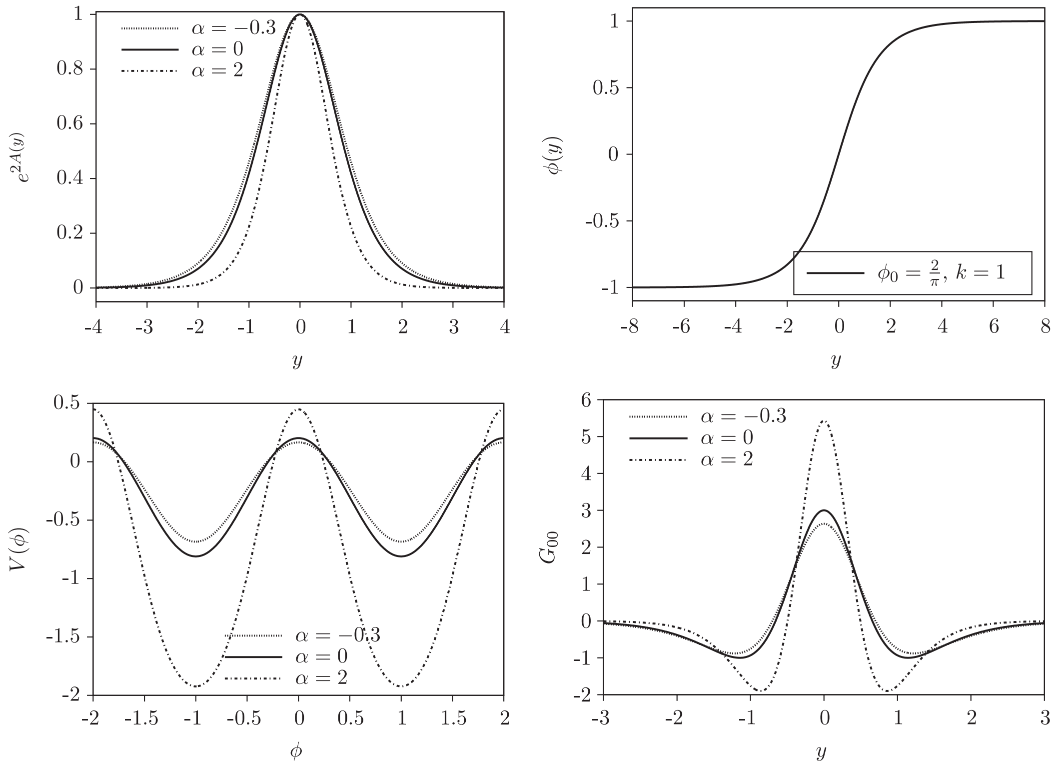

FIG. 1. Plots of the solutions and G_{00} with $k = 1$ and $\phi_0 = \frac{2}{\pi}$.

$$\partial_y A = -\frac{\kappa_5^2}{3}(W(\phi) + \alpha Y(\phi)), \tag{14}$$

where both W and Y are arbitrary functions of ϕ. Plugging the above equation into Eq. (5a) and comparing the coefficients, we immediately obtain

$$\partial_y \phi = W_\phi \tag{15}$$

and

$$Y_\phi = W_\phi^3. \tag{16}$$

From another Einstein equation (5b), we get

$$V = \frac{1}{2}W_\phi{}^2 + \frac{3}{4}\alpha W_\phi{}^4 - \frac{2}{3}\kappa_5^2(W + \alpha Y)^2. \tag{17}$$

Obviously, the construction of first-order formalism inevitably leads to noncanonical modification of the scalar potential. So, from now on, let us call the sum of all the terms in $\mathcal{L}(X, \phi)$ that contains α the α-term.

Equations (14)–(17) constitute the first-order formalism of our model: given a suitable $W(\phi)$, analytical solutions can be found by simply solving these equations. Note that such solutions usually can be embedded into a supersymmetric model [35,36].

To illustrate this, let us consider the Sine-Gordon potential:

$$W = k\phi_0^2 \sin\left(\frac{\phi}{\phi_0}\right), \tag{18}$$

which leads to a kink-like solution for the scalar field:

$$\phi = \phi_0 \arcsin(\tanh(ky)). \tag{19}$$

Without loss of generality, we take the parameter k to be positive, so that $\phi(y \to \pm\infty) = \pm\frac{\pi}{2}\phi_0 \equiv \pm v$, and $\alpha_c = -\frac{4}{3\pi^2}\frac{1}{k^2\phi_0^2}$.

Using the constraint equation (16), we get

$$Y = \frac{1}{12}k^3\phi_0^4\left(9\sin\left(\frac{\phi}{\phi_0}\right) + \sin\left(\frac{3\phi}{\phi_0}\right)\right). \tag{20}$$

Then the scalar potential can be obtained:

$$V = \frac{1}{2}k^2\phi_0^2\cos^2\left(\frac{\phi}{\phi_0}\right) + \frac{3}{4}k^4\alpha\phi_0^4\cos^4\left(\frac{\phi}{\phi_0}\right) - \frac{k^2\phi_0^2}{18}\left(6 + 5k^2\alpha\phi_0^2 + k^2\alpha\phi_0^2\cos\left(\frac{2\phi}{\phi_0}\right)\right)^2\sin^2\left(\frac{\phi}{\phi_0}\right). \tag{21}$$

For convenience, we have taken the dimensionless quantity $\phi_0^2\kappa_5^2 = 3$. Solving Eq. (14), we obtain the expression of the warp factor:

$$A = -\left(1 + \frac{2}{3}k^2\alpha\phi_0^2\right)\ \ln\ (\cosh(ky)) - \frac{1}{6}k^2\alpha\phi_0^2 + \frac{1}{6}k^2\alpha\phi_0^2\mathrm{sech}^2(ky). \tag{22}$$

The asymptotic behavior of A in the boundary of the extra dimension is

$$\lim_{y\to\infty} A = -\left(1 + \frac{2}{3}k^2\alpha\phi_0^2\right)k|y|. \tag{23}$$

Obviously, the geometry of the bulk space-time is asymptotically anti–de Sitter.

Solutions with different values of α are depicted and compared in Fig. 1. The other two parameters are fixed as $k = 1$ and $\phi_0 = \frac{2}{\pi}$, so that the solution with $\alpha > \alpha_c = -1/3$ is stable. Note that instead of studying $T_{00} = -\mathrm{e}^{2A}\mathcal{L}$, we prefer to study the zero-zero component of the Einstein tensor G_{MN}:

$$G_{00} = -3\mathrm{e}^{2A}[2(\partial_y A)^2 + \partial_y^2 A], \tag{24}$$

which is equivalent to T_{00} (up to a constant), but only depends on the warp factor A, so is much easier to calculate.

In the subsequent investigations, it is more convenient to redefine the fifth coordinate as $dy \equiv \mathrm{e}^A dr$, and to rewrite line element (4) in a conformal flat form:

$$ds^2 = \mathrm{e}^{2A(r)}(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2). \tag{25}$$

Let us denote the derivative with respect to r by a prime, for example, $A' \equiv \partial_r A$.

IV. FERMION LOCALIZATION AND THE α-TERM

The issue of trapping fermions with a nonvanished α was discussed in Ref. [24], where the analytical background solution is valid only for $0 < \alpha \ll 1$, and the numerical method was applied when α became larger. The conclusion of Ref. [24] is that the ability to trap fermions is *inversely proportional* to α. The numerical study for a large range of values of α is also consistent with this conclusion. In this section, we will study precisely how a large α would affect the localization of fermions by using the solution given in Sec. III.

As usual, we consider a bulk spin-$\frac{1}{2}$ field $\Theta(x^\mu, r)$, which couples with gravity and the background scalar in the following form:

$$S_{1/2} = \int d^5x\sqrt{-g}\bar{\Theta}(\Gamma^M D_M - \eta\phi)\Theta. \tag{26}$$

Here, $\Gamma^M = (\mathrm{e}^{-A}\gamma^\mu, \mathrm{e}^{-A}\gamma^5)$ and $D_M = \partial_M + \omega_M$ are the Γ-matrices and covariant derivative in the five-dimensional curved space-time, respectively. $\omega_M = (\frac{1}{2}A'\gamma_\mu\gamma_5, 0)$ is the spin connection (see Ref. [37] for details), and η is the Yukawa coupling. The equation of motion takes the following form:

$$\{\gamma^\mu\partial_\mu + \gamma^5(\partial_r + 2A') - \eta\mathrm{e}^A\phi\}\Theta = 0. \tag{27}$$

To obtain the four-dimensional effective action, one needs to decompose the bulk spinor field Θ into chiral Kaluza-Klein (KK) modes:

$$\Theta = \mathrm{e}^{-2A}\sum_C\sum_n \psi_{C,n}(x^\mu)f_{C,n}(r), \tag{28}$$

where n denotes different excitations of KK modes, while $C \in \{+,-\}$ reminds us that each excitation corresponds to two different chiralities. We assume $\psi_{C,n} = C\gamma^5\psi_{C,n}$, namely, $\psi_{+,n}$ and $\psi_{-,n}$ represent the right- and left-chiral spinor KK modes, respectively, and they are mutually related by four-dimensional Dirac equations:

$$\gamma^\mu\partial_\mu\psi_{C,n}(x^\rho) = m_n\psi_{-C,n}(x^\rho). \tag{29}$$

Inserting Eq. (28) into the equation of motion (27), we obtain a Schrödinger-like equation for $f_{C,n}(r)$:

$$(-\partial_r^2 + V_C(r))f_{C,n} = m_n^2 f_{C,n}; \tag{30}$$

the potential is

$$V_C = (\eta\mathrm{e}^A\phi)^2 + C\partial_r(\eta\mathrm{e}^A\phi). \tag{31}$$

Defining $\mathcal{F} \equiv \partial_r + C\eta\mathrm{e}^A\phi$, we can rewrite the Schrödinger-like equation as follows:

$$\mathcal{F}\mathcal{F}^\dagger f_{C,n} = m_n^2 f_{C,n}. \tag{32}$$

The theory of supersymmetric quantum mechanics ensures the semipositive definite of m_n^2, namely, $m_n^2 \geq 0$. In this paper, what we care about is the zero mode $f_{C,0}$ which corresponds to the massless fermion ($m_0^2 = 0$) in four-dimensional space-time. The mass of $f_{C,0}$ is assumed to be generated by the spontaneous symmetry breaking, for example, or by some other mechanisms.

The zero mode can be easily read out from Eq. (32):

$$f_{C,0}(r) \propto \exp\left(C\eta\int_0^r d\bar{r}\mathrm{e}^{A(\bar{r})}\phi(\bar{r})\right). \tag{33}$$

To trap the zero mode on the brane, we require $f_{C,0}$ to be normalizable, namely, the integration $\int dr(f_{C,0})^2$ is finite, or

$$\int dy\exp\left(-A(y) + 2C\eta\int_0^y d\bar{y}\phi(\bar{y})\right) < \infty, \tag{34}$$

as written in the y-coordinate [37]. According to Eq. (23), the integrand asymptotically behaves as

$$\left(k + \frac{2}{3}\alpha k^3 \phi_0^2 + C\eta\pi\phi_0\right)|y|, \quad \text{for } |y| \to +\infty. \tag{35}$$

Obviously, the integral converges only when

$$k + \frac{2}{3}\alpha k^3 \phi_0^2 + C\eta\pi\phi_0 < 0. \tag{36}$$

When $\eta = 0$, this condition can be fulfilled by asking $\alpha < -3/(2k^2\phi_0^2)$. However, this violates the stability condition (9). For $\eta > 0$ and $C = +$, the inequality (36) is always violated, so the right-chiral fermion is non-normalizable for positive η. On the other hand, the left-chiral fermion ($C = -$) can be normalized if

$$\eta > \frac{k}{\pi\phi_0} + \frac{2}{3\pi}\alpha k^3 \phi_0 > \frac{7k}{9\pi\phi_0} > 0. \tag{37}$$

To obtain the second inequality, we used Eq. (9).

In sum, even the smaller α can strengthen the localization of the fermion; a positive Yukawa coupling is necessary to localize the left-chiral fermion zero mode. Given a fixed coupling η, a large positive α would destroy the localization condition (37). Our results are consistent with those of Ref. [24].

V. GRAVITONS AND THE α-TERM

The localization of gravitational modes is another important issue because it relates to the reproductions and modifications of the four-dimensional Newtonian gravity. To study the localization of gravitational modes, one needs to analyze the spectrum and configurations of small perturbations $\{\delta g_{MN}, \delta\phi\}$ around the background solution $\{g_{MN}, \phi\}$. In the r-coordinate, we define the perturbed metric as follows:

$$ds^2 = \mathrm{e}^{2A(r)}(\eta_{MN} + h_{MN})dx^M dx^N, \tag{38}$$

namely, $\delta g_{MN} \equiv \mathrm{e}^{2A(r)} h_{MN}(x^\rho, r)$.

To simplify the calculation, the scalar-tensor-vector decomposition is widely applied in the study of linearization of gravitational systems:

$$h_{\mu r} = \partial_\mu F + G_\mu, \tag{39a}$$

$$h_{\mu\nu} = \eta_{\mu\nu}\Psi + \partial_\mu\partial_\nu B + 2\partial_{(\mu}C_{\nu)} + D_{\mu\nu}, \tag{39b}$$

where C_μ and G_μ are transverse vector perturbations:

$$\partial^\mu C_\mu = 0 = \partial^\mu G_\mu, \tag{40}$$

and $D_{\mu\nu}$ is a transverse and traceless (TT) perturbation:

$$\partial^\nu D_{\mu\nu} = 0 = D^\mu_\mu. \tag{41}$$

Note that all indices are raised with $\eta^{\mu\nu}$, so that $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$.

Under this decomposition, the original field perturbations can be classified into scalar ($\Xi \equiv h_{rr}, \Psi, B, F$, and $\Phi \equiv \delta\phi$), tensor ($D_{\mu\nu}$), and vector (C_μ and G_μ) modes. All these modes are functions of the bulk coordinates x^ρ and r. Each type of mode evolves independently [25], so we can discuss them separately. As in the standard model, the spectrum of the vector modes contains only a nonlocalizable zero mode. So we omit the vector modes and only consider the tensor and scalar modes.

A. Tensor mode

Let us first focus on the tensor part, for which the perturbed metric reads

$$ds^2 = \mathrm{e}^{2A(r)}[(\eta_{\mu\nu} + D_{\mu\nu})dx^\mu dx^\nu + dr^2]. \tag{42}$$

Note that the tensor mode is independent with the scalar part and in our model we only modify the scalar Lagrangian of the standard model, so the dynamical equation for the tensor mode takes the same form as the one in the standard model [7,25]:

$$\Box^{(4)} D_{\mu\nu} + D''_{\mu\nu} + 3A' D'_{\mu\nu} = 0. \tag{43}$$

Consider the following decomposition:

$$D_{\mu\nu}(x^\rho, r) = \mathrm{e}^{-3/2A}\epsilon_{\mu\nu}(x^\rho)\chi(r), \tag{44}$$

where $\epsilon_{\mu\nu}(x^\rho)$ is transverse and traceless, $\eta^{\mu\nu}\epsilon_{\mu\nu} = 0 = \partial^\mu\epsilon_{\mu\nu}$, and satisfies $\Box^{(4)}\epsilon_{\mu\nu} = m^2\epsilon_{\mu\nu}$. Then $\chi(r)$ satisfies the following Schrödinger-like equation:

$$-\chi'' + U_T(r)\chi = m^2\chi, \tag{45}$$

with

$$U_T(r) = \frac{9}{4}A'^2 + \frac{3}{2}A''. \tag{46}$$

This equation can be factorized as

$$\mathcal{J}\mathcal{J}^\dagger\chi = m^2\chi, \tag{47}$$

where

$$\mathcal{J} \equiv \partial_r + \frac{3}{2}A', \qquad \mathcal{J}^\dagger = -\partial_r + \frac{3}{2}A'. \tag{48}$$

According to the supersymmetric quantum mechanics, such a factorization implies $m^2 \geq 0$. So the model is stable against the tensor perturbation. Meanwhile, the zero mode can be easily read as

$$\chi^{(0)} \propto e^{3/2A}. \tag{49}$$

The normalization condition for the zero mode is

$$\int dr e^{3A(r)} = \int dy e^{2A(y)} < \infty. \tag{50}$$

This condition is satisfied if

$$\alpha > -\frac{3}{2}\frac{1}{k^2\phi_0^2} \equiv \alpha_t. \tag{51}$$

Recall that the stability condition is $\alpha > \alpha_c = -\frac{4}{3\pi^2}\frac{1}{k^2\phi_0^2}$. Obviously, $\alpha_c > \alpha_t$, so we can conclude that any solution that satisfies the stability condition supports a localizable tensor zero mode. As a result, four-dimensional Newtonian gravity can be reproduced in these models.

In addition to the zero mode, we have a continuum spectrum that causes a small scale correction to the Newtonian potential. According to Refs. [10,38], the correction is determined by the behavior of $U_T(r)$ at large r. One can easily prove that the asymptotic behavior of $U_T(r)$ is independent of the parameter α, so we conclude that the correction to the Newtonian potential is $\Delta\mathcal{V}_{\text{Newton}} \propto 1/r^3$ no matter what value α takes.

The next question is, does the parameter α affect the resonant spectrum of the tensor mode? To illustrate this question, let us study the following equation:

$$\mathcal{J}^\dagger\mathcal{J}\tilde{\chi} = m^2\tilde{\chi}. \tag{52}$$

This equation looks like Eq. (47), except the order of the operators is reversed. In supersymmetric quantum mechanics, $\tilde{\chi}$ is called the superpartner of χ. Except for the ground state, superpartners share the same spectrum. Thus, if $\tilde{\chi}$ has massive resonant peaks, so does χ. Expanding Eq. (52), we obtain another Schrödinger-like equation where the potential is given by

$$\tilde{U}_T(r) = \frac{9}{4}A'^2 - \frac{3}{2}A''. \tag{53}$$

The plot of $\tilde{U}_T(r)$ (Fig. 2) does not show any attractive well, so it is impossible for $\tilde{\chi}$ to have resonant modes; so does χ. From the same plot, we also see that, just like fermions, gravitons are more likely to be trapped on brane with a smaller α.

B. Scalar modes

Let us study the scalar perturbations under the longitude gauge, i.e., take $F = 0 = B$. This gauge completely fixes the gauge freedoms in the scalar section, and the perturbed metric takes a simple form:

$$ds^2 = e^{2A(r)}[\eta_{\mu\nu}(1+\Psi)dx^\mu dx^\nu + (1+\Xi)dr^2]. \tag{54}$$

Then the perturbation equations are [25]

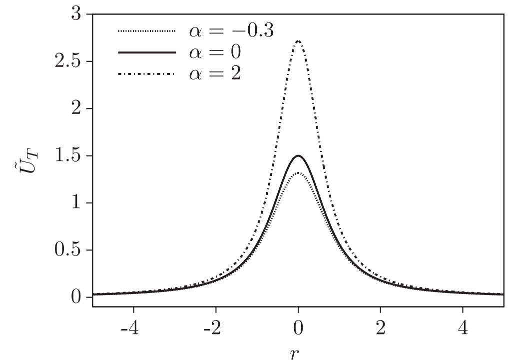

FIG. 2. Plot of $\tilde{U}_T(r)$ for $k = 1$ and $\phi_0 = \frac{2}{\pi}$.

$$-\Psi - \frac{1}{2}\Xi = 0, \tag{55}$$

$$\frac{3}{2}A'\Xi - \frac{3}{2}\Psi' = \kappa_5^2\mathcal{L}_X\phi'\Phi, \tag{56}$$

$$\frac{3}{2}\Box^{(4)}\Psi - \frac{3}{2}\Psi'' - \frac{3}{2}A'\Psi' + \kappa_5^2\phi'^2\mathcal{L}_{XX}e^{-2A}\phi'^2\Psi \\ = 2\kappa_5^2\mathcal{L}_X\phi'\Phi' - \kappa_5^2\phi'^2\mathcal{L}_{XX}e^{-2A}\phi'\Phi' + \kappa_5^2\phi'^2\mathcal{L}_{X\phi}\Phi. \tag{57}$$

Using Eqs. (55), (56), and (5), we can eliminate Ξ, Φ, and $\mathcal{L}_{X\phi}$ in Eq. (57) and obtain the following equation:

$$\Box^{(4)}\Psi + \gamma\Psi'' + \gamma\left[\partial_r \ln\left(\frac{e^{3A}}{\mathcal{L}_X(\phi')^2}\right)\right]\Psi' \\ + 2\gamma A'\left[\partial_r \ln\left(\frac{A'^2}{\mathcal{L}_X(\phi')^2}\right)\right]\Psi = 0, \tag{58}$$

where $\gamma = 1 + 2\frac{\mathcal{L}_{XX}X}{\mathcal{L}_X}$.

In the case $\mathcal{L}_X > 0$, we can rewrite the above equation into a more compact form,

$$\Box^{(4)}\hat{\Psi} + \gamma\hat{\Psi}'' - \gamma\zeta(\zeta^{-1})''\hat{\Psi} = 0, \tag{59}$$

by redefining $\Psi = e^{-3A/2}\mathcal{L}_X^{1/2}\phi'\hat{\Psi}$. Here,

$$\zeta \equiv e^{3A/2}\frac{\phi'}{A'}\mathcal{L}_X^{1/2}. \tag{60}$$

In addition, if $\gamma > 0$, then we can define a new coordinate, z, such that

$$\frac{dz}{dr} = \gamma^{-1/2}. \tag{61}$$

In the new coordinate, we can rewrite Eq. (59) as

$$\Box^{(4)}\hat{\Psi} + \ddot{\hat{\Psi}} - \frac{\dot{\gamma}}{2\gamma}\dot{\hat{\Psi}} - \zeta(\zeta^{-1})^{\cdot\cdot}\hat{\Psi} + \frac{\dot{\gamma}}{2\gamma}(\zeta^{-1})^{\cdot}\zeta\hat{\Psi} = 0, \tag{62}$$

where dots denote the derivatives with respect to z. After a further redefinition of the field $\hat{\Psi} = \gamma^{1/4}\tilde{\Psi}$, we finally obtain what we are looking for, a Schrödinger-like equation:

$$\Box^{(4)}\tilde{\Psi} + \ddot{\tilde{\Psi}} - \tilde{\Psi}\theta(\theta^{-1})^{\cdot\cdot} = 0, \tag{63}$$

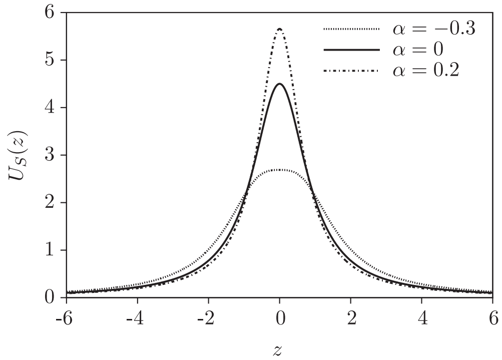

FIG. 3. Plot of $\tilde{U}_S(z) = \theta(\theta^{-1})^{\cdot\cdot}$ for $k = 1$ and $\phi_0 = \frac{2}{\pi}$.

where $\theta \equiv \gamma^{1/4}\zeta$. This equation enables us to introduce the KK decomposition

$$\tilde{\Psi} = \sum_n e^{ip_\mu^n x^\mu}\varphi_n(z). \tag{64}$$

Since $(p_\mu^n)^2 = -m_n^2$, Eq. (63) immediately reduces to the following equation for $\varphi_n(z)$:

$$-\ddot{\varphi}_n + U_S(z)\varphi_n = m_n^2\varphi_n, \tag{65}$$

where $U_S(z) = \theta(\theta^{-1})^{\cdot\cdot}$. Equivalently, Eq. (65) can be factorized as follows:

$$\mathcal{A}^\dagger\mathcal{A}\varphi_n(z) = m_n^2\varphi_n(z), \tag{66}$$

with

$$\mathcal{A} = \frac{d}{dz} + \frac{\dot{\theta}}{\theta}, \qquad \mathcal{A}^\dagger = -\frac{d}{dz} + \frac{\dot{\theta}}{\theta}. \tag{67}$$

This equation assures the positive semidefiniteness of m_n^2 and, equivalently, assures the stability of the solution. Obviously, the zero mode ($m_0^2 = 0$) takes the form $\varphi_0 \propto \theta^{-1}$. As we have pointed out in Ref. [25], the scalar zero mode is always unlocalizable, no matter what value α takes. This can also be concluded from the shape of U_S (see Fig. 3), from which we know there is no bound or resonant state in the spectrum of the scalar graviton. So, the same as the standard model, our model is free of the long range scalar fifth-force problem.[1]

[1]This is because a localizable scalar zero mode corresponds to a new long range force gauge boson, which transmits a new force we have never seen before.

VI. SUMMARY

We set an example for solving the K-brane system via the first-order formalism. A novel first-order formalism is established to solve the type I model of Ref. [22]. An exact analytical solution was obtained by taking the Sine-Gordon superpotential. We also studied the stability of the solution against linear field perturbations. The localization of the fermion and graviton was analyzed.

Our study indicates that the stability conditions demand a lower bound for the deviation parameter α. On the other hand, the requirement of localizing bulk fermions imposes an upper bound for the deviation parameter because a large deviation parameter can violate the localization condition of the fermion zero mode (37) for a given Yukawa coupling. In addition, we studied the localization of tensor and scalar gravitational perturbations. We found that the tensor zero mode is always localizable provided that the stability conditions are satisfied, while the scalar zero mode is always nonlocalizable. There is no sign for gravitational resonance either in the tensor or the scalar section.

Note that although a negative deviation parameter $\alpha_s < \alpha < 0$ does not necessarily break the linear stability of our solution, it may lead to problems when other issues such as causality and UV analyticity are considered [39]. The study of Ref. [39] indicates that α has to be positive at least in models without gravity. Obviously, to make our model a consistent low energy effective theory, it is necessary to consider constraints from other issues. But this is beyond the scope of the current work, so we leave them for our future work.

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