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Extended Object or Group Target Tracking Using Random Matrix with Nonlinear Measurements

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***Abstract*—For extended-object/group-target tracking (EOT/GTT), the random-matrix approach assumes that measurements are linear in the state and noise with a covariance being a random matrix to represent the object extension or target group. In practice, however, most measurements are nonlinear in the state and noise. This paper proposes a random-matrix approach for EOT/GTT using nonlinear measurements. First, a matched linearization (ML) is proposed to linearize nonlinear measurements. The linearized form has two parts. The first is linear in the state, and it is optimized in the sense of minimum mean square error (MMSE). The second part is linear in the extension-related noise with a preserved second moment, which is important since extension information is contained in the covariance of this noise. The linearized measurements can be incorporated into existing random-matrix algorithms after a simple conversion under certain conditions. Second, a variational Bayesian (VB) scheme is proposed for EOT/GTT using the linearized measurements. This approach can be generally applied no matter whether the linearized measurements are converted or not. The effectiveness of the proposed ML and VB approach is demonstrated by simulation results compared with existing random-matrix algorithms.**

***Index Terms*—Extended Object Tracking, Group Target Tracking, Nonlinear Estimation, Variational Bayesian Approach.**

I. Introduction

For traditional radar and sonar based tracking, most approaches consider a target as a point source of measurements due to limited sensor resolution and large sensor error relative to the target size [2], [5], [13], [30]. The kinematic state (e.g., position, velocity, and acceleration) of a target is estimated. Nevertheless, target tracking is still complicated because of the uncertainties in target motion and measurements [19]. With increased resolution of modern sensors (e.g., phased array radar), multiple (e.g., dozens of) measurements from an object can be obtained (as shown in [30] and [13]), and thus treating an object as a point mass becomes less valid [5], [13] and traditional methods do not apply well. Moreover, modern applications require more and more detailed physical information about the objects for classification, identification, tracking, etc. Here, an object may be treated as an extended object (EO) with both kinematic state and extension (e.g., size, shape, and orientation). Also, for a group of closely spaced targets in formation, target tracking faces similar challenges [13]. Knowing the extension of the group formation can be beneficial for e.g., recognition and classification. Such an extended object or target group may also originate a strongly fluctuating number of measurements [13] due to limited sensor capability, physical properties of EOs, and sensor-to-target geometry, etc. Further, one-to-one correspondence between a measurement and an originating point cannot be observed directly or easily. In this case, scattering centers of an extended object and individual targets in a group are also (partially) unresolved [5], [13], which makes tracking even more challenging.

In this paper, we consider nonlinear estimation problems in extended object tracking (EOT) or group target tracking (GTT), in which multiple nonlinear point measurements originated from partially unresolved scattering centers are obtained. Without directly considering each target within a group individually, approaches to EOT can also be applied to GTT. These approaches estimate the overall kinematics and extension of the group. For brevity, discussions will thus be focused on EOT and they are similar for GTT.

Several radar EOT approaches have been proposed [5], [13], [18], including multiple hypothesis tracking [12], probability hypothesis density filters [22], and particle filters [6]. A random hypersurface model was also proposed for EOT [4], where a nonlinear filter is used for estimation.

The approach of using a random matrix [13] appears promising because it jointly estimates the kinematic state and physical extension of an ellipsoidal object without needing measurement association. Also, the final form of this approach is computationally much simpler than the methods mentioned above. Its one-cycle computation (with one-step prediction and update being linear in the state and the extension) is comparable to that of the Kalman filter. Within the random-matrix framework, several further developments have been made, including [5], [7], [14], [15], etc. This approach estimates at time k both the random vector x_k, representing the kinematic state of the object centroid, and a symmetric positive definite (SPD) random matrix X_k, representing the object extension (simplified as an ellipsoid with size, shape, and orientation). Specifically, X_k is assumed the covariance of zero-mean Gaussian noise describing the uncertainty in the measurements caused by the object extension. This noise does not truly exist and can be referred to as the *extension noise* (versus the true-measurement noise).

Existing random-matrix algorithms assume that the measurement is linear in the kinematic state and in the extension noise. In practical tracking systems, however, measurements are nonlinear in the positions of the scattering centers of

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an EO. For example, some measurements are ranges and/or bearings of these centers and thus are nonlinear in both the state and the extension noise. In [9], [29], a nonlinear polar measurement is first converted into Cartesian coordinates and then linearized using a first-order Taylor series expansion. Some existing random-matrix algorithms are then applied for EOT using the linearized model. This approximation is similar to that used by the extended Kalman filter. In [26] and [27], a polar measurement is also converted into Cartesion coordinates and then the covariance matrix of the converted measurement noise is calculated using the approach proposed in [17].

This paper proposes a random-matrix approach to EOT using nonlinear measurements. To utilize the effectiveness and efficiency of the random-matrix approach, the nonlinear measurement function is linearized first. To do this, we not only try to find its optimized linear-in-state form but also preserve the extension information in the second moment as much as possible. For this purpose, a **matched linearization** (ML) approach is proposed to statistically linearize nonlinear measurements for EOT. The linearized function is a sum of two parts for the state and the extension noise, respectively. The first part is linear in the state, more specifically, a product of a coefficient matrix and the centroid position in the state, with coefficients obtained by optimal linearization. The second part as a linear function of the extension noise is so obtained that the linearized measurement and the original measurement have the same second moment. Based on deterministic sampling, specific calculation steps are derived to obtain the above linear coefficients under typical assumptions of existing random-matrix algorithms.

Based on the linearized measurement, existing random-matrix algorithms can be applied if the coefficient matrix of the centroid position is invertible. This condition is not always satisfied. For EOT with this difficulty, a variational Bayesian (VB) approach is proposed in this paper. A VB approach for EOT using random matrix has been proposed in [23] using the measurement model proposed in [5]. In [15], we proposed a new model and derived a Bayesian approach for EOT. In [23], the joint distribution of the state, extension, and ideal noiseless measurement are approximated as the product of their marginal distributions. In our VB approach, the posterior joint distribution of the state and the extension is considered and approximated as the product of their posterior marginal distributions. The derived estimator has a simple form, which facilitates its application.

This journal article provides significant added value to the conference version [16], including further developments, refinements, and more complete and detailed treatments. Specifically, new contents include mainly the following: (a) For the ML approach, much more detailed derivations and discussions not given in [16] are added, e.g., in Section II.B. (b) For the VB approach, a complete and detailed proof of the obtained estimation algorithm is added in the Appendix. (c) The simulation part is heavily revised. In this version, the proposed approach is compared with more closely-related existing random-matrix algorithms including the one proposed by [9] and [29] and another proposed by [26] and [27] for EOT using nonlinear measurements.

This paper makes the following contributions.

1) For EOT/GTT using nonlinear measurements, the matched linearization (ML) is proposed to statistically linearize nonlinear measurements for EOT. The ML provides an optimized form of a linearized measurement with its first two moments preserved. Extension information contained in the second moment is considered in the linearization. Such an approximation technique can be used in many other nonlinear problems. Calculation of the related coefficients in the linearized model using deterministic sampling is also provided.

2) A variational Bayesian (VB) approach to EOT/GTT using a random matrix is proposed based on the ML model. The proposed VB approach can be generally applied to EOT/GTT. A convergent iterative VB algorithm in a simple analytical form for state and extension estimation is also derived.

3) The effectiveness of the ML and the VB approach, compared with some existing algorithms for EOT using nonlinear measurements, is demonstrated by a simulation study.

This paper is organized as follows. Section II proposes the matched linearization for EOT using nonlinear measurements. The variational Bayesian scheme based on the linearized model is proposed for EOT in Section III. Section IV presents simulation results, and Section V concludes the paper.

II. Matched Linearization for EOT with Nonlinear Measurements

A. EOT Using Random Matrix

For EOT, using a random matrix to describe object extension was first proposed in [13]. It jointly estimates the kinematic state represented by a random vector x_k and the object extension represented by a $d \times d$ SPD random matrix X_k, where d is the dimension of the physical space for EOT. The extension is simplified as an ellipsoid surface defined as

$$\{y : (y - T_k x_k)^{\mathrm{T}} X_k^{-1} (y - T_k x_k) = 1\} \tag{1}$$

where y is a variable representing the points on the surface of the ellipsoid, T_k is a matrix transforming the object centroid state to its centroid position, and "T" stands for transpose.

1) State dynamic model: The following dynamic model with state x_k is assumed

$$x_k = \Phi_k x_{k-1} + w_k \tag{2}$$

where Φ_k is a transition matrix and w_k is process noise.

In [13], Φ_k and w_k are assumed as

$$\Phi_k = F_k \otimes I_d, \quad w_k \sim \mathcal{N}(0, D_k \otimes X_k) \tag{3}$$

where F_k is the dynamic matrix in the one-dimensional physical space (e.g., the Singer model used in [13]), $I_d \in \mathbb{R}^{d \times d}$ is the identity matrix, "$\otimes$" stands for the Kronecker product, $\mathcal{N}(\mu, \Sigma)$ denotes the normal distribution with mean μ and covariance matrix Σ, $D_k = \sigma_k^2 \breve{D}_k$ is the covariance matrix of the independent Gaussian process noise w_k in the one-dimensional model, σ_k^2 is the variance of the acceleration noise, and $\breve{D}_k$ is a coefficient matrix.

In [5], the above two quantities are more generally assumed. Specifically, Φ_k is assumed as a general proper matrix and

$$w_k \sim \mathcal{N}(0, Q_k) \tag{4}$$

with Q_k being a covariance matrix.

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2) Extension dynamic model: Assuming the extension is nearly time invariant, [5] and [13] adopted some equations with justification to describe extension evolution. The following evolution model was also proposed by [13]:

$$p[X_k|X_{k-1}] = \mathcal{W}\left(X_k; \delta_{k|k-1}, X_{k-1}/\delta_{k|k-1}\right) \tag{5}$$

where $\mathcal{W}(Y; a, C)$ is the density of the Wishart distribution for the SPD random matrix $Y \in \mathbb{R}^{d\times d}$, defined as

$$\mathcal{W}(Y; a, C) = c^{-1}\,|C|^{-\frac{1}{2}a}\,|Y|^{\frac{1}{2}(a-d-1)}\,\mathrm{etr}(-C^{-1}Y/2) \tag{6}$$

with $a > d-1$. Here etr(·) = exp(trace(·)), c is the normalization factor (c and c_k will always be normalization factors in the sequel). These methods consider only evolution of the object size, but not shape nor orientation (e.g., $\delta_{k|k-1}$ in (5) is a scalar). In [14] and [15], we proposed the following model:

$$p[X_k|X_{k-1}] = \mathcal{W}(X_k; \delta_k, A_k X_{k-1} A_k^{\mathrm{T}}) \tag{7}$$

where $\delta_k > d-1$ is the degrees of freedom, and the invertible matrix $A_k \in \mathbb{R}^{d\times d}$ describes the evolution mode: i) scalar δ_k can describe the dependence of the extension on size over time; ii) A_k can describe the dependence of the extension on orientation (if A_k is a rotation matrix), size (e.g, $A_k = \lambda I_d$), or shape. Note that (7) with $A_k = I_d/\sqrt{\delta_k}$ and $\delta_k = \delta_{k|k-1}$ reduces to (5).

In [8], an extension evolution model was proposed and it can describe the phenomenon that the extension orientation of an object changes with its centroid turn motions.

3) Measurement model: Let $Z_k = \{z_k^r\}_{r=1}^{n_k}$ denote a set of n_k vector-valued position measurements at time k. In [13] and [5], the measurement model is assumed as

$$z_k^r = \tilde{H}_k x_k + v_k^r,\ r = 1, ..., n_k \tag{8}$$

where $\tilde{H}_k = H_k \otimes I_d$ with $H_k = [1, 0, 0]$ being the measurement matrix in one-dimensional physical space (assuming that only positions are measured and the state in one-dimensional space is [position, velocity, acceleration]^{T}), and v_k^r is white Gaussian noise independent across r. v_k^r is assumed in [13] as

$$v_k^r \sim \mathcal{N}(0, X_k) \tag{9}$$

It is called extension noise with covariance X_k representing the object extension. To consider both the true-measurement error and the uncertainties in X_k simultaneously, [5] assumed

$$v_k^r \sim \mathcal{N}(0, \lambda X_k + R_k) \tag{10}$$

where λ is a scalar describing the effect of X_k, and R_k is the covariance matrix of the true-measurement error. λ is used such that the Gaussian distribution in (10) is a good approximation of a more realistic distribution assumption (e.g., $\lambda = 1/4$ provides a good Gaussian approximation (10) for scattering centers uniformly distributed across the object extent) [5].

Model (10) indicates that the measurements have two sources of uncertainty: the extension and the noise of the true measurement. Although (10) is better than (9), it is not as handy as (9) for estimation. Thus, [5] proposed some approximations. Based on (10), [23] proposed a more rigorous estimator using the VB approach by approximating the joint distribution of the state, extension, and noiseless measurement as the product of their marginal distributions. A detailed comparison of these two approaches can be found in [24].

To describe the observation distortion of the extension, in [15] we proposed a model in the form of (8) with

$$v_k^r \sim \mathcal{N}(0, B_k X_k B_k^{\mathrm{T}}) \tag{11}$$

where the invertible matrix $B_k \in \mathbb{R}^{d\times d}$ describes distortion of the observed extension (embodied by the covariance matrix of multiple measurements) from the actual one. Model (11) is convenient for deriving a Bayesian estimator in a simple form, as given in [14] and [15]. It can also incorporate (10) approximately. Assume $X_k \approx \bar{X}_{k|k-1} \triangleq E[X_k|Z^{k-1}]$ and $Z^{k-1} = \{Z_1, ..., Z_{k-1}\}$. Then

$$\lambda X_k + R_k \approx B_k X_k B_k^{\mathrm{T}} \tag{12}$$

where $B_k \triangleq (\lambda \bar{X}_{k|k-1} + R_k)^{1/2} \bar{X}_{k|k-1}^{-1/2}$. Using (12), the estimator in [15] can be used for EOT having true-measurement noise.

Based on (2) with (4) and (8) with (10), an approach to EOT is proposed by [5]. In this approach, the joint distribution $p(x_k, X_k|Z^k)$ of x_k and X_k is obtained as $p(x_k, X_k|Z^k) = \mathcal{N}(x_k; \hat{x}_k, P_k)\mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k)$ from $p(x_{k-1}, X_{k-1}|Z^{k-1}) = \mathcal{N}(x_{k-1}; \hat{x}_{k-1}, P_{k-1})\mathcal{IW}(X_{k-1}; \hat{\alpha}_{k-1}, \hat{X}_{k-1})$. Here, $\mathcal{IW}(Y; a, C)$ is the inverse Wishart (IW) density, defined as

$$\mathcal{IW}(Y; a, C) = c^{-1}\,|C|^{\frac{a-d-1}{2}}\,|Y|^{-\frac{a}{2}}\,\mathrm{etr}(-CY^{-1}/2) \tag{13}$$

with the degrees of freedom $a > 2d$ and

$$E[Y] = C/(a-2d-2), \quad a-2d-2 > 0 \tag{14}$$

Based on (2), (7), (8), and (11), a Bayesian approach to estimating x_k and X_k was obtained in [14]. It obtains $p(x_k, X_k|Z^k) = \mathcal{N}(x_k; \hat{x}_k, P_k \otimes X_k)\mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k)$ from $p(x_{k-1}, X_{k-1}|Z^{k-1}) = \mathcal{N}(x_{k-1}; \hat{x}_{k-1}, P_{k-1} \otimes X_{k-1})\mathcal{IW}(X_{k-1}; \hat{\alpha}_{k-1}, \hat{X}_{k-1})$ using Z_k. Equivalently, the estimator calculates $\{\hat{x}_k, P_k, \hat{\alpha}_k, \hat{X}_k\}$ using $\{\hat{x}_{k-1}, P_{k-1}, \hat{\alpha}_{k-1}, \hat{X}_{k-1}\}$ and Z_k. One cycle of this estimator is summarized in Table I.

Table I: The Bayesian algorithm [14] [15]

State	$\hat{x}_{k\|k-1} = (F_k \otimes I_d)\hat{x}_{k-1}$ $P_{k\|k-1} = F_k P_{k-1} F_k^{\mathrm{T}} + D_k$ $\hat{x}_k = \hat{x}_{k\|k-1} + (K_k \otimes I_d)G_k$ $P_k = P_{k\|k-1} - K_k S_k K_k^{\mathrm{T}}$ $S_k = H_k P_{k\|k-1} H_k^{\mathrm{T}} + \frac{\|B_k\|^{2/d}}{n_k}$ $K_k = P_{k\|k-1} H_k^{\mathrm{T}} S_k^{-1}$ $G_k \triangleq \bar{z}_k - (H_k \otimes I_d)\hat{x}_{k\|k-1}$
	$E[x_k\|Z^k] = \hat{x}_k$
Extension	$\hat{X}_{k\|k-1} = \frac{\delta_k}{\lambda_{k-1}}(\hat{\alpha}_{k\|k-1} - 2d - 2)A_k \hat{X}_{k-1} A_k^{\mathrm{T}}$ $\lambda_{k-1} = \hat{\alpha}_{k-1} - 2d - 2$ $\hat{\alpha}_{k\|k-1} = \frac{2\delta_k(\lambda_{k-1}+1)(\lambda_{k-1}-1)(\lambda_{k-1}-2)}{\lambda_{k-1}^2(\lambda_{k-1}+\delta_k)} + 2d + 4$ $\hat{X}_k = \hat{X}_{k\|k-1} + N_k + B_k^{-1}\bar{Z}_k B_k^{-\mathrm{T}}$ $\hat{\alpha}_k = \hat{\alpha}_{k\|k-1} + n_k$ $N_k = S_k^{-1} G_k G_k^{\mathrm{T}}$
	$E[X_k\|Z^k] = \hat{X}_k/(\hat{\alpha}_k - 2d - 2)$

* $\bar{z}_k$ and $\bar{Z}$ are given in (69).

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B. Matched Linearization for EOT with Nonlinear Measurements

As shown above, the existing random-matrix EOT approach assumes that the measurement is linear in the state and the extension noise. In practice, measurements are nonlinear. Generally, a nonlinear measurement model has the form:

$$z_k^r = h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) \tag{15}$$

for the rth of a total of n_k measurements at time k, where $x_k^{\mathrm{p}} = \tilde{H} x_k$ is the object centroid position, and

$$u_k^r \sim \mathcal{N}(0, \lambda X_k), \quad \tilde{v}_k^r \sim \mathcal{N}(0, R_k) \tag{16}$$

are white and they describe effects of the extension and the true-measurement noise, respectively. x_k^{p}, u_k^r, and v_k^r are assumed mutually independent conditioned on X_k and Z^{k-1}.

Example 1: Let $x_k^{\mathrm{p}} = [\mathsf{x}, \mathsf{y}]_k^{\mathrm{T}}$, $u_k^r = [u_x^r, u_y^r]_k^{\mathrm{T}}$, and $\tilde{v}_k^r = [\tilde{v}_r^r, \tilde{v}_\theta^r]_k^{\mathrm{T}}$. The range and angle measurement of a scattering center can be

$$z_k^r = \begin{bmatrix} \bar{r} \\ \bar{\theta} \end{bmatrix}_k^r = \begin{bmatrix} \sqrt{(\mathsf{x}+u_x^r)^2 + (\mathsf{y}+u_y^r)^2} + \tilde{v}_{\bar{r}}^r \\ \arctan(\frac{\mathsf{y}+u_y^r}{\mathsf{x}+u_x^r}) + \tilde{v}_{\bar{\theta}}^r \end{bmatrix}_k \tag{17}$$

where the sensor is assumed located in the origin.

For a nonlinear function of a random vector x

$$y = g(x) \tag{18}$$

the optimal linearization of $g(x)$ in the MMSE sense can be obtained as [21]

$$L(x) = \bar{g}(x) + C_{gx} C_x^{-1}(x - \bar{x}) \tag{19}$$

where $\bar{x} \triangleq E[x]$, $\bar{g}(x) \triangleq E[g(x)]$, $C_{gx} \triangleq \mathrm{cov}(g(x), x)$, $C_x \triangleq \mathrm{cov}(x)$. The linearization error $e = g(x) - L(x)$ is zero mean and has the covariance

$$\mathrm{cov}(e) = C_g - C_{gx} C_x^{-1} C_{gx}^{\mathrm{T}} \tag{20}$$

For a dynamic problem, the above linearization can be applied to both the dynamic and the measurement functions, and then a statistically linearized system is obtained. For dynamic estimation, existing popular nonlinear filters can be viewed as linear MMSE estimators using different moment-approximation techniques, e.g., the unscented transformation (UT) [11] and the Gaussian Hermite quadrature (GHQ) [1]. Their integration with linear MMSE estimation can be considered as special cases of this statistical linearization [20].

Inspired by the above optimal linearization, we consider linearizing the nonlinear function in (15) as

$$h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) \approx L_h(x_k^{\mathrm{p}}, \check{v}_k^r) = H x_k^{\mathrm{p}} + G \check{v}_k^r + b_k \tag{21}$$

with H, G, b_k to be determined. Here x_k^{p} is the same as in (15). Noise $\check{v}_k^r$ differs from u_k^r and $\tilde{v}_k^r$ in (15) and we assume:

1) $\check{v}_k^r$ is white and Gaussian distributed:

$$\check{v}_k^r \sim \mathcal{N}(0, \tilde{B}_k X_k \tilde{B}_k^{\mathrm{T}}) \tag{22}$$

where $\tilde{B}_k \in \mathbb{R}^{d \times d}$ is an unknown invertible matrix.

2) $\check{v}_k^r$ depends on u_k^r and $\tilde{v}_k^r$ but $\check{v}_k^r$ and x_k^{p} are uncorrelated conditioned on Z^{k-1}.

Remark 1: (a) $\check{v}_k^r$ with X_k in its covariance is used to preserve the information of the extension represented by X_k after linearization, and this information is contained in the second moment of the original measurement described by (15).

(b) The effect of u_k^r and $\tilde{v}_k^r$ on the measurement covariance is replaced by that of $\check{v}_k^r$. Specifically, the second moments of the measurement before and after the linearization are guaranteed to be identical by designing a proper $G\check{v}_k^r$.

According to (19) and treating x_k^{p} and $\check{v}_k^r$ as the driving variables for the measurement, we can obtain the optimal linearization as

$$L_h(x_k^{\mathrm{p}}, \check{v}_k^r) = \bar{h} + C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1}(x_k^{\mathrm{p}} - \bar{x}_k^{\mathrm{p}}) + C_{h\check{v}^r} C_{\check{v}^r}^{-1} \check{v}_k^r \tag{23}$$

under the assumption that $\check{v}_k^r$ is zero-mean and independent of x_k^{p}. Here,

$$\bar{x}_k^{\mathrm{p}} \triangleq E[x_k^{\mathrm{p}} | Z^{k-1}] \tag{24}$$

$$C_{\check{v}^r} \triangleq \mathrm{cov}(\check{v}_k^r | Z^{k-1}) \tag{25}$$

$$\bar{h} \triangleq E[h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) | Z^{k-1}] \tag{26}$$

$$C_{hx^{\mathrm{p}}} \triangleq \mathrm{cov}(h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r), x_k^{\mathrm{p}} | Z^{k-1}) \tag{27}$$

$$C_{x^{\mathrm{p}}} \triangleq \mathrm{cov}(x_k^{\mathrm{p}} | Z^{k-1}) \tag{28}$$

$$C_{h\check{v}^r} \triangleq \mathrm{cov}(h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r), \check{v}_k^r | Z^{k-1}) \tag{29}$$

Their calculation will be discussed later.

$L_h(x_k^{\mathrm{p}}, \check{v}_k^r)$ is unbiased because $E[L_h(x_k^{\mathrm{p}}, \check{v}_k^r) | Z^{k-1}] = \bar{h} = E[h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) | Z^{k-1}]$ according to (23) and (26). Its covariance can be obtained as

$$\mathrm{cov}(L_h(x_k^{\mathrm{p}}, \check{v}_k^r) | Z^{k-1}) = C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}} + C_{h\check{v}^r} C_{\check{v}^r}^{-1} C_{h\check{v}^r}^{\mathrm{T}} \tag{30}$$

A proof of (30) is given in Appendix A. According to Remark 1, the second moment of the measurement is also to be preserved. Setting $\mathrm{cov}(L_h(x_k^{\mathrm{p}}, \check{v}_k^r) | Z^{k-1}) = C_h \triangleq \mathrm{cov}(h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) | Z^{k-1})$, we have

$$\begin{aligned} &\mathrm{cov}(L_h(x_k^{\mathrm{p}}, \check{v}_k^r) | Z^{k-1}) = C_h \\ \Rightarrow &C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}} + C_{h\check{v}^r} C_{\check{v}^r}^{-1} C_{h\check{v}^r}^{\mathrm{T}} = C_h \\ \Leftrightarrow &C_{h\check{v}^r} C_{\check{v}^r}^{-1} C_{h\check{v}^r}^{\mathrm{T}} = C_{h|x^{\mathrm{p}}} \triangleq C_h - C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}} \end{aligned} \tag{31}$$

Assume z_k^r and $\check{v}_k^r$ have the same dimension (e.g., (8) and (17)). Then $C_{h\check{v}^r}$ can be obtained based on (31) as

$$C_{h\check{v}^r} = C_{h|x^{\mathrm{p}}}^{1/2} C_{\check{v}^r}^{\mathrm{T}/2} \tag{32}$$

and thus

$$C_{h\check{v}^r} C_{\check{v}^r}^{-1} = C_{h|x^{\mathrm{p}}}^{1/2} C_{\check{v}^r}^{-1/2} \tag{33}$$

From (22), we have

$$C_{\check{v}^r} \triangleq E[\check{v}_k^r (\check{v}_k^r)^{\mathrm{T}} | Z^{k-1}] = \tilde{B}_k \bar{X}_{k|k-1} \tilde{B}_k^{\mathrm{T}} \tag{34}$$

where $\bar{X}_{k|k-1} \triangleq E[X_k | Z^{k-1}]$. A proof of (34) is given in Appendix B.

Then (33) can be obtained as

$$C_{h\check{v}^r} C_{\check{v}^r}^{-1} = C_{h|x^{\mathrm{p}}}^{1/2} C_{\check{v}^r}^{-1/2} = C_{h|x^{\mathrm{p}}}^{1/2} \bar{X}_{k|k-1}^{-1/2} \tilde{B}_k^{-1} \tag{35}$$

The linearized $L_h(x_k^{\mathrm{p}}, \check{v}_k^r)$ can be obtained by substituting (35) into (23) as

$$L_h(x_k^{\mathrm{p}}, \check{v}_k^r) = \bar{h} + C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1}(x_k^{\mathrm{p}} - \bar{x}_k^{\mathrm{p}}) + C_{h|x^{\mathrm{p}}}^{1/2} \bar{X}_{k|k-1}^{-1/2} \tilde{B}_k^{-1} \check{v}_k^r \tag{36}$$

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Note that $\tilde{B}_k$ is still unknown in (36). Actually, since $\check{v}_k^r \sim \mathcal{N}(0, \tilde{B}_k X_k \tilde{B}_k^{\mathrm{T}})$ by (22), $\check{v}_k^r$ can be equivalently rewritten as

$$\check{v}_k^r = \tilde{B}_k v_k^r \tag{37}$$

where $v_k^r \sim \mathcal{N}(0, X_k)$.

Then the matched linearization can be finally obtained by subsitituion of (37) into (36) as

$$L_h(x_k^{\mathrm{p}}, v_k^r) = \bar{h} + C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1}(x_k^{\mathrm{p}} - \bar{x}_k^{\mathrm{p}}) + C_{h|x^{\mathrm{p}}}^{1/2} \bar{X}_{k|k-1}^{-1/2} v_k^r \tag{38}$$

Remark 2: (a) Eq. (38) is the final form of linearized $h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r)$. It has the same first and second moments as the original nonlinear function $h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r)$ (15). Thus, the above linearization is named **matched linearization** (ML).

(b) The physical meaning of zero-mean v_k^r is not important since only its covariance X_k needs to be considered and estimated.

Example 2: As proposed in [5], one reasonable measurement equation is

$$z_k^r = \tilde{H}_k x_k + v_k^r = x_k^{\mathrm{p}} + v_k^r,\ v_k^r \sim \mathcal{N}(0, \lambda X_k + R_k) \tag{39}$$

which can be rewritten as

$$\begin{aligned} z_k^r &= h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) = x_k^{\mathrm{p}} + u_k^r + \tilde{v}_k^r \\ u_k^r &\sim \mathcal{N}(0, \lambda X_k),\ \tilde{v}_k^r \sim \mathcal{N}(0, R_k) \end{aligned} \tag{40}$$

The ML equation of (40) can be obtained using (38) as

$$z_k^r = H x_k^{\mathrm{p}} + G v_k^r,\ v_k^r \sim \mathcal{N}(0, X_k) \tag{41}$$

with

$$\begin{aligned} H &= C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} = I_d \\ G &= C_{h|x^{\mathrm{p}}}^{1/2} \bar{X}_{k|k-1}^{-1/2} = (\lambda \bar{X}_{k|k-1} + R_k)^{1/2} \bar{X}_{k|k-1}^{-1/2} \end{aligned} \tag{42}$$

because

$$C_{hx^{\mathrm{p}}} = \mathrm{cov}(h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r), x_k^{\mathrm{p}} | Z^{k-1}) = C_{x^{\mathrm{p}}} \tag{43}$$

$$C_{h|x^{\mathrm{p}}} \triangleq C_h - C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}} = \lambda \bar{X}_{k|k-1} + R_k \tag{44}$$

Note that (42) is the same as (12), which justifies that (12) is an ML form of (39). Proofs of (43) and (44) are given in Appendices C and D, respectively.

C. Coefficient Calculation for Matched Linearization

In (38), the following quantities are to be calculated (note that $C_{h|x^{\mathrm{p}}} = C_h - C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}}$ according to (31)):

$$\bar{X}_{k|k-1},\ \bar{h},\ C_h,\ C_{hx^{\mathrm{p}}},\ C_{x^{\mathrm{p}}} \tag{45}$$

These quantities are all expectations conditioned on Z^{k-1}, as shown in (24)–(29), which rely on the predicted joint distribution $p[x_k, X_k | Z^{k-1}]$.

At time k, existing methods can obtain the predicted distribution as follows.

(a) In [13], [14], and [15]:

$$\begin{aligned} &p[x_k, X_k | Z^{k-1}] = p[x_k | X_k, Z^{k-1}] p[X_k | Z^{k-1}] \\ &= \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1} \otimes X_k) \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \end{aligned} \tag{46}$$

(b) In [5]:

$$\begin{aligned} p[x_k, X_k | Z^{k-1}] =& \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\ &\times \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \end{aligned} \tag{47}$$

According to (46) or (47), $\bar{X}_{k|k-1} \triangleq E[X_k | Z^{k-1}]$ in (45) can be obtained as

$$\bar{X}_{k|k-1} \triangleq E[X_k | Z^{k-1}] = \hat{X}_{k|k-1} / (\hat{\alpha}_{k|k-1} - 2d - 2)$$

based on $p[X_k | Z^{k-1}] = \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1})$ according to (14). The other four quantities in (45) are all expectations of functions of x_k^{p}, u_k^r, and $\tilde{v}_k^r$, as defined by (24)–(29). For any function $f(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r)$ of them, we have

$$\begin{aligned} &E[f(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) | Z^{k-1}] \\ &= \int f(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) p[x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r | Z^{k-1}] \mathrm{d}x_k^{\mathrm{p}} \mathrm{d}u_k^r \mathrm{d}\tilde{v}_k^r \\ &= \int f(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}] p[\tilde{v}_k^r] \mathrm{d}x_k^{\mathrm{p}} \mathrm{d}u_k^r \mathrm{d}\tilde{v}_k^r \end{aligned} \tag{48}$$

Eq. (48) holds because $\tilde{v}_k^r$ is independent of x_k^{p}, u_k^r, and Z^{k-1}. Here, the key is to obtain $p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}]$ since $p[\tilde{v}_k^r]$ is given by (16). This can be done according to (46) or (47) respectively, as follows.

(a) Based on (46), we have

$$p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}] = \mathcal{T}(x_k^{\mathrm{a}}; \hat{\alpha}_{k|k-1} + s^{\mathrm{a}}, \hat{x}_{k|k-1}^{\mathrm{a}}, P_{k|k-1}^{\mathrm{a}} \otimes \hat{X}_{k|k-1}) \tag{49}$$

where $x_k^{\mathrm{a}} \triangleq [(x_k^{\mathrm{p}})^{\mathrm{T}}, (u_k^r)^{\mathrm{T}}]^{\mathrm{T}}$, $s^{\mathrm{a}} = s^{\mathrm{p}} - s^{\mathrm{p}} d - 2d - 2$ with $s^{\mathrm{p}} = 1$ being the dimension of x_k^{p} in one direction of a physical space ($s^{\mathrm{p}} = 1$ because x_k^{p} is the position of an object), $\mathcal{T}(\cdot)$ is the multivariate t-distribution, and

$$\hat{x}_{k|k-1}^{\mathrm{a}} \triangleq \begin{bmatrix} \hat{x}_{k|k-1}^{\mathrm{p}} \\ \mathbf{0}_{d \times 1} \end{bmatrix},\ P_{k|k-1}^{\mathrm{a}} \triangleq \begin{bmatrix} P_{k|k-1}^{\mathrm{p}} & 0 \\ 0 & \lambda \end{bmatrix} \tag{50}$$

Then $p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}]$ can be approximated as a Gaussian distribution by moment matching:

$$p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}] \approx \mathcal{N}(x_k^{\mathrm{a}}; \hat{x}_{k|k-1}^{\mathrm{a}}, \frac{P_{k|k-1}^{\mathrm{a}} \otimes \hat{X}_{k|k-1}}{\hat{\alpha}_{k|k-1} + s^{\mathrm{a}}}) \tag{51}$$

The first two moments of x_k^{a} in (51) are the same as those of (49). They are obtained according to the properties of the multivariate t-distribution.

(b) Based on (47), we have

$$\begin{aligned} p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}] =& \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}}) \\ &\times \mathcal{T}(u_k^r; \hat{\alpha}_{k|k-1} - 2d, 0, \lambda \hat{X}_{k|k-1}) \end{aligned} \tag{52}$$

Then $p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}]$ can also be approximated as a Gaussian distribution by moment matching:

$$\begin{aligned} p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}] \approx& \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}}) \\ &\times \mathcal{N}(u_k^r; 0, \lambda \hat{X}_{k|k-1} / (\hat{\alpha}_{k|k-1} - 2d - 2)) \end{aligned} \tag{53}$$

Proofs of Eqs. (49) and (52) are given in Appendices F and G, respectively.

Note $p[\tilde{v}_k^r] = \mathcal{N}(\tilde{v}_k^r; 0, R_k)$ in (48). And $p[x_k^{\mathrm{p}}, u_k^r | Z^{k-1}]$ in (48) is also approximated as a Gaussian distribution in (51) or (53). Then the expectation $E[f(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) | Z^{k-1}]$ in (48) can be easily calculated by deterministic sampling under the

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Gaussian assumption, e.g., the unscented transformation (UT) [11] or the Gauss-Hermite quadrature (GHQ) [1] rule. For this purpose, the expectation (48) can be equivalently written in the following more compact form:

$$\begin{aligned} E[f(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r)|Z^{k-1}] =& E[f(x_k^{\mathrm{A}})|Z^{k-1}] \\ =& \int f(x_k^{\mathrm{A}}) p(x_k^{\mathrm{A}}|Z^{k-1}) \mathrm{d}x_k^{\mathrm{A}} \end{aligned}$$

where $x_k^{\mathrm{A}} \triangleq [(x_k^{\mathrm{p}})^{\mathrm{T}}, (u_k^r)^{\mathrm{T}}, (\tilde{v}_k^r)^{\mathrm{T}}]^{\mathrm{T}}$ and $p(x_k^{\mathrm{A}}|Z^{k-1}) = \mathcal{N}(x_k^{\mathrm{A}}; \bar{x}_k^{\mathrm{A}}, P_k^{\mathrm{A}})$ with

$$\bar{x}_k^{\mathrm{A}} = \begin{bmatrix} \hat{x}_{k|k-1}^{\mathrm{p}} \\ \mathbf{0}_{2d\times 1} \end{bmatrix}, P_k^{\mathrm{A}} = \begin{bmatrix} \frac{P_{k|k-1}^{\mathrm{a}} \otimes \hat{X}_{k|k-1}}{\hat{\alpha}_{k|k-1}+s^{\mathrm{a}}} & \mathbf{0}_{2d\times d} \\ \mathbf{0}_{d\times 2d} & R_k \end{bmatrix}$$

for (46), and

$$\bar{x}_k^{\mathrm{A}} = \begin{bmatrix} \hat{x}_{k|k-1}^{\mathrm{p}} \\ \mathbf{0}_{2d\times 1} \end{bmatrix}, P_k^{\mathrm{A}} = \begin{bmatrix} P_{k|k-1}^{\mathrm{p}} & \mathbf{0}_{d\times d} & \mathbf{0}_{d\times d} \\ \mathbf{0}_{d\times d} & \frac{\lambda \hat{X}_{k|k-1}}{\hat{\alpha}_{k|k-1}-2d-2} & \mathbf{0}_{d\times d} \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times d} & R_k \end{bmatrix}$$

for (47).

Remark 3: With the above calculation, a linearized function $L_h(x_k^{\mathrm{p}}, v_k^r)$ in (38) can be obtained. If $C_{hx^{\mathrm{p}}}$ is invertible, some existing random-matrix algorithms [13], [14], and [15] can be used to estimate x_k and X_k. In this case, the measurement function (38) should be converted as follows:

$$\begin{aligned} & z_k^r = \bar{h}(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r) + C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1}(x_k^{\mathrm{p}} - \bar{x}_k^{\mathrm{p}}) + C_{h|x^{\mathrm{p}}}^{1/2} \bar{X}_{k|k-1}^{-1/2} v_k^r \\ & \Rightarrow \tilde{z}_k^r = x_k^{\mathrm{p}} + C_{x^{\mathrm{p}}} C_{hx^{\mathrm{p}}}^{-1} C_{h|x^{\mathrm{p}}}^{1/2} \bar{X}_{k|k-1}^{-1/2} v_k^r \end{aligned} \tag{54}$$

where

$$\tilde{z}_k^r \triangleq C_{x^{\mathrm{p}}} C_{hx^{\mathrm{p}}}^{-1}(z_k^r - \bar{h}(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r)) + \bar{x}_k^{\mathrm{p}} \tag{55}$$

Note that $x_k^{\mathrm{p}} = \tilde{H}_k x_k$ with $\tilde{H}_k$ of (8), and thus (54) can be used by the approaches based on (8) and (9) or (11). For EOT, the existing algorithms can use the converted measurement $\tilde{z}_k^r$ (54) to satisfy the assumptions on the measurement function (8).

III. Variational Bayesian Approach to EOT Using Random Matrix

Consider the following general model

$$\begin{cases} x_k = \Phi_k x_{k-1} + w_k, \ w_k \sim \mathcal{N}(0, Q_{k-1}) \\ p[X_k|X_{k-1}] = \mathcal{W}(X_k; \delta_k, A_k X_{k-1} A_k^{\mathrm{T}}) \\ z_k^r = \tilde{H}_k x_k + v_k^r, \ v_k^r \sim \mathcal{N}(0, B_k X_k B_k^{\mathrm{T}}) \end{cases} \tag{56}$$

where $r = 1, ..., n_k$. Let $Z_k = \{z_k^r\}_{r=1}^{n_k}$ and $Z^k = \{Z_\kappa\}_{\kappa=1}^k$.

Note that the original model (2) and (8) must be in the following special forms:

$$\Phi_k = F_k \otimes I_d, \ \tilde{H}_k = H_k \otimes I_d, \ Q_{k-1} = D_k \otimes X_k \tag{57}$$

but (56) does not have such restrictions. And the matched linearization (38) is a special case of (56). In [25], a variational Bayesian (VB) adaptive Kalman filter was proposed to estimate the state and the covariance matrix of the measurement noise. However, this filter does not solve the above estimation problem.

For recursive estimation, we make the following usual assumption for the random-matrix approach:

$$\begin{aligned} & p[x_{k-1}, X_{k-1}|Z^{k-1}] \\ & = \mathcal{N}(x_{k-1}; \hat{x}_{k-1}, P_{k-1}) \mathcal{IW}(X_{k-1}; \hat{\alpha}_{k-1}, \hat{X}_{k-1}) \end{aligned} \tag{58}$$

with $\hat{x}_{k-1}$, P_{k-1}, $\hat{\alpha}_{k-1}$, and $\hat{X}_{k-1}$ to be obtained recursively based on model (56) and measurements.

A. One-step Prediction

Under assumption (58), for model (56) we directly obtain the density of the predicted state as

$$p[x_k|Z^{k-1}] = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \tag{59}$$

where

$$\hat{x}_{k|k-1} = \Phi_k \hat{x}_{k-1}, \ P_{k|k-1} = \Phi_k P_{k-1} \Phi_k^{\mathrm{T}} + Q_{k-1} \tag{60}$$

For extension prediction, we have [15]

$$p[X_k|Z^{k-1}] = \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \tag{61}$$

with $\hat{\alpha}_{k|k-1}$ and $\hat{X}_{k|k-1}$ given in Table I, as proven in [15].

Then the predicted joint density is

$$\begin{aligned} & p[x_k, X_k|Z^{k-1}] = p[x_k|Z^{k-1}] p[X_k|Z^{k-1}] \\ & = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \end{aligned} \tag{62}$$

B. Update Using Variational Bayesian Approximation

In VB approximation, the posterior joint density of the state and the extension is approximated by the product of two functions as

$$p[x_k, X_k|Z^k] \approx Q_x(x_k) Q_X(X_k) \tag{63}$$

where $Q_x(x_k)$ and $Q_X(X_k)$ are obtained by minimizing the following KL divergence [28] between the approximating density and the true density:

$$D_{\mathrm{KL}}(Q_x(x_k) Q_X(X_k) || p[x_k, X_k|Z^k]) \tag{64}$$

where $D_{\mathrm{KL}}(p(y)||q(y)) = \int p(y) \ln(\frac{p(y)}{q(y)}) \mathrm{d}y$. Then $Q_x(x_k)$ and $Q_X(X_k)$ can be obtained as [28] [25]

$$Q_x(x_k) \propto e^{\int \ln(p[Z_k, x_k, X_k|Z^{k-1}]) Q_X(X_k) \mathrm{d}X_k} \tag{65}$$

$$Q_X(X_k) \propto e^{\int \ln(p[Z_k, x_k, X_k|Z^{k-1}]) Q_x(x_k) \mathrm{d}x_k} \tag{66}$$

Actually, (63) indicates that the posterior joint density is approximated by the product of the posterior marginal densities of x_k and X_k. Iteratively using (65) and (66), $Q_x(x_k)$ and $Q_X(X_k)$ can be calculated after initialization, given in the next subsection. Based on model (56), we have

$$p[Z_k, x_k, X_k|Z^{k-1}] = p[Z_k|x_k, X_k] p[x_k, X_k|Z^{k-1}] \tag{67}$$

where $p[x_k, X_k|Z^{k-1}]$ is given by (62) and

$$\begin{aligned} p[Z_k|x_k, X_k] \propto & \mathcal{N}(\bar{z}_k; \tilde{H}_k x_k, (B_k X_k B_k^{\mathrm{T}})/n_k) \\ & \times \mathcal{W}(\bar{Z}_k; n_k - 1, B_k X_k B_k^{\mathrm{T}}) \end{aligned} \tag{68}$$

with

$$\bar{z}_k = \frac{1}{n_k} \sum_{r=1}^{n_k} z_k^r, \quad \bar{Z}_k = \sum_{r=1}^{n_k} (z_k^r - \bar{z}_k)(z_k^r - \bar{z}_k)^{\mathrm{T}} \tag{69}$$

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according to Eq. (16) in [15].

Substituting (62) and (68) into (67) and then into (65) and (66) yields

$$Q_x(x_k) = \mathcal{N}(x_k; \hat{x}_k, P_k) \tag{70}$$

$$Q_X(X_k) = \mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k) \tag{71}$$

with (denoting the dimension of x_k by n_x)

$$\hat{x}_k = \hat{x}_{k|k-1} + K(\bar{z}_k - \tilde{H}_k \hat{x}_{k|k-1}) \tag{72}$$

$$P_k = (I_{n_x} - K\tilde{H}_k^{\mathrm{T}})P_{k|k-1},\ K = P_{k|k-1}\tilde{H}_k^{\mathrm{T}} S_k^{-1} \tag{73}$$

$$S_k = \tilde{H}_k P_{k|k-1}\tilde{H}_k^{\mathrm{T}} + B_k \hat{X}_k B_k^{\mathrm{T}}/(n_k(\hat{\alpha}_k - d - 1)) \tag{74}$$

$$\hat{\alpha}_k = \hat{\alpha}_{k|k-1} + n_k \tag{75}$$

$$\hat{X}_k = \hat{X}_{k|k-1} + B_k^{-1}[\bar{Z}_k + n_k(\bar{z}_k - \tilde{H}_k\hat{x}_k)(\cdot)^{\mathrm{T}} + n_k \tilde{H}_k P_k \tilde{H}_k^{\mathrm{T}}]B_k^{-\mathrm{T}} \tag{76}$$

where $(\cdot)$ means the term right before it. A proof of (70)–(76) is given in Appendix G.

C. One Cycle of VB Approach to EOT

For recursive estimation, the above VB-based EOT approach at k obtains $p[x_k, X_k|Z^k] = \mathcal{N}(x_k; \hat{x}_k, P_k)\mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k)$ based on $p[x_{k-1}, X_{k-1}|Z^{k-1}]$ in the same form (58). One cycle of this approach is:

1) Prediction

Calculate $\{\hat{x}_{k|k-1}, P_{k|k-1}\}$ using (60) and $\{\hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}\}$ following the corresponding steps in Table I. Then the predicted density is $p[x_k, X_k|Z^{k-1}] = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})\mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1})$.

2) Update (N iterations, N is designed a priori)

A pseudocode description of the VB update procedure is:

algorithm VB update **is**
input: $\hat{x}_{k|k-1}, P_{k|k-1}, \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}, N, \bar{z}_k, \bar{Z}_k$
output: $\hat{x}_k, P_k, \hat{\alpha}_k, \hat{X}_k$
$\hat{\alpha}_k^{(0)} \leftarrow \hat{\alpha}_{k|k-1}$, $\hat{X}_k^{(0)} \leftarrow \hat{X}_{k|k-1}$, $i \leftarrow 1$
while $i \leq N$ **do**
  $\{\hat{\alpha}_k, \hat{X}_k\} \leftarrow \{\hat{\alpha}_k^{(i-1)}, \hat{X}_k^{(i-1)}\}$
  $\hat{x}_k^{(i)} \leftarrow$ (72), $P_k^{(i)} \leftarrow$ (73)
  $\{\hat{x}_k, P_k\} \leftarrow \{\hat{x}_k^{(i)}, P_k^{(i)}\}$
  $\hat{\alpha}_k^{(i)} \leftarrow$ (75), $\hat{X}_k^{(i)} \leftarrow$ (76)
  $i \leftarrow i + 1$
$\hat{x}_k \leftarrow \hat{x}_k^{(N)}$, $P_k \leftarrow P_k^{(N)}$, $\hat{\alpha}_k \leftarrow \hat{\alpha}_k^{(N)}$, $\hat{X}_k \leftarrow \hat{X}_k^{(N)}$
return $\hat{x}_k, P_k, \hat{\alpha}_k, \hat{X}_k$

Here, "$\leftarrow$" means "is assigned as". Note that the posterior density $p[x_k, X_k|Z^k] = \mathcal{N}(x_k; \hat{x}_k, P_k)\mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k)$. The needed state estimate $E[x_k|Z^k]$ is $\hat{x}_k$ and the extension estimate $E[X_k|Z^k]$ is $\hat{X}_k/(\hat{\alpha}_k - 2d - 2)$.

Remark 4: (a) The above iteration will converge due to the general properties of the VB approach, as proven in [28].

(b) The termination condition can also be designed by comparing the difference between the obtained results at two consecutive iterations, since the above iteration will converge.

(c) Only few iterations are needed for effective estimation, to be demonstrated by simulation results given next.

IV. Simulation Study

In this section, the above VB approach for EOT with nonlinear measurements is compared with the approach recently proposed in [9] [29] and the one proposed in [26] [27]. We don't know any other random matrix algorithms specifically considering nonlinear measurements.

In [9] [29], nonlinear polar measurements with measurement equation (17) are considered. The measurement is first converted into the Cartesian coordinates:

$$z_k^r = [\bar{r}_k^r \cos(\bar{\theta}_k^r)\ \bar{r}_k^r \sin(\bar{\theta}_k^r)]^{\mathrm{T}} \tag{77}$$

Then the converted measurement is written in the following form [9] [29]

$$z_k^r = \tilde{H}_k x_k + v_k^r,\ v_k^r \sim \mathcal{N}(0, \lambda X_k + R_k(p_k)) \tag{78}$$

where $\tilde{H}_k$ is the same as in (8). Assume the noise in the polar coordinates is

$$\tilde{v}_k^r = [\tilde{v}_{\bar{r}}^r, \tilde{v}_{\bar{\theta}}^r]_k^{\mathrm{T}} \sim \mathcal{N}(0, \mathrm{diag}(\sigma_{\bar{r}}^2, \sigma_{\bar{\theta}}^2)), \tag{79}$$

where $\tilde{v}_{\bar{r}}^r$ and $\tilde{v}_{\bar{\theta}}^r$ are true-measurement errors of range and angle, respectively, as defined in (17). Then $R_k(p_k)$ is given as [9] [29]

$$R_k(p_k) = J(p_k)\mathrm{diag}(\sigma_{\bar{r}}^2, \sigma_{\bar{\theta}}^2)J(p_k)^{\mathrm{T}} \tag{80}$$

$$J(p_k) = \begin{bmatrix} \cos(\bar{\theta}) & -\bar{r}\sin(\bar{\theta}) \\ \sin(\bar{\theta}) & \bar{r}\cos(\bar{\theta}) \end{bmatrix}_k$$

where p_k represents the true position of a scattering center, and range $\bar{r}$ and bearing $\bar{\theta}$ is calculated using p_k. Eq. (80) is based on a first-order approximation of the polar-to-Cartesian conversion of the polar measurement noise, given in [3].

In [26] [27], nonlinear polar measurements are also converted into the form (78) with $R_k(p_k)$ given as

$$R_k(p_k) \approx \frac{(\sigma_{\bar{r}}^2 - \bar{r}_k^2\sigma_{\bar{\theta}}^2)}{2} \begin{bmatrix} b + \cos(2\bar{\theta}) & \sin(2\bar{\theta}) \\ \sin(2\bar{\theta}) & b - \cos(2\bar{\theta}) \end{bmatrix}_k \tag{81}$$

$$b = (\sigma_{\bar{r}}^2 + \bar{r}_k^2\sigma_{\bar{\theta}}^2)/(\sigma_{\bar{r}}^2 - \bar{r}_k^2\sigma_{\bar{\theta}}^2)$$

In practice, p_k is unknown and it is thus approximated by $\hat{p}_k = \tilde{H}_k\hat{x}_{k|k-1}$ with $\hat{x}_{k|k-1}$ being the one-step prediction of the centroid state [9] [29]. Based on the converted measurement, existing random matrix estimation algorithms can be applied: in [29] and [27], the one of [5] is applied using (78) for nonlinear estimation. Thus, we refer to the approaches based on [5] using (80) and (81) as the current EOT approach 1 (EOT-C1) and the current EOT approach 2 (EOT-C2), respectively.

In our variational Bayesian approach for EOT using ML (VB-EOT-ML), the ML measurement model (38) is used. The matched linearization is based on the converted measurement (77) without using (78). Multiple versions of this approach with different numbers of iterations N are compared to show how N affects estimation performance. All these algorithms use the same GHQ rule with the same number of quadrature points to calculate the moments needed in models (38) and (54). Three quadrature points are used in each dimension of the augmented random vector (including x_k^{p}, u_k^r, and $\tilde{v}_k^r$ in (48)) for moment calculation.

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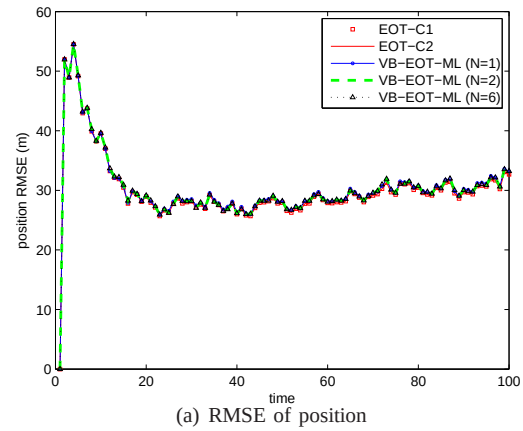
The extended object is an ellipse with diameters of 340m and 80m (roughly the size of an aircraft carrier of the Nimitz-class). Starting from the position (-10000m, 5000m), the object moves at a constant velocity (v, v) and the speed $\sqrt{2}v$ was assumed to be 27 knots. The trajectory of the object is similar to the first non-maneuvering stage of the EOT scenario in [15]. The sampling period $T_{\mathrm{s}} = 10$s. Measurements are ranges and bearings described by Eq. (17). The scattering centers are uniformly distributed over the extension X_k. $u_k^r = [u_x^r, u_y^r]_k^{\mathrm{T}}$ is the location of a scattering center relative to the centroid. Thus, according to [5], the distribution of u_k^r can be approximated by $\mathcal{N}(0, \lambda X_k)$ with $\lambda = 1/4$. In (79), $\sigma_{\bar{r}} = 50$m and different values of $\sigma_{\bar{\theta}}$ are considered to test the effectiveness of the proposed approach, because the bearing is highly nonlinear in the estimand. And x_k^{p}, u_k^r, and v_k^r are mutually independent of each other. The number of measurements at each scan is assumed Poisson distributed with mean λ_p. To test estimation performance related to the number of measurements, scenarios with $\lambda_p = 5$, 10, and 15 are simulated.

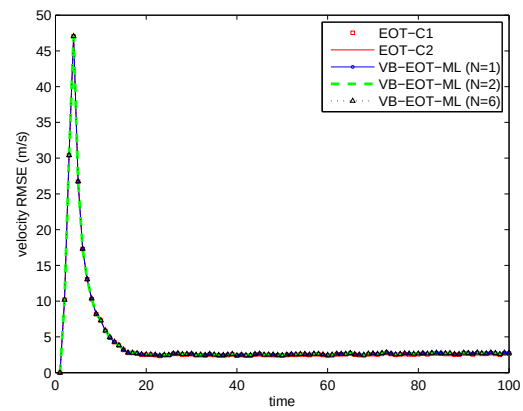
Root-mean-square errors (RMSE) and average Gaussian Wasserstein distances (GWD) over $N_s = 300$ Monte Carlo runs are compared. The RMSE of extension estimation is calculated as in [5]. The GWD proposed in [31] is adopted here because it is capable of evaluating the joint performance of state and extension estimation for ellipsoidal objects. The results are shown in Figure 1 for $\sigma_{\bar{\theta}} = 0.01$rad with $\lambda_p = 10$, in Figure 2 for $\sigma_{\bar{\theta}} = 0.1$rad with $\lambda_p = 10$, in Figure 3 for $\sigma_{\bar{\theta}} = 0.1$rad with $\lambda_p = 5$, and in Figure 4 for $\sigma_{\bar{\theta}} = 0.1$rad with $\lambda_p = 15$.

As shown in the figures, the VB-EOT-ML consistently outperforms EOT-C1 and EOT-C2 for extension estimation and joint estimation (reflected by average GWD) for different λ_p, especially when the bearing noise level is high. This indicates that the proposed ML approach is more effective than those adopted in EOT-C1 and EOT-C2, given by (80) and (81), respectively. EOT-C1 and EOT-C2 perform almost identically, and they become less valid as the bearing noise level increases, as shown in Figures 1(c), 2(c), 3(c), and 4(c). The consistent performance of the VB-EOT-ML illustrates that the extension information contained in the measurement is indeed reserved after the matched linearization. All approaches have similar performance in kinematic state estimation. This indicates that the state estimation is not largely affect by the extension estimation in all approaches. One possible reason is that EOT-C1 and EOT-C2 using the estimator in [5] have an underlying assumption that the kinematic state is independent of the extension, as shown by Eq. (14) in [5]. The variational approximation of the VB-EOT-ML optimally decouples the posterior densities of the state and the extension, so the estimation error of one does not affect that of the other. Actually, in this scenario, the kinematic state is independent of the extension. Note that if the kinematic state and the extension are dependent, the compared approaches may have worse performance than approaches utilizing this correlation information. Since the extension estimation is a key characteristic of EOT, the comparison results demonstrated the effectiveness of the proposed approach.

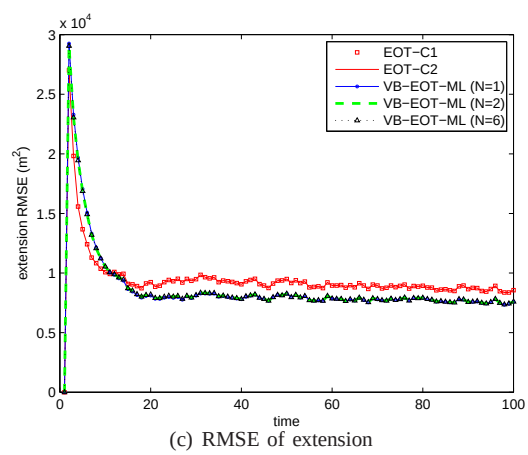
As can also be observed, the VB-EOT-ML algorithm with



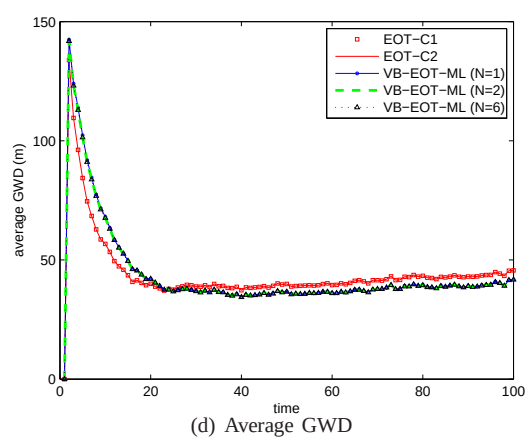
(a) RMSE of position



(b) RMSE of velocity



(c) RMSE of extension



(d) Average GWD

Figure 1: RMSE and Average GWD ($\sigma_{\bar{\theta}} = 0.01$rad, $\lambda_p = 10$)

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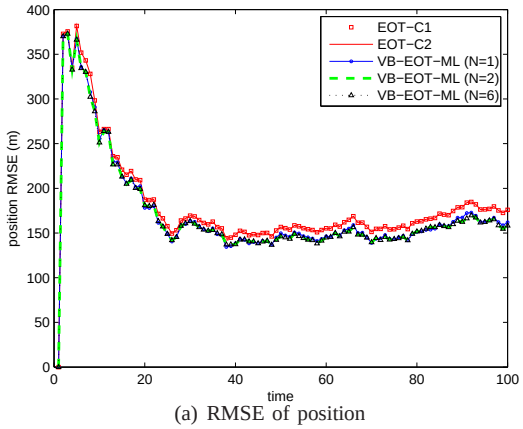
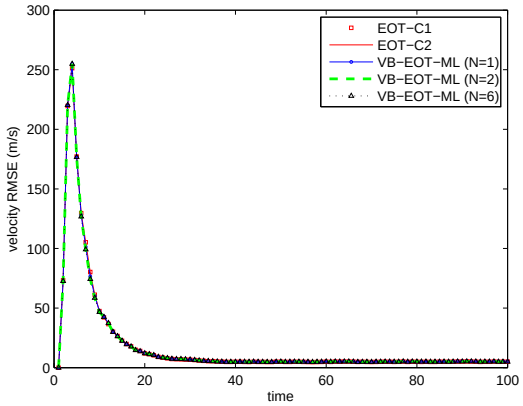
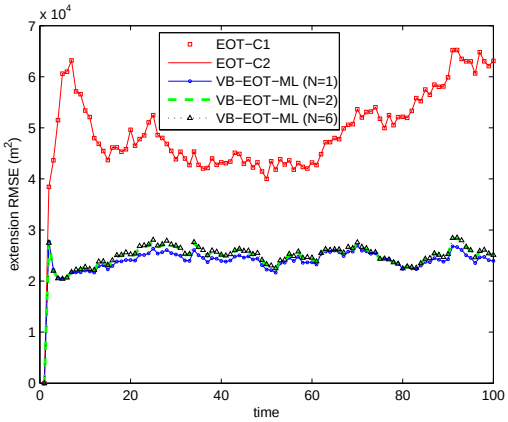


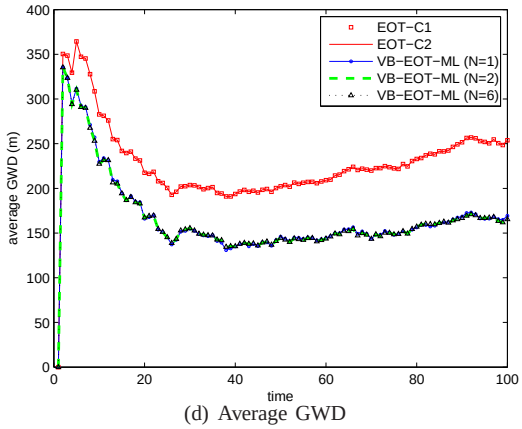
(a) RMSE of position



(b) RMSE of velocity

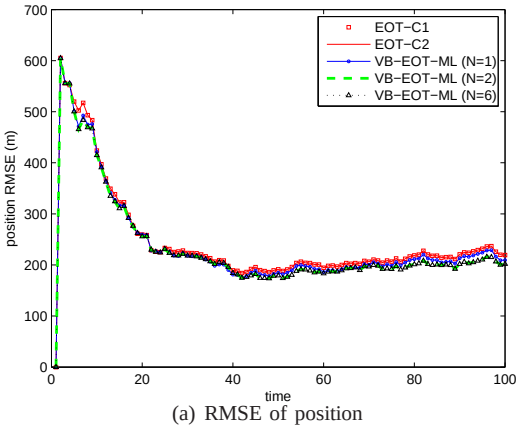


(c) RMSE of extension

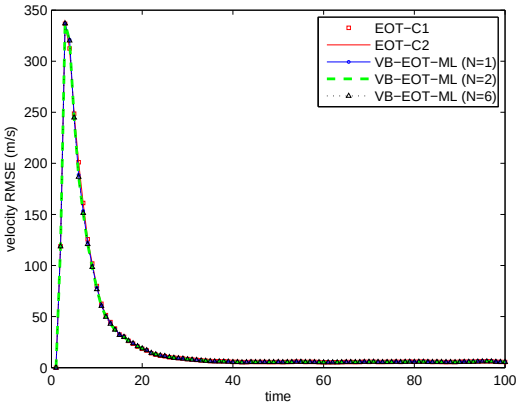


(d) Average GWD

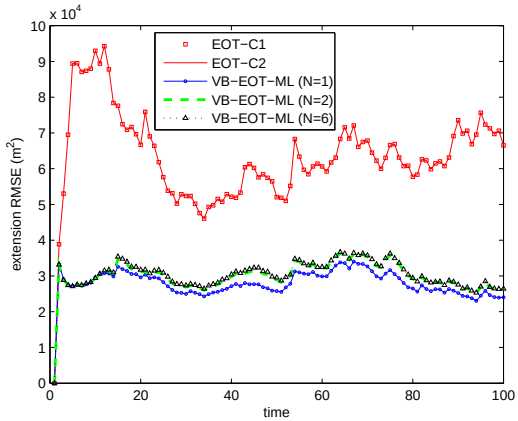
Figure 2: RMSE and Average GWD ($\sigma_{\bar{\theta}} = 0.1$rad, $\lambda_p = 10$)



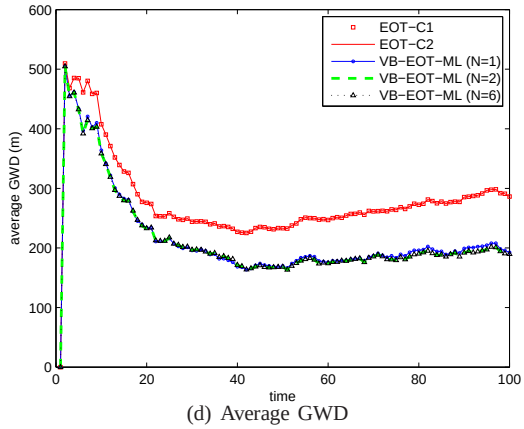
(a) RMSE of position



(b) RMSE of velocity



(c) RMSE of extension



(d) Average GWD

Figure 3: RMSE and Average GWD ($\sigma_{\bar{\theta}} = 0.1$rad, $\lambda_p = 5$)

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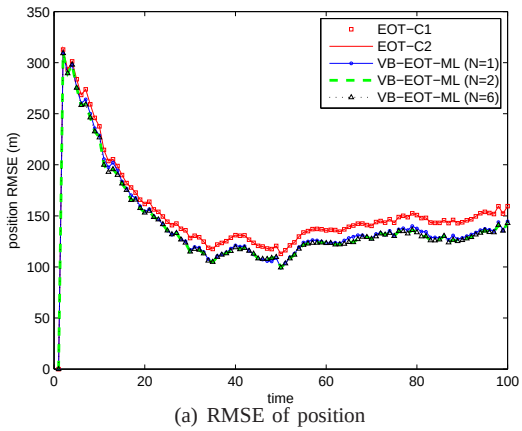
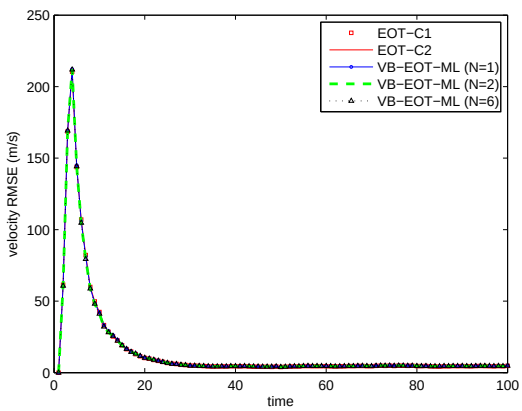
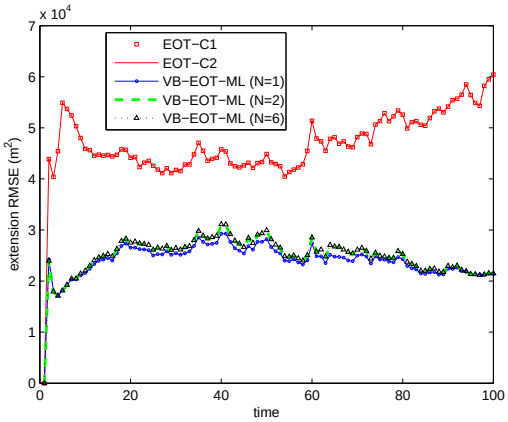


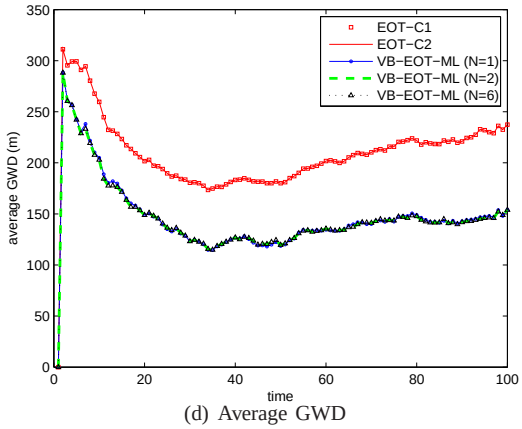
(a) RMSE of position



(b) RMSE of velocity



(c) RMSE of extension



(d) Average GWD

Figure 4: Average GWD ($\sigma_{\bar{\theta}} = 0.1$rad, $\lambda_p = 15$)

$N = 1, 2, 6$ iterations performs almost identically, especially for joint estimation as shown in Figures 1(d), 2(d), 3(d), and 4(d). This indicates that the number of iterations does not affect the final results significantly in this scenario.

The simulation results thus demonstrate the effectiveness of the proposed ML and VB approach for EOT with nonlinear measurements.

V. Conclusion

For EOT and GTT with nonlinear measurements, the so-called matched linearization is proposed to linearize the nonlinear measurement considering special properties of the random matrix. It is linear in both the state and the measurement noise, where the optimal weighting matrix of the latter is modified to preserve the second moment of the measurement. With it, the optimal linearized form of the kinematic state is obtained while the first two moments of the nonlinear measurement are preserved. This is important since the second moment contains important information of the extension. This linearization approach can be generally applied since it does not depend on specific structures of nonlinear measurement functions. Moreover, it can be numerically more stable than the methods based on function linearization techniques, e.g., Taylor series expansions, since only the first two moments of the related quantities are needed. And these moments can also be obtained easily by existing deterministic sampling methods.

Based on the ML model, a variational Bayesian approach to estimating the state and the extension is proposed. The VB approach can be generally applied and this iterative approach also has a simple analytical form. The iteration in the approach is convergent, which is guaranteed by the general properties of the VB approximation. Simulation experiments show that the number of iteration does not affect estimation results significantly, especially for joint estimation. That is, the approach converges very quickly.

Simulation results demonstrated the effectiveness of the proposed linearization and estimation approaches compared with two newly proposed random matrix approaches for nonlinear estimation, especially when the true-measurement noise level is high. In summary, the proposed approach and algorithm can be generally applied to EOT and GTT problems with different types of nonlinear measurements.

Appendix

A. Calculation of cov$(L_h(x_k^p, \check{v}_k^r)|Z^{k-1})$ (30)

According to (23), cov$(L_h(x_k^p, \check{v}_k^r)|Z^{k-1})$ can be given as

$$\begin{aligned}
&\mathrm{cov}(L_h(x_k^{\mathrm{p}}, \check{v}_k^r)|Z^{k-1}) = E[(L_h(x_k^{\mathrm{p}}, \check{v}_k^r) - \bar{h})(\cdot)^{\mathrm{T}}|Z^{k-1}] \\
&= E[(C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1}(x_k^{\mathrm{p}} - \bar{x}_k^{\mathrm{p}}) + C_{h\check{v}^r} C_{\check{v}^r}^{-1} \check{v}_k^r)(\cdot)^{\mathrm{T}}|Z^{k-1}] \\
&= E[(C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1}(x_k^{\mathrm{p}} - \bar{x}_k^{\mathrm{p}}))(\cdot)^{\mathrm{T}}|Z^{k-1}] \\
&\qquad + E[(C_{h\check{v}^r} C_{\check{v}^r}^{-1} \check{v}_k^r)(\cdot)^{\mathrm{T}}|Z^{k-1}] \qquad (82) \\
&= C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}} + C_{h\check{v}^r} C_{\check{v}^r}^{-1} C_{h\check{v}^r}^{\mathrm{T}} \qquad (83)
\end{aligned}$$

Eq. (82) holds due to the assumed conditional uncorrelatedness of x_k^{p} and $\check{v}_k^r$.

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B. Calculation of $C_{\check{v}^r}$ (34)

According to (22), $C_{\check{v}^r}$ can be obtained as

$$\begin{aligned} C_{\check{v}^r} &\triangleq E[\check{v}_k^r(\check{v}_k^r)^{\mathrm{T}}|Z^{k-1}] \\ &= E\{E[\check{v}_k^r(\check{v}_k^r)^{\mathrm{T}}|X_k, Z^{k-1}]|Z^{k-1}\} \\ &= E[\tilde{B}_k X_k \tilde{B}_k^{\mathrm{T}}|Z^{k-1}] = \tilde{B}_k \bar{X}_{k|k-1} \tilde{B}_k^{\mathrm{T}} \end{aligned}$$

The second equation above follows from the total expectation theorem. The third equation is obtained using (22).

C. Calculation of $C_{hx^{\mathrm{p}}}$ (43)

Based on (40), $C_{hx^{\mathrm{p}}}$ can be calculated as

$$\begin{aligned} C_{hx^{\mathrm{p}}} &= \mathrm{cov}(h(x_k^{\mathrm{p}}, u_k^r, \tilde{v}_k^r), x_k^{\mathrm{p}}|Z^{k-1}) \\ &= \mathrm{cov}((x_k^{\mathrm{p}} + u_k^r + \tilde{v}_k^r), x_k^{\mathrm{p}}|Z^{k-1}) \\ &= E[(x_k^{\mathrm{p}} + u_k^r + \tilde{v}_k^r)(x_k^{\mathrm{p}} - \hat{x}_{k|k-1}^{\mathrm{p}})^{\mathrm{T}}|Z^{k-1}] \qquad (84) \\ &= E[E[(x_k^{\mathrm{p}} + u_k^r + \tilde{v}_k^r)(x_k^{\mathrm{p}} - \hat{x}_{k|k-1}^{\mathrm{p}})^{\mathrm{T}}|X_k, Z^{k-1}]|Z^{k-1}] \\ &= E[E[x_k^{\mathrm{p}}(x_k^{\mathrm{p}} - \hat{x}_{k|k-1}^{\mathrm{p}})^{\mathrm{T}}|X_k, Z^{k-1}]|Z^{k-1}] \qquad (85) \\ &= E[x_k^{\mathrm{p}}(x_k^{\mathrm{p}} - \hat{x}_{k|k-1}^{\mathrm{p}})^{\mathrm{T}}|Z^{k-1}] = \mathrm{cov}(x_k^{\mathrm{p}}|Z^{k-1}) = C_{x^{\mathrm{p}}} \end{aligned}$$

Eq. (84) with $\hat{x}_{k|k-1}^{\mathrm{p}} \triangleq E[x_k^{\mathrm{p}}|Z^{k-1}]$ holds by the definition of covariance. Eq. (85) is due to the assumed independence of x_k^{p}, u_k^r, and $\tilde{v}_k^r$ conditioned on X_k and Z^{k-1}.

D. Calculation of $C_{h|x^{\mathrm{p}}}$ (44)

According to (31), $C_{h|x^{\mathrm{p}}}$ in (42) is

$$\begin{aligned} C_{h|x^{\mathrm{p}}} &\triangleq C_h - C_{hx^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{hx^{\mathrm{p}}}^{\mathrm{T}} \\ &= C_{x^{\mathrm{p}}} + C_{v^r} - C_{x^{\mathrm{p}}} C_{x^{\mathrm{p}}}^{-1} C_{x^{\mathrm{p}}}^{\mathrm{T}} \qquad (86) \\ &= C_{v^r} = \mathrm{cov}(v_k^r|Z^{k-1}) \\ &= \int (\lambda X_k + R_k) p[X_k|Z^{k-1}] \mathrm{d}X_k \qquad (87) \\ &= \lambda \bar{X}_{k|k-1} + R_k \end{aligned}$$

Eq. (86) holds because $C_h = C_{x^{\mathrm{p}}} + C_{v^r}$ and $C_{hx^{\mathrm{p}}} = C_{x^{\mathrm{p}}}$ according to (43). Eq. (87) is due to (39).

E. Calculation of $p[x_k^p, u_k^r|Z^{k-1}]$ (49)

According to (46), we have

$$\begin{aligned} &p[x_k^{\mathrm{p}}, u_k^r|Z^{k-1}] = E[p[x_k^{\mathrm{p}}, u_k^r|X_k, Z^{k-1}]|Z^{k-1}] \\ &= \int \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}} \otimes X_k) \mathcal{N}(u_k^r; 0, \lambda X_k) \\ &\qquad \times p[X_k|Z^{k-1}] \mathrm{d}X_k \qquad (88) \\ &= \int \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}} \otimes X_k) \mathcal{N}(u_k^r; 0, \lambda \otimes X_k) \qquad (89) \\ &\qquad \times p[X_k|Z^{k-1}] \mathrm{d}X_k \\ &= \int \mathcal{N}(x_k^{\mathrm{a}}; \hat{x}_{k|k-1}^{\mathrm{a}}, P_{k|k-1}^{\mathrm{a}} \otimes X_k) \qquad (90) \\ &\qquad \times \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \mathrm{d}X_k \\ &= \mathcal{T}(x_k^{\mathrm{a}}; \hat{\alpha}_{k|k-1} + s^{\mathrm{a}}, \hat{x}_{k|k-1}^{\mathrm{a}}, P_{k|k-1}^{\mathrm{a}} \otimes \hat{X}_{k|k-1}) \qquad (91) \end{aligned}$$

Eq. (88) follows from the independence of x_k^{p} and u_k^r conditioned on X_k and Z^{k-1}. Eq. (89) holds because λ is a scalar and thus $\lambda X_k = \lambda \otimes X_k$. Eq. (90) is obtained by merging the first two terms of the integrand in (89) into a single Gaussian. Eq. (91) holds according to the proof given by Appendix B for Eq. (66) in [13], the definition given by Eq. (4.2.1) on page 134 of [10], and the statement above Eq. (67) in [15].

F. Calculation of $p[x_k^p, u_k^r|Z^{k-1}]$ (52)

According to (47), we have

$$\begin{aligned} &p[x_k^{\mathrm{p}}, u_k^r|Z^{k-1}] \\ &= \int \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}}) \mathcal{N}(u_k^r; 0, \lambda X_k) p[X_k|Z^{k-1}] \mathrm{d}X_k \\ &= \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}}) \\ &\qquad \times \int \mathcal{N}(u_k^r; 0, \lambda X_k) \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \mathrm{d}X_k \\ &= \mathcal{N}(x_k^{\mathrm{p}}; \hat{x}_{k|k-1}^{\mathrm{p}}, P_{k|k-1}^{\mathrm{p}}) \mathcal{T}(u_k^r; \hat{\alpha}_{k|k-1} - 2d, 0, \lambda \hat{X}_{k|k-1}) \end{aligned} \qquad (92)$$

Eq. (92) holds as analyzed above for Eq. (91).

G. Proof of the VB approach (70)–(76)

For recursive estimation, the posterior distribution should be in the same form as the prior distribution (58). Thus, we attempt to find an updated distribution having the following form

$$p[x_k, X_k|Z^k] = \mathcal{N}(x_k; \hat{x}_k, P_k) \mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k) \qquad (93)$$

Then comparing the VB approximation (63) with (93), we have

$$Q_x(x_k) = \mathcal{N}(x_k; \hat{x}_k, P_k) \qquad (94)$$

$$Q_X(X_k) = \mathcal{IW}(X_k; \hat{\alpha}_k, \hat{X}_k) \qquad (95)$$

Thus, updating the distribution using VB approximation amounts to obtaining $\hat{x}_k, P_k, \hat{\alpha}_k$, and $\hat{X}_k$ in (94) and (95).

Substituting (68) and (62) into (67) yields

$$\begin{aligned} &p[Z_k, x_k, X_k|Z^{k-1}] \\ &\propto \mathcal{N}(\bar{z}_k; \tilde{H}_k x_k, \frac{B_k X_k B_k^{\mathrm{T}}}{n_k}) \mathcal{W}(\bar{Z}_k; n_k - 1, B_k X_k B_k^{\mathrm{T}}) \\ &\quad \times \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \mathcal{IW}(X_k; \hat{\alpha}_{k|k-1}, \hat{X}_{k|k-1}) \\ &\propto |X_k|^{-\frac{\hat{\alpha}_{k|k-1}+n_k}{2}} \mathrm{etr}[-\frac{1}{2} B_k^{-\mathrm{T}} X_k^{-1} B_k^{-1} \bar{Z}_k - \frac{1}{2} \hat{X}_{k|k-1} X_k^{-1} \\ &\quad - \frac{1}{2} (\bar{z}_k - \tilde{H}_k x_k)^{\mathrm{T}} (B_k X_k B_k^{\mathrm{T}}/n_k)^{-1} (\bar{z}_k - \tilde{H}_k x_k) \\ &\quad - \frac{1}{2} (x_k - \hat{x}_{k|k-1})^{\mathrm{T}} P_{k|k-1}^{-1} (x_k - \hat{x}_{k|k-1})] \end{aligned} \qquad (96)$$

Eq. (96) is obtained by the definitions of Gaussian, Wishart, and inverse Wishart distributions [10].

To obtain $Q_x(x_k)$, substituting (96) into (65) yields

$$\begin{aligned} \ln Q_x(x_k) &= \int \ln(p[Z_k, x_k, X_k|Z^{k-1}]) Q_X(X_k) \mathrm{d}X_k + c_x^1 \\ &= -\frac{n_k}{2} (\bar{z}_k - \tilde{H}_k x_k)^{\mathrm{T}} B_k^{-\mathrm{T}} (X_k^{-1})_X B_k^{-1} (\bar{z}_k - \tilde{H}_k x_k) \\ &\quad - \frac{1}{2} (x_k - \hat{x}_{k|k-1})^{\mathrm{T}} P_{k|k-1}^{-1} (x_k - \hat{x}_{k|k-1}) + c_x^2 \end{aligned} \qquad (97)$$

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where c_x^1 and c_x^2 are constants, and

$$(X_k^{-1})_X \triangleq \int X_k^{-1} Q_X(X_k) \mathrm{d}X_k = (\hat{\alpha}_k - d - 1)\hat{X}_k \quad (98)$$

Eq. (98) follows from (95) and properties of the inverse Wishart distribution [10]. Substituting (98) into (97) and after a simple factorization, we have

$$\ln Q_x(x_k) = -\frac{1}{2}(x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \hat{x}_k) + c_x^3 \quad (99)$$

where c_x^3 is a proper constant, and $\hat{x}_k$ and P_k are given in (72) and (73), respectively. Eq. (99) indicates that $Q_x(x_k)$ is a Gaussian density function in the form of (94) with $\hat{x}_k$ of (72) and P_k of (73).

To obtain $Q_X(X_k)$, substituting (96) into (65) yields

$$\begin{aligned}
\ln Q_X(X_k) &= \int \ln(p[Z_k, x_k, X_k | Z^{k-1}]) Q_x(x_k) \mathrm{d}x_k + c_X^1 \\
&= -\frac{1}{2}(\hat{\alpha}_{k|k-1} + n_k) \ln |X_k| + c_X^2 \\
&- \frac{1}{2}\mathrm{tr}(\hat{X}_{k|k-1} X_k^{-1}) - \frac{1}{2}\mathrm{tr}(B_k^{-\mathrm{T}} X_k^{-1} B_k^{-1} \bar{Z}_k) \\
&- \frac{1}{2}\mathrm{tr}[(\bar{z}_k - \tilde{H}_k x_k)^{\mathrm{T}} (B_k X_k B_k^{\mathrm{T}}/n_k)^{-1} (\bar{z}_k - \tilde{H}_k x_k)]_x \\
&= -\frac{1}{2}(\hat{\alpha}_{k|k-1} + n_k) \ln |X_k| + c_X^2 \\
&- \frac{1}{2}\mathrm{tr}((\hat{X}_{k|k-1} + B_k^{-1} \bar{Z}_k B_k^{-\mathrm{T}}) X_k^{-1}) \\
&- \frac{n_k}{2}\mathrm{tr}([(\bar{z}_k - \tilde{H}_k x_k)(\bar{z}_k - \tilde{H}_k x_k)^{\mathrm{T}}]_x (B_k X_k B_k^{\mathrm{T}})^{-1}) \quad (100)
\end{aligned}$$

where c_X^1 and c_X^2 are constants, $\mathrm{tr}(\cdot) = \mathrm{trace}(\cdot)$, and $[\cdot]_x \triangleq \int [\cdot] Q_x(x_k) \mathrm{d}x_k$. Eq. (100) follows from properties of the trace operation. In (100), we have

$$\begin{aligned}
&[(\bar{z}_k - \tilde{H}_k x_k)(\bar{z}_k - \tilde{H}_k x_k)^{\mathrm{T}}]_x \\
&= [(\bar{z}_k - \tilde{H}_k \hat{x}_k + \tilde{H}_k \hat{x}_k - \tilde{H}_k x_k)(\cdot)^{\mathrm{T}}]_x \\
&= [(\bar{z}_k - \tilde{H}_k \hat{x}_k)(\cdot)^{\mathrm{T}}]_x + [(\tilde{H}_k x_k - \tilde{H}_k \hat{x}_k)(\cdot)^{\mathrm{T}}]_x \quad (101) \\
&= (\bar{z}_k - \tilde{H}_k \hat{x}_k)(\cdot)^{\mathrm{T}} + \tilde{H}_k [(x_k - \hat{x}_k)(\cdot)^{\mathrm{T}}]_x \tilde{H}_k^{\mathrm{T}} \\
&= (\bar{z}_k - \tilde{H}_k \hat{x}_k)(\cdot)^{\mathrm{T}} + \tilde{H}_k P_k \tilde{H}_k^{\mathrm{T}} \quad (102)
\end{aligned}$$

where $(\cdot)$ repeats the term right before it. Eq. (101) holds because $[(\bar{z}_k - \tilde{H}_k \hat{x}_k)(\tilde{H}_k x_k - \tilde{H}_k \hat{x}_k)^{\mathrm{T}}]_x = 0$ with $[x_k]_x \triangleq \int x_k Q_x(x_k) \mathrm{d}x_k = \hat{x}_k$ according to (94). Eq. (102) follows from

$$\begin{aligned}
[(x_k - \hat{x}_k)(\cdot)^{\mathrm{T}}]_x &= \int (x_k - \hat{x}_k)(\cdot)^{\mathrm{T}} Q_x(x_k) \mathrm{d}x_k \\
&= \int (x_k - \hat{x}_k)(\cdot)^{\mathrm{T}} \mathcal{N}(x_k; \hat{x}_k, P_k) \mathrm{d}x_k = P_k
\end{aligned}$$

Substituting (102) into (100) and after a simple factorization, we have

$$\ln Q_X(X_k) = -\frac{1}{2}\hat{\alpha}_k \ln |X_k| - \frac{1}{2}\mathrm{tr}(\hat{X}_k X_k^{-1}) + c_X^3 \quad (103)$$

where c_X^3 is a proper constant, and $\hat{\alpha}_k$ and $\hat{X}_k$ are given in (75) and (76), respectively. Eq. (99) indicates that $Q_x(x_k)$ is an inverse Wishart density function in the form of (95) with $\hat{\alpha}_k$ of (75) and $\hat{X}_k$ of (76).

This completes the proof.

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