Times Series Analysis (III) – Models For Stationary Time Series
Moving Average Process (MA)

Jianyong Sun
School of Mathematics and Statistics
Xi'an Jiaotong University

19th Sep., 2017

Recap

- Trends – Stochastic and deterministic trends $Y_t = \mu_t + X_t$ with $\{X_t\}$ the stochastic component and $E(X_t) = 0$
  - constant mean: $\mu_t = \mu$. With different $\{X_t\}$, estimate, and assess its precision or $Var(\mu)$
  - linear trend, seasonal means model and cosine trend model
  - residue analysis: to analyze how well our assumption on $\{X_t\}$: stationarity, normality, independence
  - Sample ACF, QQ-plot on residue.

Definition

A *general linear process*, $\{Y_t\}$, is one that can be represented as a weighted linear combination of present and past white noise terms as:

$$Y_t = e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + \cdots$$

such that

$$\sum_{i=1}^{\infty} \Psi_i^2 < \infty$$

and $\{e_t\}$ is a sequence of i.i.d. zero-mean, random variables.

General Linear Processes

One important nontrivial example:

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

Here we let $\Psi_j = \phi^j$ for $j \leq 0$ and $|\phi| < 1$

General Linear Processes

$$
\begin{aligned}
E(Y_t) &= E\left(\sum_{k=0}^{\infty} \phi^k e_{t-k}\right) = 0 \\
Var(Y_t) &= Var\left(\sum_{k=0}^{\infty} \phi^k e_{t-k}\right) \\
&= \sum_{k=0}^{\infty} \phi^{2k} Var(e_{t-k}) = \sigma_e^2 \sum_{k=0}^{\infty} \phi^{2k} \\
&= \frac{\sigma_e^2}{1-\phi^2}
\end{aligned}
$$

General Linear Processes

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov\left(\sum_{k=0}^{\infty} \phi^k e_{t-k}, \sum_{k=0}^{\infty} \phi^k e_{t-k-1}\right) \\ &= \sum_{k=1}^{\infty} Cov(\phi^k e_{t-k}, \phi^{k-1} e_{t-k}) \\ &= \sum_{k=1}^{\infty} \phi^{2k-1}\sigma_e^2 = \phi\sigma_e^2 \sum_{k=0}^{\infty} \phi^{2k} = \frac{\phi\sigma_e^2}{1-\phi^2} \end{aligned}$$

Thus

$$\begin{aligned} Corr(Y_t, Y_{t-1}) &= Cov(Y_t, Y_{t-1})/\sqrt{Var(Y_t)Var(Y_{t-1})} \\ &= \left[\frac{\phi\sigma_e^2}{1-\phi^2}\right] / \left[\frac{\sigma_e^2}{1-\phi^2}\right] = \phi \end{aligned}$$

General Linear Processes

Similarly, we can find

$$Cov(Y_t, Y_{t-k}) = \frac{\phi^k \sigma_e^2}{1-\phi^2}$$

Thus

$$Corr(Y_t, Y_{t-k}) = \phi^k$$

Observations

- General linear process is **stationary** since
- $E(Y_t) = 0$ and $\gamma_k = Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{i=0}^{\infty} \Psi_i \Psi_{i+k}, k \geq 0$

Moving Average Processes of order q – MA(q)

Assume a finite number of Ψ's are nonzero, we have the so-called moving average process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

Accordingly,

$$Y_{t+1} = e_{t+1} - \theta_1 e_t - \theta_2 e_{t-1} - \cdots - \theta_q e_{t-q+1}$$

First considered by Slutsky (1972) and Wold (1938).

Lag-Operator

$$\begin{aligned} LX_t &= X_{t-1} \\ L^iX_t &= X_{t-i} \end{aligned}$$

Example

$$\begin{aligned} L^2X_t &= LLX_t \\ &= LX_{t-1} = X_{t-2} \end{aligned}$$

Difference Operator

$$\begin{aligned} \Delta X_t &= X_t - X_{t-1} \\ \Delta^i X_t &= (1-L)^i X_t \end{aligned}$$

Example

$$\begin{aligned} \Delta^2 X_t &= (1-L)^2 X_t \\ &= (1-2L+L^2)X_t \\ &= \underbrace{\underbrace{(X_t - X_{t-1})}_{\Delta X_t} - \underbrace{(X_{t-1} - X_{t-2})}_{\Delta X_{t-1}}}_{\Delta\Delta X_t} \end{aligned}$$

MA(q) process thus has the form

$$
\begin{aligned}
Y_t &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \\
&= (1 - \theta_1 L^1 - \cdots - \theta_q L^q) e_t \\
&= \theta(L) e_t
\end{aligned}
$$

The First-Order MA

The model

$$Y_t = e_t - \theta e_{t-1}$$

and

$$E(Y_t) = 0 \text{ and } \; Var(Y_t) = (1+\theta^2)\sigma_e^2$$

and

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) \\ &= Cov(-\theta e_{t-1}, e_{t-1}) = -\theta\sigma_e^2 \\ Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) = 0 \end{aligned}$$

Therefore, for $k > 1$, $Cov(Y_t, Y_{t-k}) = 0$there has no correlation beyond lag 1.

The First-Order MA

summary

$$
\begin{aligned}
E(Y_t) &= 0 \\
\gamma_0 &= Var(Y_t) = \sigma_e^2(1+\theta^2) \\
\gamma_1 &= -\theta\sigma_e^2 \\
\rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{-\theta}{1+\theta^2} \\
\gamma_k &= \rho_k = 0 \text{ for } \ k \geq 2.
\end{aligned}
$$

The First-Order MA

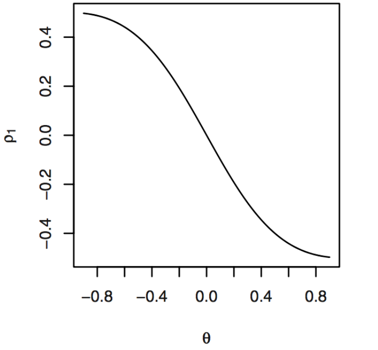

Figure: Lag 1 Autocorrelation of an MA(1) process for Different θ

Invertibility

What if we replace θ by $1/\theta$? Then we have:

$$\rho_1 = ?$$

If we knew that an $MA(1)$ process had $\rho_1 = 0.4$, we still could not tell the precise value of θ.

The First-Order MA

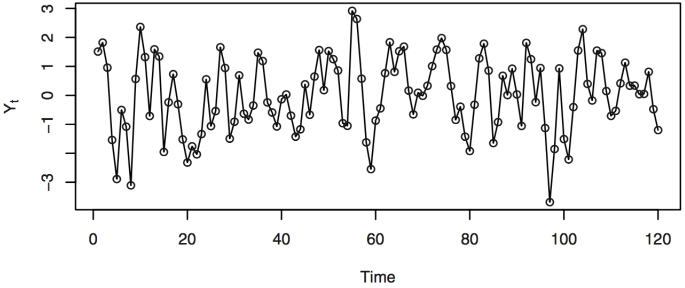

Figure: Time plot of an MA(1) Process with $\theta = -0.9$ ($\rho_1 = \frac{-\theta}{1+\theta^2} = 0.4972$)

The First-Order MA

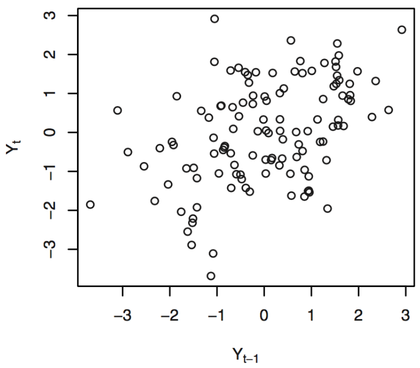

Figure: Plot of Y_t versus Y_{t-1} for MA(1) Series.

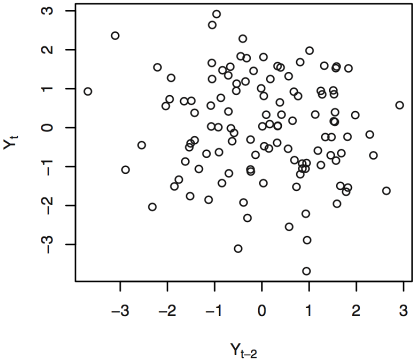

Figure: Plot of Y_t versus Y_{t-2} for MA(1) Series.

The First-Order MA

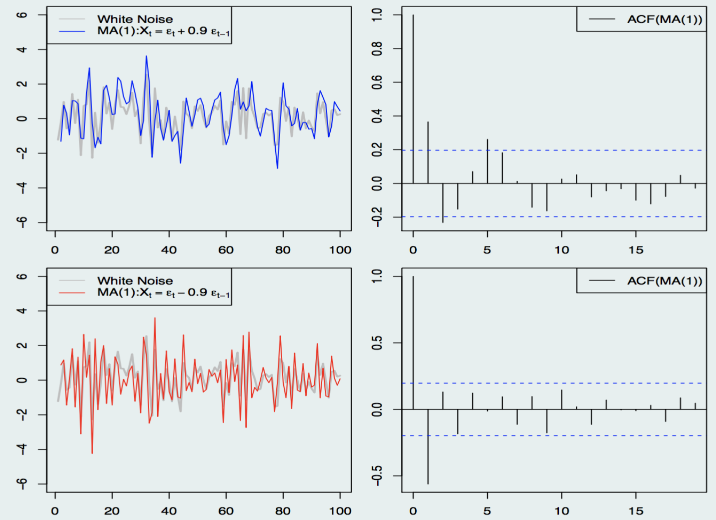

Figure: MA(1) with sample ACF.

The Second-Order MA Process

Consider a moving average process of order 2:

$$Y_t = \theta(L)e_t$$

where $\theta(L) = 1 - \theta_1 L - \theta_2 L^2$

The Second-Order MA Process

$$
\begin{aligned}
\gamma_0 &= Var(Y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&= (1 + \theta_1^2 + \theta_2^2)\sigma_e^2 \\
\gamma_1 &= Cov(Y_t, Y_{t-1}) \\
&= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_1 e_{t-2}, -\theta_2 e_{t-2}) \\
&= [-\theta_1 + \theta_1\theta_2]\sigma_e^2 \\
\gamma_2 &= Cov(Y_t, Y_{t-2}) \\
&= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\
&= Cov(-\theta_2 e_{t-2}, e_{t-2}) \\
&= -\theta_2\sigma_e^2
\end{aligned}
$$

The Second-Order MA Process

Thus

$$
\begin{aligned}
\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\rho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\rho_k &= ?
\end{aligned}
$$

The Second-Order MA Process

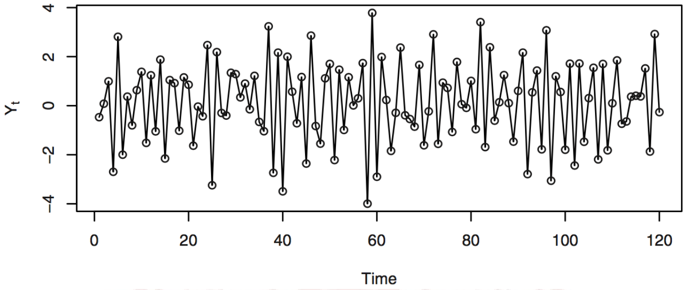

Figure: Time Plot of an MA(2): $Y_t = e_t - e_{t-1} + 0.6e_{t-2}$.

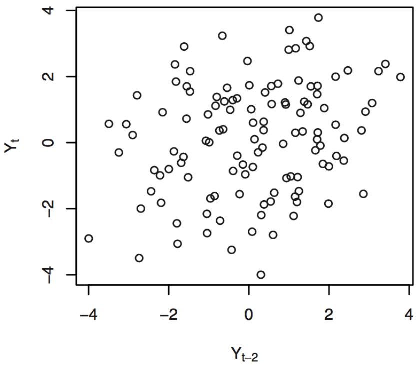

Figure: Plot of Y_t versus Y_{t-2} for MA(2) process.

The Second-Order MA Process

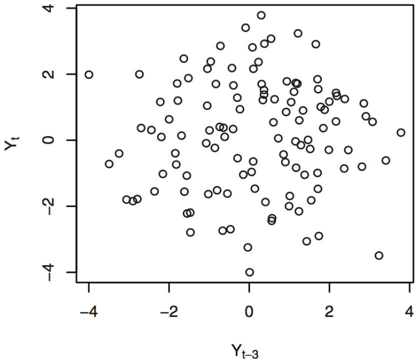

Figure: Plot of Y_t versus Y_{t-3} for MA(2) process.

The Second-Order MA Process

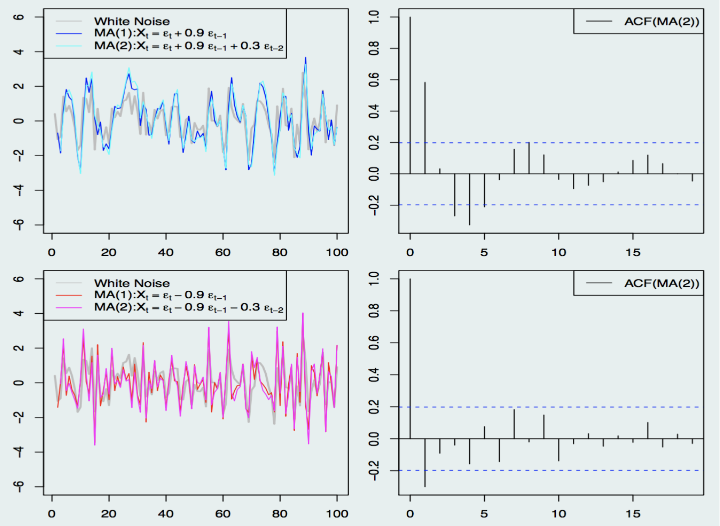

Figure: MA(2) process with sample ACF.

General MA(q) Process

For a general MA(q) process

$$Y_t = \theta(L)e_t$$

where $\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \cdots - \theta_q L^q$, we have:

$$\gamma_0 = (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma_e^2$$

and

$$\rho_k = \begin{cases} \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2} & \textit{for } k = 1, 2, \cdots, q \\ 0 & \text{for } k > q \end{cases}$$

Questions?

A Brief Introduction to Difference Equations, referred to Mickens (1990).

Suppose that we have a sequence of numbers $u_0, u_1, \cdots$, such that

$$u_n - \alpha u_{n-1} = 0, \alpha \neq 0, n = 1, 2, \cdots$$

This equation represents a homogeneous difference equation of order 1.

To solve this equation $u_n - \alpha u_{n-1} = 0$, we write:

$$\begin{aligned} u_1 &= \alpha u_0 \\ u_2 &= \alpha u_1 = \alpha^2 u_0 \\ &\vdots \\ u_n &= \alpha u_{n-1} = \alpha^n u_0 \end{aligned}$$

Given an initial condition $u_0 = c$, we may solve $u_n = \alpha^n c$.
We may write the difference equation as $(1 - \alpha L)u_n$ where L is the lag operator. Let the polynomial associated with it as

$$\alpha(z) = 1 - \alpha z$$

Suppose its root is z_0, we know the solution to the difference equation with initial condition $u_0 = c$, is:

$$u_n = \alpha^n c = (z_0^{-1})^n c.$$

Now suppose that the sequence satisfies

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0, \alpha_2 \neq 0, n = 2, 3, \cdots$$

Its corresponding polynomial is

$$\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2$$

Two roots z_1 and z_2, that is $\alpha(z_1) = \alpha(z_2) = 0$.

Difference Equations

The first case: suppose that $z_1 \neq z_2$. The general solution to the order-2 difference equation is:

$$u_n = c_1 {z_1}^{-n} + c_2 {z_2}^{-n}$$

where c_1 and c_2 depend on initial conditions.
The claim that a solution can be verified by direct substitution.

$$\begin{aligned}(c_1 {z_1}^{-n} + c_2 {z_2}^{-n}) - \alpha_1 (c_1 {z_1}^{-(n-1)} + c_2 {z_2}^{-(n-1)}) - \alpha_2 (c_1 {z_1}^{-(n-2)} + c_2 {z_2}^{-(n-2)}) \\ = c_1 {z_1}^{-n} (1 - \alpha_1 z_1 - \alpha_2 z_1^2) + c_2 {z_2}^{-n} (1 - \alpha_2 z_1 - \alpha_2 z_2^2) = 0\end{aligned}$$

Given two initial conditions u_0 and u_1, we may solve for c_1 and c_2:

$$u_0 = c_1 + c_2 \text{ and } u_1 = c_1 {z_1}^{-1} + c_2 {z_2}^{-1}$$

The second case: suppose that $z_1 = z_2 (= z_0)$. A general solution is:

$$u_n = z_0^{-n}(c_1 + c_2 n)$$

This can be verified by direct substitution:

$$\begin{aligned} z_0^{-n}(c_1+c_2 n)-\alpha_1 z_0^{-(n-1)}[c_1+c_2(n-1)]-\alpha_2 z_0^{-(n-2)}[c_1+c_2(n-2)] \\ = z_0^{-n}(c_1 + c_2 n)(1 - \alpha_1 z_0 - \alpha_2 z_0^2) + c_2 z_0^{-n+1}(\alpha_1 + 2\alpha_2 z_0) \\ = c_2^{-n+1}(\alpha_1 + 2\alpha_2 z_0) = 0 \end{aligned}$$

To show that $\alpha_1 + 2\alpha_2 z_0$, write $1 - \alpha_1 z - \alpha_2 z^2 = (1 - z_0^{-1} z)^2$. Take derivatives w.r.t. z:

$$(\alpha_1 + 2\alpha_2 z) = 2 z_0^{-1}(1 - z)^{-1} z$$

Given two initial conditions u_0 and u_1, we can solve for c_1 and c_2:

$$u_0 = c_1, u_1 = (c_1 + c_2) z_0^{-1}$$