Times Series Analysis (IV) – Models For Stationary Time Series
Autoregressive Process (AR)

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Recap

- MA $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_p e_{t-p}$ with $\{e_t\}$ the stochastic component and $E(X_t) = 0$
  - General linear process
  - variance, autocovariance, autocorrelation
  - MA(1) and MA(2)
  - MA(p)

Definition

A p-th order autoregressive process, $\{Y_t\}$, satisfy the equation:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 X_{t-2} + \phi_p Y_{t-p} + e_t$$

e_t is independent of $Y_{t-1}, Y_{t-2}, \cdots$

Explanation

- Y_t is a linear combination of the p most recent past values
- e_t is usually called 'innovation'. It explains everything new that cannot be explained by the past values.

Yule (1962) is the first to carry out work on autoregressive processes.

The First-Order Autoregressive Process: AR(1)

Assume the series is stationary and satisfies:

$$Y_t = \phi Y_{t-1} + e_t$$

Also we assume that the process mean has been subtracted out so that the series mean is zero.

The First-Order Autoregressive Process: AR(1)

If we take variances of both side, we obtain:

$$\gamma_0 = \phi^2 \gamma_0 + \sigma_e^2$$

This gives us

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi^2}$$

The immediate implication is that $\phi^2 < 1$, i.e. $|\phi| < 1$.

The First-Order Autoregressive Process: AR(1)

Now compute $Cov(Y_t, Y_{t-k})$, multiply both sides by Y_{t-k}:

$$Y_t Y_{t-k} = \phi Y_{t-1} Y_{t-k} + e_t Y_{t-k}$$

Take expected values,

$$\begin{aligned} E(Y_t Y_{t-k}) &= \phi E(Y_{t-1} Y_{t-k}) + E(e_t Y_{t-k}) \\ \gamma_k &= \phi \gamma_{k-1} + E(e_t Y_{t-k}) \end{aligned}$$

Note that we assume e_t is independent of previous variables Y_{t-k}, so

$$\gamma_k = \phi \gamma_{k-1} \text{ for } k = 1, 2, 3, \cdots$$

The First-Order Autoregressive Process: AR(1)

$$\gamma_1 = \phi\gamma_0 = \frac{\phi\sigma_e^2}{1-\phi^2}$$

$$\gamma_2 = \phi\gamma_1 = \frac{\phi^2\sigma_e^2}{1-\phi^2}$$

$$\vdots$$

$$\gamma_k = \frac{\phi^k\sigma_e^2}{1-\phi^2}$$

Thus

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k \text{ for } k = 1, 2, 3, \cdots$$

The First-Order Autoregressive Process: AR(1)

Observations

- $|\phi| < 1$, the magnitude of the ACF decrease exponentially as the number of lags k
- If $0 < \phi < 1$, all ACF are positive, while if $-1 < \phi < 0$, we have successive positive and negative ACF with their magnitudes exponentially decreased.

The First-Order Autoregressive Process: AR(1)

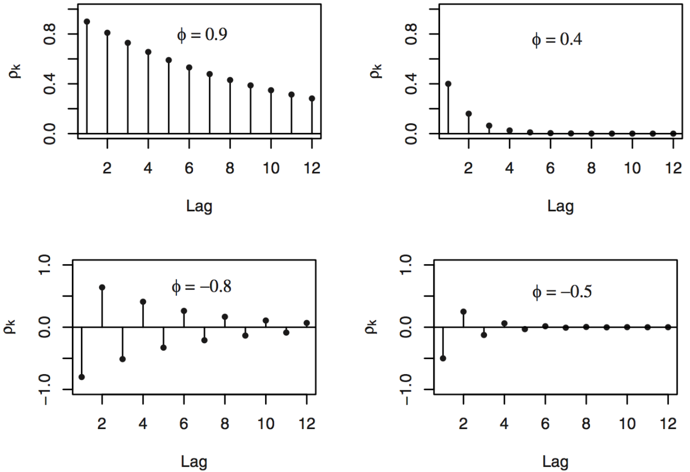


Figure: ACF for several AR(1) Models.

The First-Order Autoregressive Process: AR(1)

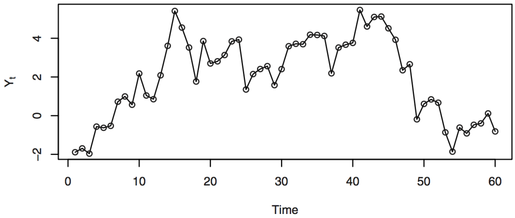
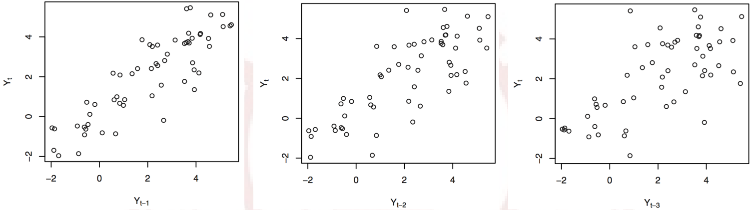


Figure: a sample of AR(1) with $\phi = 0.9$.

Observations

- It infrequently crosses its theoretical mean of zero.
- There is a lot of inertia in the series: it remains on the same side of the mean for extended periods.
- The illusion of trends is due to the strong autocorrelation of neighboring values of the series.

The First-Order Autoregressive Process: AR(1)



(a) (b) (c)

Figure: a. Plot of Y_t versus Y_{t-1} for AR(1) series with $\phi = 0.9$; b. Plot of Y_t versus Y_{t-2} for AR(1) series with $\phi = 0.9$; c.Plot of Y_t versus Y_{t-3} for AR(1) series with $\phi = 0.9$

General Linear Process Version of the AR(1) Model

Express AR(1) as a General Linear Process

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
Y_{t-1} &= \phi Y_{t-2} + e_{t-1} \\
Y_t &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\
&= e_t + \phi e_{t-1} + \phi^2 Y_{t-2}
\end{aligned}
$$

Repeat the substitution $k-1$ times, we get

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k} \quad (1)$$

General Linear Process Version of the AR(1) Model

Assume $|\phi| < 1$ and let $k \to \infty$, we obtain

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots$$

Recall: General Linear Process

$$Y_t = e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + \cdots$$

with $\Psi_j = \phi^j$

Stationarity of an AR(1) Process

Subject to e_t be independent of $Y_{t-1}, Y_{t-2}, \cdots$ and that $\sigma_e^2 > 0$, AR(1) is stationary if and only if $|\phi| < 1$.

Stationarity Condition

The requirement $|\phi| < 1$ is called the stationarity condition for the AR(1) process defined as

$$Y_t = \phi Y_{t-1} + e_t$$

with

$$\begin{aligned} \gamma_k &= Cov(Y_t, Y_{t-k}) = \frac{\phi^k \sigma_e^2}{1 - \phi^2} \\ \rho_k &= Corr(Y_t, Y_{t-k}) = \phi^k \end{aligned}$$

Three Ways to derive the ACF

- General Linear form: $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots$
- Recursive form: $Y_t = \phi Y_{t-1} + e_t$
- Eq. (1):
  $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k}$

Multiply Y_{t-k} on both sides of Eq. 1, and take expected values

$$\begin{aligned} E(Y_t Y_{t-k}) &= E(Y_{t-k} e_t + \phi Y_{t-k} e_{t-1} + \cdots + \phi^k Y_{t-k} Y_{t-k}) \\ &= \phi^k E(Y_{t-k} Y_{t-k}) = \phi^k Var(Y_t) = \phi^k \gamma_0 \end{aligned}$$

The Second-Order Autoregressive Process

Consider an autoregressive process of order 2:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

where we assume that e_t is independent of $Y_{t-1}, Y_{t-2}, \cdots$. Note

$$\begin{aligned} e_t &= Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} \\ &= (1 - \phi_1 L^1 - \phi_2 L^2) Y_t \end{aligned}$$

To discuss **stationarity**, consider the **AR characteristic polynomial:**

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2$$

and the corresponding **AR characteristic equation:**

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

Stationarity of the AR(2) Process

- It may be shown that, subject to the condition that e_t is independent of $Y_{t-1}, Y_{t-2}, \cdots$, a stationary solution to the AR(2) exists if and only if the roots of the AR characteristic equation exceed 1 in absolute value (modulus).
- We sometimes say that the roots should lie outside the unit circle in the complex plane.
- The statement is applicable to the p-th order case without change.

Stationarity of the AR(2) Process

For AR(2), it has stationary solutions if and only if
$\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1$ and $|\phi_2| < 1$

Proof

Roots for a order-2 polynomial is $\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$, let

$$
\begin{aligned}
G_1 &= \frac{2\phi_2}{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}} = \frac{2\phi_2}{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}} \left[\frac{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}} \right] \\
&= \frac{\phi_1 - \sqrt{\phi^2 + 4\phi_2}}{2}
\end{aligned}
$$

Similarly, we have

$$
G_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}
$$

Proof Cont.

I. **Real Roots**: $|G_i| < 1$ for $i = 1, 2$ if and only if

$$-1 < \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} < \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$

Consider the first inequality, $-2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2}$ if and only if $\sqrt{\phi_1^2 + 4\phi_2} < \phi_1 + 2 \to \phi_2 < \phi_1 + 1$. Similarly, the inequality $\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2$ leads to $\phi_2 + \phi_1 < 1$

II. **Complex Roots:** Now $\phi_1^2 + 4\phi_2 < 0$. Here G_1 and G_2 will be complex conjugates, $|G_1| = |G_2| < 1$ if and only if $|G_1|^2 < 1$. $|G_1|^2 = [\phi_1^2 + (-\phi_1^2 - 4\phi_2)]/4 = -\phi_2$, so that $\phi_2 > -1$. This together with the inequality $\phi_1^2 + 4\phi_2 < 0$ defines the part of the stationarity region for complex roots.

Stationarity of the AR(2) Process

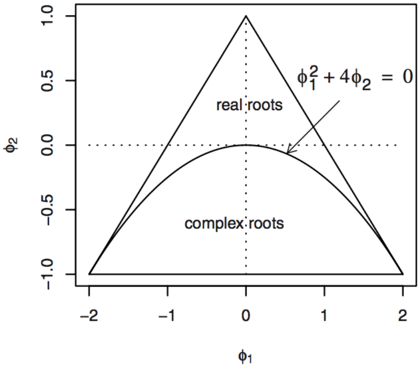


Figure: Stationary Parameter Region for AR(2) Process.

The Autocorrelation Function for the AR(2) Process

To derive the ACF for AR(2), consider the recursive formulation of AR(2): $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$, multiply both sides by Y_{t-k} and take expectations. Assuming stationarity, zero-mean and e_t is independent of $Y_{t-1}, Y_{t-2}, \cdots$, we get

$$E[Y_t Y_{t-k}] = E[\phi_1 Y_{t-1} Y_{t-k}] + E[\phi_2 Y_{t-2} Y_{t-k}] + E[Y_{t-k} e_t]$$

i.e.

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \text{ for } \ k = 1, 2, 3, \cdots$$

Hence

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \text{ for } \ k = 1, 2, 3, \cdots$$

Yule-Walker equations

Note $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get

$$\rho_1 = \frac{\phi_1}{1-\phi_2}, \rho_2 = \frac{\phi_2(1-\phi_2)+\phi_1^2}{1-\phi_2}, \cdots$$

It is desirable to have an explicit formula for ρ_k.
If the roots are distinct, i.e. $G_1 \neq G_2$, it can be shown that

$$\rho_k = \frac{(1-G_2^2)G_1^{k+1} - (1-G_1^2)G_2^{k+1}}{(G_1 - G_2)(1+G_1 G_2)}$$

The Autocorrelation Function for the AR(2) Process

If the roots are complex, then $z_1 = \bar{z}_2$ are complex conjugate pair, then $c_2 = \bar{c}_1$ and

$$\rho_k = c_1 z_1^{-k} + \bar{c}_1 \bar{z}_1^{-k}$$

If write c_1 and z_1 in polar coordinates, e.g. $z_1 = |z_1| e^{i\theta}$ where θ is tne angle whose tangent is the ratio of the imaginary and real part of z_1; the range θ is $[-\pi, \pi]$. Using the fact that $e^{i\alpha} + e^{-i\alpha} = 2cos(\alpha)$, then the solution has the form

$$\rho_k = a|z_1|^{-k} cos(k\theta + b)$$

Therefore:

$$\rho_k = R^k \frac{\sin(\Theta k + \Phi)}{\sin(\Phi)}$$

where $R = \sqrt{-\phi_2}$ and Θ and Φ are defined by $\cos(\Theta) = \phi_1/(2\sqrt{-\phi_2})$ and $\tan(\Phi) = [(1 - \phi_2)/(1 + \phi_2)]$

The Autocorrelation Function for the AR(2) Process

If the roots are equal, we have

$$\rho_k = \left(1 + \frac{1+\phi_2}{1-\phi_2}k\right)\left(\frac{\phi_1}{2}\right)^k$$

Observations:

- the ACF can assume a wide variety of shapes.
- the magnitude of ρ_k dies out exponentially fast as the lag k increases
- In the case of complex roots, ρ_k displays a damped sine wave behavior with damping factor $R, 0 \leq R < 1$, frequency Θ and phase Φ.

The Autocorrelation Function for the AR(2) Process

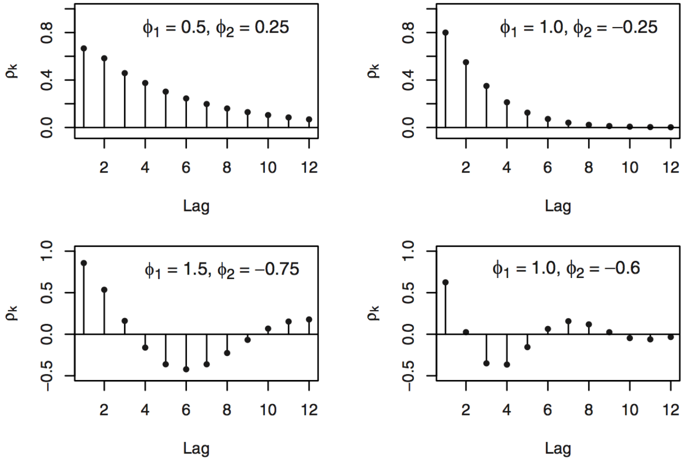


Figure: ACF for several AR(2) Models.

The Autocorrelation Function for the AR(2) Process

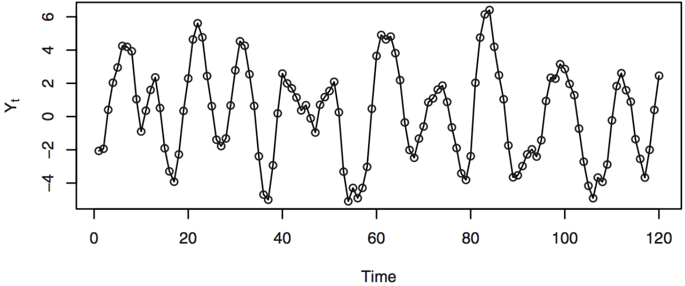


Figure: Time Plot of an AR(2) Series with $\phi_1 = 1.5$ and $\phi_2 = -0.75$.

The Variance for the AR(2) Model

$$AR(2): Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Taking the variance of both sides:

$$\gamma_0 = (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2$$

Note that

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}$$

Setting $k = 1$, we have $\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$. This gives

$$\begin{aligned}\gamma_0 &= \frac{(1-\phi_2)\sigma_e^2}{(1-\phi_2)(1-\phi_1^2-\phi_2^2) - 2\phi_2\phi_1^2} \\ &= \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2}\end{aligned}$$

The Ψ-weights of AR(2)

To represent AR(2) in the general linear form

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

we could resort to the lag-operator. Note that

$$\begin{aligned} Y_t &= (\psi_0 + \psi_1 L^1 + \psi_2 L^2 + \cdots) e_t \\ Y_t(1 - \phi_1 L^1 - \phi_2 L^2) &= e_t \Rightarrow \\ Y_t &= (1 - \phi_1 L^1 - \phi_2 L^2)^{-1} e_t \end{aligned}$$

This indicates

$$(\psi_0 + \psi_1 L^1 + \psi_2 L^2 + \cdots)(1 - \phi_1 L^1 - \phi_2 L^2) = 1$$

The Ψ-weights of AR(2)

Expand, we obtain:

$$\begin{aligned} \Psi_0 &= 1 \\ \Psi_1 - \phi_1 \Psi_0 &= 0 \\ \Psi_j - \phi_1 \Psi_{j-1} - \phi_2 \Psi_{j-2} &= 0 \text{ for } \; j = 2, 3, \cdots \end{aligned}$$

Using results from difference equation, we have: for $G_1 \neq G_2$

$$\psi_j = \frac{G_1^{j+1} - G_2^{j+1}}{G_1 - G_2}$$

If the roots are complex, we may obtain the results

$$\Psi_j = R^j \left\{ \frac{\sin[(j+1)\Theta]}{\sin(\Theta)} \right\}$$

with damping factor R and frequency Θ. If roots are the same,

$$\Psi_j = (1+j)\phi_1^j$$

The General AR(p) Process

Consider the p-th order AR process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

with **AR characteristic polynomial**:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

and corresponding **AR characteristic equation**:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$

In notation of lag-operator, we have:

$$\phi(L) Y_t = e_t$$

If we can do inversion on $\phi(L)$, we obtain

$$Y_t = \phi(L)^{-1} e_t$$

Note that this inversion will produce an infinite sum on e_{t-i}, i.e.

$$Y_t := \sum_{i=0}^{\infty} c_i e_{t-i} \rightarrow \text{MA}(\infty) \text{ process}$$

Now the question is: can we do the inversion?

The General AR(p) Process

Let's compute the moments of Y_t using the infinite sum:

$$E(Y_t) = \phi(L)^{-1}E[e_t] = 0 \quad \Rightarrow \quad \phi(L) \neq 0$$
$$Var(Y_t) = \phi(L)^{-2}Var(e_t) \quad \Rightarrow \quad \phi(L)^{-2} > 0$$

Using the fundamental theorem of algebra, $\phi(z)$ can be factored as

$$\phi(z) = (1 - r_1^{-1}z)(1 - r_2^{-1}z)\cdots(1 - r_p^{-1}z)$$

where $r_1, \cdots, r_p$ are the p roots of $\phi(z)$.

- To guarantee $\phi(L) > 0$, we need to ask $|r_i| > 1$ for all $1 \leq i \leq p$. That is, **all p roots of the AR characteristic equation lie outside the unit circle.**
- Note the fact that

$$\sum_{i=1}^{\infty} \phi^i = \frac{1}{1-\phi}$$

in case $|\phi| < 1$.

The General AR(p) Process

Theorem

The linear AR(p) process is strictly stationary if and only if $|r_i| > 1$ for all i, where $|r_i|$ is the modulus of the complex number r_i

Example

Are these AR processes stationary?

$$\begin{aligned} Y_t &= 0.7Y_{t-1} - 0.1Y_{t-2} + e_t \\ Y_t &= 1.5Y_{t-1} + Y_{t-2} + e_t \end{aligned}$$

The General AR(p) Process

The following two inequalities are necessary for stationarity. That is, for the roots to be greater than 1 in modulus, it is necessary but not sufficient that both

$$\begin{aligned} \phi_1 + \phi_2 + \cdots \phi_p &< 1 \\ |\phi_p| &< 1. \end{aligned}$$

The General AR(p) Process

Assuming stationarity and zero means, multiply Y_{t-k}, take expectations, and divide by γ_0, we obtain

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p} \text{ for } k \geq 1$$

Putting $k = 1, 2, \cdots$ and p and using $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get the general **Yule-Walker equations**:

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \cdots + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \cdots + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \cdots + \phi_p
\end{aligned}$$

Solving these linear equations, we can obtain numerical values for $\rho_1, \rho_2, \cdots, \rho_p$ given $\phi_1, \cdots, \phi_p$.

Noting that

$$E(e_t Y_t) = E(e_t(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t) = E(e_t^2) = \sigma_e^2$$

We may multiply Y_t by Y_t, take expectations, and find

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p + \sigma_e^2$$

Using $\rho_k = \gamma_k / \gamma_0$, we have

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2 - \cdots - \phi_p \rho_p}$$

The General AR(p) Process

- Explicit solutions for ρ_k are essentially impossible in this generality
- But we know that ρ_k will be a linear combination of exponentially decaying terms (in case of real roots) and damped sine wave terms (in case of complex roots)
- Assuming stationarity, the process can be expressed in the general linear process, but the Ψ-coefficient are complicated functions of the parameters $\phi_1, \cdots, \phi_p$

Questions?