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Characteristic Inequalities of Uniformly Convex and Uniformly Smooth Banach Spaces

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Let X be a real Banach space with dual X^* and moduli of convexity and smoothness $\delta_X(\varepsilon)$ and $\rho_X(\tau)$, respectively. For $1<p<\infty$, J_p denotes the duality mapping from X into 2^{X^*} with gauge function t^{p-1} and j_p denotes an arbitrary selection for J_p. Let $\mathscr{A}=\{\phi: R^+ \to R^+: \phi(0)=0,\ \phi(t)$ is strictly increasing and there exists $c>0$ such that $\phi(t)\geqslant c\delta_X(t/2)\}$ and $\mathscr{F}=\{\varphi: R^+\to R^+: \varphi(0)=0,\ \varphi$ is convex, nondecreasing and there exists $K>0$ such that $\varphi(\tau)\leqslant K\rho_X(\tau)\}$. It is proved that X is uniformly convex if and only if there is a $\phi\in\mathscr{A}$ such that

$$\|x+y\|^p \geqslant \|x\|^p + p\langle j_p x, y\rangle + \sigma_\phi(x,y) \qquad \forall x, y\in X$$

and X is uniformly smooth if and only if there is a $\varphi\in\mathscr{F}$ such that

$$\|x+y\|^p \leqslant \|x\|^p + p\langle j_p x, y\rangle + \sigma_\varphi(x,y) \qquad \forall x, y\in X,$$

where, for given function f, $\sigma_f(x,y)$ is defined by

$$\sigma_f(x,y) = p\int_0^1 \frac{(\|x+ty\| \vee \|x\|)^p}{t} f\left(\frac{t\|y\|}{\|x+ty\|\vee\|x\|}\right) dt.$$

These inequalities which have various applications can be regarded as general Banach space versions of the well-known polarization identity occurring in Hilbert spaces.

0. INTRODUCTION

Among all Banach spaces the Hilbert spaces are generally regarded as the ones with the simplest and perhaps most immediately and clearly

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discernible geometric structure. This observation is supported and indeed characterised by the availability of the parallelogram law

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2) \tag{$*$}$$

or equivalently the polarisation identity

$$\|x+y\|^2=\|x\|^2+2\mathrm{Re}\langle x, y\rangle+\|y\|^2. \tag{$**$}$$

With this understanding we shall say that Hilbert spaces are spaces with the best structure. The reason for saying this is that problems posed in such spaces can be analysed in a comparatively straightforward and easy manner. However, in applications many problems do not fall naturally into spaces with this best structure. Therefore it is natural to ask what spaces are nearest to spaces with the best structure in the sense that their geometric structure can be characterised by similar relations to $(*)$ and $(**)$.

Generalisations of $(*)$ and $(**)$ to L^p spaces are known [3, 12, 13, 24, 25]. The main results can be summarised as follows: Let $\lambda, \mu \in [0, 1]$ be arbitrary real numbers such that $\lambda+\mu=1$. Then

(i) Lim inequalities [13, Theorem 1; 25, pp. 3–85].

$$\|\lambda x+\mu y\|^p+g(\mu)\|x-y\|^p \leqslant \lambda\|x\|^p+\mu\|y\|^p, \qquad 2\leqslant p<\infty$$

$$\|\lambda x+\mu y\|^p+g(\mu)\|x-y\|^p \geqslant \lambda\|x\|^p+\mu\|y\|^p, \qquad 1<p\leqslant 2$$

for any $x, y \in L^p$, where

$$g(\mu)=\lambda\mu\frac{1+x(\lambda\wedge\mu)^{p-1}}{(1+x(\lambda\wedge\mu))^{p-1}}$$

and $x(\mu)$ for $0\leqslant\mu\leqslant 1/2$ is the unique solution of

$$\lambda x^{p-1}-\mu-(\lambda x-\mu)^{p-1}=0, \qquad \frac{\mu}{\lambda}\leqslant x\leqslant 1.$$

When $\lambda=\mu=1/2$, these inequalities reduce to the Clarkson inequalities

$$\|x+y\|^p+\|x-y\|^p\leqslant 2^{p-1}(\|x\|^p+\|y\|^p), \qquad 2\leqslant p<\infty$$

$$\|x+y\|^p+\|x-y\|^p\geqslant 2^{p-1}(\|x\|^p+\|y\|^p), \qquad 1<p\leqslant 2.$$

(ii) Characteristic inequalities [24]. Let $p\in(1,\infty)$ and $x, y\in L^p$. Then

$$\|\lambda x+\mu y\|^2+(p-1)\lambda\mu\|x-y\|^2<\lambda\|x\|^2+\mu\|y\|^2, \qquad x\neq y, \text{ iff } p<2$$

$$\|\lambda x+\mu y\|^2+(p-1)\lambda\mu\|x-y\|^2=\lambda\|x\|^2+\mu\|y\|^2, \qquad \text{iff } p=2$$

$$\|\lambda x+\mu y\|^2+(p-1)\lambda\mu\|x-y\|^2>\lambda\|x\|^2+\mu\|y\|^2, \qquad x\neq y, \text{ iff } p>2$$

or equivalently,

$$\|x+y\|^2 > \|x\|^2 + 2\langle Jx, y\rangle + (p-1)\|y\|^2, \qquad \|x\|\,\|y\| \neq 0, \text{ iff } p<2$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle Jx, y\rangle + (p-1)\|y\|^2, \qquad \text{iff } p=2$$

$$\|x+y\|^2 < \|x\|^2 + 2\langle Jx, y\rangle + (p-1)\|y\|^2, \qquad \|x\|\,\|y\| \neq 0, \text{ iff } p>2,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalised duality pairing and $J: L^p \to L^q$ is the normalised duality mapping defined by

$$Jx = \{x^* \in L^q : \langle x^*, x\rangle = \|x^*\|\,\|x\|, \|x^*\| = \|x\|\}, \qquad q = (p-1)^{-1}p.$$

These inequalities contain those developed by Kay [12], Bynum and Drew [3], and Ishikawa [10], respectively, as special cases.

More generally, Reich in [19] established the following version of (**)

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle Jx, y\rangle + \sigma_x(\|y\|)$$

in Banach spaces whose dual spaces are uniformly convex, where J is again the normalised duality mapping of the spaces and $\sigma_x(\|y\|)$ is given by

$$\sigma_x(\|y\|) = \max\{\|x\|, 1\}\,\|y\|\,\beta(\|y\|)$$

with

$$\beta(t) = \sup\{t^{-1}[\|x+ty\|^2 - \|x\|^2 - 2\langle Jx, y\rangle] : \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

Recently, Prus and Smarzewski [18] proved the inequality

$$\|\lambda x + \mu y\|^q + dW_q(\mu)\|x-y\|^q \leqslant \lambda\|x\|^q + \mu\|y\|^q$$

in uniformly convex Banach spaces with the moduli of convexity of power type $q \geqslant 2$, where

$$W_q(\mu) = \lambda\mu^q + \mu\lambda^q$$

and d is a positive constant. In these versions, no characteristic relation between the inequality developed and the underlying Banach spaces is reported. In particular, no explicit expression for $\sigma_x(\|y\|)$ in Reich's inequality is given.

In this paper we shall present a characteristic version of (**) in Banach spaces possessing the property of either uniform convexity or uniform smoothness. More precisely, with $J_p: X \to X^*$ being an appropriate duality mapping, $\sigma_p: X \times X \to [0, \infty)$ a definite functional, and $p \in (1, \infty)$, we prove that an inequality of the form

$$\|x+y\|^p \geqslant \|x\|^p + p\langle J_p x, y\rangle + \sigma_p(x, y) \tag{0.1}$$

characterises a Banach space X of uniform convexity, and an inequality of the form

$$\|x+y\|^p \leqslant \|x\|^p + p\langle J_p x, y\rangle + \sigma_p(x, y) \tag{0.2}$$

characterises a Banach space X of uniform smoothness. Unlike Reich's version, the functional $\sigma_p(x, y)$ in our version can be expressed explicitly in terms of either the moduli of convexity or the smoothness of the underlying Banach spaces. Accordingly, obtaining a specialization of the characteristic inequality to particular spaces reduces to computing the moduli of convexity or smoothness of the space. The inequalities (0.1) and (0.2) presented here provide a reasonable reflection of the well-known fact that there is a complete duality between uniformly convex and uniformly smooth Banach spaces (see, e.g., [16, Proposition 1.e.2]).

The inequalities developed here have applications in a number of different fields. Examples of the use of particular forms of these inequalities can be found, for instance, in [1, 3–5, 10–15, 17–21, 24–26]. Further applications will be given in [27–29].

1. Preliminary

Let X be a real Banach space and X^* be its dual space. $B(X)$ and $S(X)$ denote respectively the unit ball and the unit sphere in X. For any pair $x \in X$ and $x^* \in X^*$, $x^*(x)$ is denoted by $\langle x^*, x\rangle$.

Let

$$\delta_X(\varepsilon) = \inf\{1 - \|\tfrac{1}{2}(x+y)\| : x, y \in S(X), \|x-y\| \geqslant \varepsilon\}$$

and

$$\rho_X(\tau) = \tfrac{1}{2}\sup\{\|x+y\| + \|x-y\| - 2 : x \in S(X), \|y\| \leqslant \tau\}.$$

The functions δ_X: $[0, 2] \to [0, 1]$ and ρ_X: $[0, \infty) = R^+ \to R^+$ are respectively referred to as the moduli of convexity and smoothness of X. Recall that X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for any $\varepsilon \in (0, 2]$ and X is uniformly smooth if $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$. It is known [16, p. 63; 7; 9] that all Hilbert spaces and the Banach spaces l^p, L^p, and W_m^p $(1 < p < \infty)$ all are both uniformly convex and uniformly smooth and

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - (1/4)\varepsilon^2} \tag{1.1}$$

$$\delta_{l^p}(\varepsilon)=\delta_{L^p}(\varepsilon)=\delta_{W_m^p}(\varepsilon)=\begin{cases}\dfrac{p-1}{8}\,\varepsilon^2+o(\varepsilon^2)>\dfrac{p-1}{8}\,\varepsilon^2, & 1<p<2\\[2mm] 1-\left[1-\left(\dfrac{\varepsilon}{2}\right)^p\right]^{1/p}>\dfrac{1}{p}\left(\dfrac{\varepsilon}{2}\right)^p, & p\geqslant 2\end{cases} \tag{1.2}$$

$$\rho_H(\tau)=(1+\tau^2)^{1/2}-1 \tag{1.3}$$

$$p_{l^p}(\tau)=\rho_{L^p}(\tau)=\rho_{W_m^p}(\tau)=\begin{cases}(1+\tau^p)^{1/p}-1<\dfrac{1}{p}\,\tau^p, & 1<p<2\\[2mm] \dfrac{p-1}{2}\,\tau^2+o(\tau^2)<\dfrac{p-1}{2}\,\tau^2, & p\geqslant 2.\end{cases} \tag{1.4}$$

We will need the following basic properties of the functions $\delta_X(\varepsilon)$ and $\rho_X(\tau)$ [7, 16]:

(δ1) $\delta_X(0)=0$, $\delta_X(\varepsilon)\leqslant 1$ (<1 if X is uniformly convex);

(δ2) $\delta_X(\varepsilon)$ is continuous and nondecreasing and, $\delta_X(\varepsilon)$ is strictly increasing if and only if X is uniformly convex;

(δ3) $\delta_X(\varepsilon)\leqslant\delta_H(\varepsilon)$;

(δ4) if X is uniformly convex, then $\delta_X(\varepsilon)=:\delta_X(\varepsilon)/\varepsilon$ is nondecreasing.

(ρ1) $\rho_X(0)=0$, $\rho_X(\tau)\leqslant\tau$;

(ρ2) $\rho_X(\tau)$ is convex, continuous, and nondecreasing;

(ρ3) $\rho_X(\tau)/\tau$ is nondecreasing;

(ρ4) $\rho_X(\tau)\geqslant\rho_H(\tau)$;

(ρ5) $\rho_{X^*}(\tau)=\sup\{\tau\varepsilon/2-\delta_X(\varepsilon)\colon 0\leqslant\varepsilon\leqslant 2\}$;

(ρ6) $\rho_X(\tau)$ is equivalent to a decreasing function, namely, there exists a positive constant c so that $\rho_X(\eta)/\eta^2\leqslant c\rho_X(\tau)/\tau^2$ whenever $\eta\geqslant\tau>0$.

Let $\varphi\colon R^+\to R^+$, $\varphi(0)=0$, be a continuous, strictly increasing function (such a function is said to be a gauge function). The mapping $J_\varphi\colon X\to 2^{X^*}$ defined by

$$J_\varphi=\{x^*\in X^*\colon\langle x^*,x\rangle=\|x^*\|\,\|x\|,\ \|x^*\|=\varphi(\|x\|)\}$$

is called the duality mapping with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t)=t$, denoted by J, is referred to as the normalised duality mapping. We will use the following properties of duality mappings which are established in [6], [7], [8], and [23], respectively:

(J1) $J=I$ if and only if X is a Hilbert space;

(J2) X is uniformly smooth if and only if J (and hence J_φ) is single valued and uniformly continuous on any bounded subset of X;

(J3) J is surjective if and only if X is reflexive;

(J4) $J_\varphi(\lambda x) = \text{sign}(\lambda)(\varphi(|\lambda|\ \|x\|)/\|x\|)\,Jx$, $\lambda \in R^1$;

(J5) $J_\varphi x \subset \partial\Phi(\|x\|)$, where $\partial\Phi(\|x\|)$ is the subgradient of $\Phi(\|\cdot\|)$ at x and Φ is given by

$$\Phi(t) = \int_0^t \varphi(s)\,ds.$$

Moreover, if X is reflexive, then $J_\varphi x = \partial\Phi(\|x\|)$.

In what follows, the notation J_p is used to denote the duality mapping with gauge function $\varphi(t) = t^{p-1}$, j_p denotes an arbitrary selection for J_p (namely $j_p x \in J_p x$ for every $x \in X$). For arbitrarily real numbers a and b we always let

$$a \vee b = \max(a, b), \qquad a \wedge b = \min(a, b)$$

and further, $\lambda, \mu \in [0, 1]$, $p, q \in (1, \infty)$ are always assumed to be such that

$$\lambda + \mu = 1, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Also, given a multi-valued mapping $F: X \to 2^{X^*}$, $D(F)$, $R(F)$, $G(F)$, and F^{-1} will always denote its domain, range, graph, and inverse, which are respectively defined by

$$\begin{aligned}
D(F) &= \{x \in X : Fx \neq \phi\}\\
R(F) &= \{x^* \in X^*, x^* \in Fx, x \in D(F)\}\\
G(F) &= \{[x, x^*] \in X \times X^* : x \in D(F), x^* \in Fx\}\\
F^{-1}x^* &= \{x \in D(F) : x^* \in Fx\}.
\end{aligned}$$

2. Characteristic Inequalities of Uniformly Convex Banach Spaces

Let X be a real Banach space with modulus of convexity $\delta_X(\varepsilon)$, $p > 1$ be an arbitrarily real number, and

$$\begin{aligned}\mathscr{A} = \{\phi : R^+ \to R^+ : \phi(0) = 0,\ &\phi(t) \text{ is strictly increasing and there}\\ &\text{is positive constant } K \text{ such that } \phi(t) \geqslant K\,\delta_X(t/2)\}.\end{aligned}$$

We now establish the main result of this section:

THEOREM 1. *The following statements are all equivalent*:

(i) *X is uniformly convex*;

(ii) *there exists a function $\phi_p \in \mathscr{A}$ such that*

$$\langle j_p x - j_p y, x - y \rangle \geqslant (\|x\| \vee \|y\|)^p \phi_p\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right); \tag{2.1}$$

(iii) *there exists a function $\phi_p \in \mathscr{A}$ such that*

$$\|x+y\|^p \geqslant \|x\|^p + p\langle j_p x, j \rangle + \sigma_p(x, y)$$

for any x, $y \in X$, where

$$\sigma_p(x, y) = p \int_0^1 \frac{(\|x+ty\| \vee \|x\|)^p}{t} \phi_p\left(\frac{t\|y\|}{\|x+ty\| \vee \|y\|}\right) dt. \tag{2.2}$$

To prove this theorem we need the following lemmas which are also interesting in their own right.

LEMMA 1. *For every x, $y \in S(X)$ and $t \in (0, 1)$, let $\varepsilon = \|x - y\| \neq 0$. Then*

$$\|\lambda x + \mu t y\| \leqslant \lambda + \mu t - 2(\lambda \wedge \mu) t\, \delta_X(\varepsilon).$$

Proof. The inequality follows trivially when x and y are linearly independent. Suppose that x and y are linearly dependent and denote by E the subspace spanned by the elements x, y, and the zero element. Then the element $\lambda x + \mu t y$ belongs to E. Let z be the intersection of the vector $x - y$ and the ray $\tau(\lambda x + \mu t y)$, $\tau \geqslant 0$, in the subspace E. Then there exist real numbers α and β such that

$$z = \alpha(\lambda x + \mu t y), \qquad \alpha \geqslant 0 \tag{2.3}$$

$$z = \beta x + (1-\beta)(\lambda x + \mu y), \qquad 0 \leqslant \beta \leqslant 1. \tag{2.4}$$

Since x and y are linearly dependent, it follows that

$$\alpha\lambda = \beta + (1-\beta)\lambda$$

$$\alpha\mu t = \mu(1-\beta).$$

Solve these equations and find

$$\alpha = (\lambda + \mu t)^{-1} \qquad \text{and} \qquad \beta = (\lambda + \mu t)^{-1} \lambda(1-t).$$

Consequently, from (2.3) and (2.4), it follows that

$$\begin{aligned} \|\lambda x + \mu t y\| &= \alpha^{-1}\|z\| = (\lambda + \mu t)\, \|\beta x + (1-\beta)(\lambda x + \mu y)\| \\ &\leqslant (\lambda + \mu t)[\beta\|x\| + (1-\beta)\, \|\lambda x + \mu y\|] \\ &= (1-t)\,\lambda + t\,\|\lambda x + \mu y\|. \end{aligned} \tag{2.5}$$

Furthermore, we observe that the function of the form $f(s, w) = s^{-1}(\|x+sw\| - \|x\|)$, $s>0$, is nondecreasing in s and for every fixed s and w in X. Hence the definition of the modulus of convexity implies that

$$\begin{aligned}\|\lambda x+\mu y\| &= 1+\mu[\|x+\mu(y-x)\| - \|x\|]/\mu \\ &= 1+\mu f(\mu, y-x) \leqslant 1+\mu f(\tfrac{1}{2}, y-x) \\ &= 1-2\mu[1-\tfrac{1}{2}\|x+y\|] \leqslant 1-2\mu\,\delta_X(\varepsilon)\end{aligned}$$

whenever $\mu \leqslant 1/2$ and, similarly, that

$$\|\lambda x+\mu y\| \leqslant 1-2\lambda\,\delta_X(\varepsilon) \qquad \text{whenever} \quad \lambda \leqslant \tfrac{1}{2}.$$

Combining these inequalities with (2.5), we have that

$$\begin{aligned}\|\lambda x+\mu t y\| &\leqslant (1-t)\lambda + t[1-2(\mu \wedge \lambda)\,\delta_X(\varepsilon)] \\ &= \lambda+\mu t-2(\mu \wedge \lambda)t\,\delta_X(\varepsilon)\end{aligned}$$

and the proof is complete.

LEMMA 2. *X is uniformly convex if and only if, for each $p \in (1, \infty)$, there exists a strictly increasing function $\delta_p(\lambda, \mu, \cdot)$: $R^+ \to R^+$, $\delta_p(\lambda, \mu, 0) = 0$, such that*

$$\|\lambda x+\mu y\|^p + (\|x\| \vee \|y\|)^p\,\delta_p\left(\lambda, \mu, \frac{\|x-y\|}{\|x\| \vee \|y\|}\right) \leqslant \lambda\,\|x\|^p + \mu\,\|y\|^p \tag{2.6}$$

for every $x, y \in X$.

Proof. When the inequality (2.6) is satisfied, for any $x, y \in S(X)$, $\|x-y\| \geqslant \varepsilon$, we take $\lambda = \mu = 1/2$ in the inequality and find

$$\left\|\frac{x+y}{2}\right\|^p + \delta_p\left(\frac{1}{2}, \frac{1}{2}, \varepsilon\right) \leqslant 1$$

which implies that

$$\delta_X(\varepsilon) \geqslant 1-[1-\delta_p(\tfrac{1}{2}, \tfrac{1}{2}, \varepsilon)]^{i/p} > 0$$

namely, X is uniformly convex. Conversely, suppose that X is uniformly convex. We then construct a function $\delta_p(\lambda, \mu, \cdot)$ so that the inequality (2.6) is fulfilled. For this purpose, we first define a function φ_p: $[0, 1] \times [0, 1] \times R^+$ by

$$\varphi_p(\lambda, \mu, \varepsilon) = \min\{f_p^{(1)}(\lambda, \mu, \varepsilon), f_p^{(2)}(\lambda, \mu, \varepsilon)\}, \tag{2.7}$$

where

$$f_p^{(1)}(\lambda, \mu, \varepsilon) = \lambda + \mu\left(1 - \frac{\varepsilon}{2}\right)^p - \left(1 - \mu\frac{\varepsilon}{2}\right)^p \tag{2.8}$$

$$\begin{aligned} &f_p^{(2)}(\lambda, \mu, \varepsilon) \\ &\quad = \lambda[1 - \mu\lambda^{p-1}/\{\mu^{1/p-1} - [\mu - 2(\mu \wedge \lambda)\,\delta_X(\varepsilon/2)]^{p(p-1)^{-1}}\}^{p-1}]. \end{aligned} \tag{2.9}$$

We show that with the function φ_p so defined the inequality

$$\|\lambda x + \mu y\|^p + \varphi_p(\lambda, \mu, \varepsilon) \leqslant \lambda\|x\|^p + \mu\|y\|^p \tag{2.10}$$

holds for every $x \in S(X)$ and $y \in B(X)$. Indeed, let $t_0 = \|y\|$, $\bar{y} = t_0^{-1}y$, $\varepsilon = \|x - y\|$, and $\bar{\varepsilon} = \|x - \bar{y}\|$. We consider the function g defined by

$$g(t) = \lambda + \mu t^p - [\lambda + \mu t - 2(\mu \wedge \lambda)t\,\delta_X(\bar{\varepsilon})]^p, \qquad 0 \leqslant t \leqslant 1.$$

Form Lemma 1, we then have

$$\begin{aligned} g(t_0) &\leqslant \lambda + \mu t_0^p - \|\lambda x + \mu t_0\,\bar{y}\|^p = \lambda + \mu t_0^p - \|\lambda x + \mu y\|^p \\ &= \lambda\|x\|^p + \mu\|y\|^p - \|\lambda x + \mu y\|^p. \end{aligned} \tag{2.11}$$

We now distinguish two possible cases:

Case I. $t_0 \leqslant 1 - \varepsilon/2$. Then the strictly decreasing monotonicity of the function $(\lambda + \mu t^p) - (\lambda + \mu t)^p$ implies that

$$g(t_0) \geqslant \lambda + \mu t_0^p - (\lambda + \mu t_0)^p \geqslant f_p^{(1)}(\lambda, \mu, \varepsilon).$$

From (2.11), therefore, (2.10) follows directly.

Case II. $t_0 > 1 - \varepsilon/2$. Then $\bar{\varepsilon} = \|x - \bar{y}\| \geqslant \|x - y\| - \|y - \bar{y}\| = \varepsilon - (1 - t_0) \geqslant \varepsilon/2$. By the property ($\delta 2$), it follows that

$$g(t_0) \geqslant \lambda + \mu t_0^p - \left[\lambda + \mu t_0 - 2(\mu \wedge \lambda)t_0\,\delta_X\left(\frac{\varepsilon}{2}\right)\right]^p =: h(t_0).$$

We find that the function $h(t_0)$, $0 \leqslant t_0 \leqslant 1$, attains its minimum at

$$t_* = \lambda\left[\mu - 2(\mu \wedge \lambda)\,\delta_X\left(\frac{\varepsilon}{2}\right)\right]^{(p-1)^{-1}} \Bigg/ \left\{\mu^{(p-1)^{-1}} - \left[\mu - 2(\mu \wedge \lambda)\,\delta_X\left(\frac{\varepsilon}{2}\right)\right]^{p/(p-1)}\right\}$$

which is the unique solution of the equation

$$h'(t_0)=p\left\{\mu t_0^{p-1}-\left[\lambda+\mu t_0-2(\mu\wedge\lambda)t_0\,\delta_X\left(\frac{\varepsilon}{2}\right)\right]^{p-1}\right.$$
$$\left.\times\left[\mu-2(\mu\wedge\lambda)\,\delta_X\left(\frac{\varepsilon}{2}\right)\right]\right\}=0$$

or equivalently

$$\left[\lambda+\mu t_0-2(\mu\wedge\lambda)t_0\,\delta_X\left(\frac{\varepsilon}{2}\right)\right]^p$$
$$=\frac{\mu t_0^{p-1}[\lambda+\mu t_0-2(\mu\wedge\lambda)t_0\,\delta_X(\varepsilon/2)]}{\mu-2(\mu\wedge\lambda)\,\delta_X(\varepsilon/2)}.$$

Hence

$$\inf_{0\leqslant t_0\leqslant 1}h(t_0)=h(t_*)=\lambda+\mu t_*^p-\frac{\mu t_*^{p-1}[\lambda+\mu t_*-2(\mu\wedge\lambda)t_*\,\delta_X(\varepsilon/2)]}{\mu-2(\mu\wedge\lambda)\,\delta_X(\varepsilon/2)}$$
$$=\lambda-\lambda\mu t_*^{p-1}[\mu-2(\mu\wedge\lambda)\,\delta_X(\varepsilon/2)]^{-1}=f_p^{(2)}(\lambda,\mu,\varepsilon).$$

Consequently, (2.10) follows.

It is easy to see that the function $\delta_p(\lambda,\mu,\varepsilon)$ is strictly increasing in ε. Therefore, (2.10) implies that the inequality (2.6) is fulfilled for

$$\delta_p(\lambda,\mu,\varepsilon)=\min\{\phi_p(\lambda,\mu,\varepsilon),\phi_p(\mu,\lambda,\varepsilon)\}. \tag{2.12}$$

With this, the proof is completed.

LEMMA 3. *Let $\delta_p(\lambda,\mu,\varepsilon)$ be given as in* (2.12). *Then*

$$\limsup_{\mu\to 0}\delta_p(\lambda,\mu,\varepsilon)/\mu+\limsup_{\lambda\to 0}\delta_p(\lambda,\mu,\varepsilon)/\lambda\geqslant K_p\,\delta_X\left(\frac{\varepsilon}{2}\right),$$

where K_p is a positive constant defined by

$$K_p=4(2+\sqrt{3})\min\left\{\frac{1}{2}p(p-1)\wedge 1,\left(\frac{1}{2}p\wedge 1\right)(p-1),\right.$$
$$\left.(p-1)[1-(\sqrt{3}-1)^{p(p-1)^{-1}}],1-\left[1+\frac{(2-\sqrt{3})p}{p-1}\right]^{1-p}\right\}. \tag{2.13}$$

Proof. Since $\delta_p(\lambda,\mu,\varepsilon)$ is symmetric with respect to λ and μ, it is seen that

$$\limsup_{\mu\to 0}\delta_p(\lambda,\mu,\varepsilon)/\mu+\limsup_{\lambda\to 0}\delta_p(\lambda,\mu,\varepsilon)/\lambda$$
$$\geqslant 2\min\{\lim_{\mu\to 0}f_p^{(i)}(\lambda,\mu,\varepsilon)/\mu,\ \lim_{\lambda\to 0}f_p^{(i)}(\lambda,\mu,\varepsilon)/\lambda : i=1,2\}.$$

From (2.8) and (2.9), we calculate

$$\lim_{\mu\to 0} f_p^{(1)}(\lambda, \mu, \varepsilon)/\mu = \left(1-\frac{\varepsilon}{2}\right)^p - \left(1-\frac{p}{2}\varepsilon\right) =: g_1(\varepsilon)$$

$$\lim_{\lambda\to 0} f_p^{(1)}(\lambda, \mu, \varepsilon)/\lambda = 1-\left[1+\frac{p-1}{2}\varepsilon\right]\left(1-\frac{\varepsilon}{2}\right)^{p-1} =: g_2(\varepsilon)$$

$$\lim_{\mu\to 0} f_p^{(2)}(\lambda, \mu, \varepsilon)/\mu = (p-1)\left[1-\left(1-2\delta_X\left(\frac{\varepsilon}{2}\right)\right)^{p(p-1)^{-1}}\right] = h_1(\delta)$$

$$\lim_{\lambda\to 0} f_p^{(2)}(\lambda, \mu, \varepsilon)/\lambda = 1-\left[1+\frac{2p}{p-1}\delta_X\left(\frac{\varepsilon}{2}\right)\right]^{1-p} =: h_2(\delta),$$

where $\delta = \delta_X(\varepsilon/2)$. Then, using the fact that the property ($\delta 3$) implies

$$\delta = \delta_X\left(\frac{\varepsilon}{2}\right) \leqslant \delta_H\left(\frac{\varepsilon}{2}\right) = 1-\left[1-\frac{1}{4}\left(\frac{\varepsilon}{2}\right)^2\right]^{1/2}$$

which yields

$$\left(\frac{\varepsilon}{2}\right)^2 \geqslant 4\delta(2-\delta) \geqslant (4+2\sqrt{3})\,\delta \quad \text{and} \quad \delta \leqslant \frac{1}{2}(2-\sqrt{3}) \quad \text{for any } \varepsilon\in(0,2],$$

we consequently obtain

$$g_1(\varepsilon)/\delta \geqslant (4+2\sqrt{3}) \inf_{0\leqslant\varepsilon\leqslant 2} g_1(\varepsilon)\Big/\left(\frac{\varepsilon}{2}\right)^2 = (4+2\sqrt{3})(p-1)\left(\frac{p}{2}\wedge 1\right)$$

$$g_2(\varepsilon)/\delta \geqslant (4+2\sqrt{3}) \inf_{0\leqslant\varepsilon\leqslant 2} g_2(\varepsilon)\Big/\left(\frac{\varepsilon}{2}\right)^2 = (4+2\sqrt{3})\left(\frac{p(p-1)}{2}\wedge 1\right)$$

$$\begin{aligned} h_1(\delta)/\delta &\geqslant \inf_{0\leqslant\delta\leqslant(1/2)(2-\sqrt{3})} h_1(\delta)/\delta \\ &= (4+2\sqrt{3})(p-1)\,[1-(\sqrt{3}-1)^{p(p-1)^{-1}}] \end{aligned}$$

$$h_2(\delta)/\delta \geqslant \inf_{0\leqslant\delta\leqslant(1/2)(2-\sqrt{3})} h_2(\delta)/\delta = (4+2\sqrt{3})\left[1-\left(1+\frac{(2-\sqrt{3})p}{p-1}\right)^{1-p}\right].$$

From these inequalities, Lemma 3 readily follows.

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Since the duality mapping J_p is positively homogeneous from (J4), we can assume $\|x\| \vee \|y\| = 1$ without loss of generality. Thus, applying Lemma 2, we have

$$\|\lambda x + \mu y\|^p + \delta_p(\lambda, \mu, \|x-y\|) \leqslant \lambda\|x\|^p + \mu\|y\|^p \qquad \forall x, y\in X$$

which in particular implies

$$\|x+\mu(y-x)\|^p - \|x\|^p \leqslant \mu(\|y\|^p - \|x\|^p) - \delta_p(\lambda, \mu, \|x-y\|) \quad (2.14)$$

$$\|y+\lambda(x-y)\|^p - \|y\|^p \leqslant \lambda(\|x\|^p - \|y\|^p) - \delta_p(\lambda, \mu, \|x-y\|). \quad (2.15)$$

Since X is uniformly convex, X is also reflexive [16, Proposition 1.e.3]. It follows from the property (J5) that $J_p x = \partial((1/p)\|x\|^p)$ for every $x \in X$. Accordingly, the inequalities (2.14) and (2.15) further imply

$$p\langle j_p x, y-x\rangle \leqslant \|y\|^p - \|x\|^p - \limsup_{\mu \to 0} \delta_p(\lambda, \mu, \|x-y\|)/\mu$$

and

$$p\langle j_p y, x-y\rangle \leqslant \|x\|^p - \|y\|^p - \limsup_{\lambda \to 0} \delta_p(\lambda, \mu, \|x-y\|)/\lambda.$$

Combining these two inequalities then yeilds

$$\langle j_p x - j_p y. x-y\rangle \geqslant \frac{1}{p}\Big[\limsup_{\mu \to 0} \delta_p(\lambda, \mu, \|x-y\|)/\mu + \limsup_{\lambda \to 0} \delta_p(\lambda, \mu, \|x-y\|)/\lambda\Big].$$

From Lemma 3, we thus obtain

$$\langle j_p x - j_p y, x-y\rangle \geqslant K_p\, \delta_X(\|x-y\|/2),$$

that is, (ii) is established for $\phi_p(t) = K_p\, \delta_X(t/2)$.

(ii) $\Rightarrow$ (iii). Let $\Phi(t) = (1/p)\|x+ty\|^p$ and $0 = t_0 < t_1 < \cdots < t_N = 1$ an arbitrary partition of the interval [0, 1]. By the property (J5), we have

$$\frac{1}{p}\|x+y\|^p - \frac{1}{p}\|x\|^p = \Phi(1) - \Phi(0) = \sum_{n=0}^{N-1} (\Phi(t_{k+1}) - \Phi(t_k))$$
$$\geqslant \sum_{k=0}^{N-1} \langle j_p(x+t_k y), y\rangle (t_{k+1} - t_k).$$

It follows from (ii) that

$$\|x+y\|^p - \|x\|^p - p\langle j_p, y\rangle$$
$$\geqslant p \sum_{k=0}^{n-1} \langle j_p(x+t_k y) - j_p x, y\rangle (t_{k+1} - t_k)$$
$$\geqslant p \sum_{k=0}^{N-1} \frac{(\|x+t_k y\| \vee \|x\|)^p}{t_k}\, \phi_p\left(\frac{t_k\|y\|}{\|x+t_k y\| \vee \|x\|}\right)(t_{k+1} - t_k).$$

Since ϕ_p is strictly increasing and $\|x+ty\| \vee \|x\|$ is continuous in t, the function

$$S(t)=\frac{(\|x+ty\| \vee \|x\|)^p}{t}\,\phi_p\left(\frac{t\,\|y\|}{\|x+ty\| \vee \|x\|}\right)$$

is integrable (in the sense of Riemann). We therefore have

$$\begin{aligned}\|x+y\|^p-\|x\|^p-p\langle j_p x, y\rangle &\geqslant \lim_{N\to\infty} p\sum_{k=0}^{N-1} S(t_k)(t_{k+1}-t_k)\\ &= p\int_0^1 S(t)\,dt\end{aligned}$$

which establishes (iii).

(iii) $\Rightarrow$ (i). For any $x, y\in S(X)$, the inequality in (iii) implies that

$$\begin{aligned}0&=\|x+(y-x)\|^2-\|x\|^2\\ &\geqslant 2\langle jx, y-x\rangle+2\int_0^1 \frac{(\|x+t(y-x)\| \vee \|x\|)^2}{t}\\ &\quad\times\phi_2\left(\frac{t\,\|y-x\|}{\|x+t(y-x)\| \vee \|x\|}\right)dt\\ &\geqslant 2\langle jx, y-x\rangle+2\int_0^1 t^{-1}\phi_2(t\,\|y-x\|)\,dt\\ &=2\langle jx, y-x\rangle+2\int_0^{\|x-y\|}\phi_2(\eta)/\eta\,d\eta.\end{aligned}$$

Hence, from the increasing monotonicity of ϕ_2, we find

$$1-\langle jx, y\rangle \geqslant \int_0^{\varepsilon}\phi_2(\eta)/\eta\,d\eta \qquad \text{whenever} \quad \|x-y\|\geqslant\varepsilon.$$

This, combined with the property ($\delta 2$) and the fact $\phi_p(\eta)\in\mathscr{A}$ implies that there exists a strictly increasing function $Y(\varepsilon)$ (for example, $Y(\varepsilon)=K_2\int_0^{\varepsilon}\delta_X(\eta/2)/\eta\,d\eta$) such that $\langle jx, y\rangle\geqslant 1-Y(\varepsilon)$ for any $x, y\in S(X)$, $\|x-y\|\geqslant\varepsilon$. Consequently, [20, Lemma 2.4] implies that X is uniformly convex. This completes the proof.

Remark 1. It is seen from the proof of Theorem 1 that we have the specific inequalities

$$\langle j_p x-j_p y, x-y\rangle\geqslant K_p(\|x\| \vee \|y\|)^p\,\delta_X\left(\frac{\|x-y\|}{2(\|x\| \vee \|y\|)}\right) \tag{2.16}$$

$$\|x+y\|^p\geqslant\|x\|^p+p\langle j_p x, y\rangle+\sigma_p(x, y)$$

with

$$\sigma_p(x, y) = pK_p \int_0^1 \frac{\|x+ty\| \vee \|x\|}{t} \delta_X\left(\frac{t\|y\|}{2(\|x+ty\| \vee \|x\|)}\right) dt \qquad (2.17)$$

which hold for every x, y in a uniformly convex Banach space, where K_p is as defined in Lemma 3. In particular, when X has modulus of convexity of power type m ($m > 1$), (2.16) and (2.17) imply that for some positive constant K

$$\langle j_m x - j_m y, x - y \rangle \geqslant K\|x - y\|^m \qquad (2.16)'$$

$$\|x + y\|^m \geqslant \|x\|^m + m\langle j_m x, y\rangle + K\|y\|^m. \qquad (2.17)'$$

These inequalities, together with (3.2) and (3.8), generalise the L^p ($1 < p < \infty$) characteristic inequalities in [24], which in turn extend the well-known polarisation identity in Hilbert spaces. We notice that the inequalities (2.16)′ and (2.17)′ are homogeneous in the sense that all terms in the expression have the same power ($\langle j_m x, y\rangle$ is naturally regarded as having power m, because $\langle j_m x, x\rangle = \|x\|^m$). This favourable feature plays an important role similar to the L^p characteristic inequalities (see, e.g., [4, 5, 11, 17, 21, 24]). This also explains the advantage of using the generalised duality mappings J_p rather than the normalised one. For example, (2.16) and (2.16)′ extend and improve the results in [1, Theorem 1; 22, Proposition 2.11].

Remark 2. Recall that a possibly multi-valued mapping A from X into X^* is said to be strictly monotone (respectively, strongly monotone) if, $\langle f - g, x - y\rangle > 0$ for every $[x, f]$, $[y, g] \in G(A)$ and $x \neq y$ (respectively, there exists a strictly increasing function $\phi: R \to R^+$, $\phi(0) = 0$, such that $\langle f - g, x - y\rangle \geqslant \phi(\|x - y\|)\|x - y\|$ for any $[x, f]$, $[y, g] \in G(A)$). Furthermore, a strongly monotone operator is said to be uniformly monotone if $\phi(t) = Kt$ for some positive constant K. It is shown in [23, Lemma 2.12; 25, Theorem 3.4.2] that the strict monotonicity of the normalised duality mapping J on $B(X)$ characterises the strict convexity of X (where X is said to be strictly convex if any x, $y \in S(X)$, $x \neq y$, imply that $(1/2)\|x + y\| < 1$). We observe that (2.16) implies

$$\langle jx - jy, x - y\rangle \geqslant K_2(\|x\| \vee \|y\|)^2 \delta_X\left(\frac{\|x - y\|}{2(\|x\| \vee \|y\|)}\right).$$

By using the fact that the function $\delta_X(\varepsilon) = \sup\{\varepsilon\tau/2 - \rho_{X^*}(\tau) : \tau \geqslant 0\}$ is the maximal convex function majorised by $\delta_X(\varepsilon)$ and $\tilde{\delta}_X(\varepsilon)/\varepsilon^2$ is equivalent to an increasing function (see [16, Proposition 1.e.6]), we get (noticing that $\|x\| \vee \|y\| \leqslant 1$)

$$\langle jx - jy, x - y \rangle \geqslant K_2 \|x - y\|^2 \tilde{\delta}_X \left(\frac{\|x - y\|}{2(\|x\| \vee \|y\|)} \right) \Big/ \left(\frac{\|x - y\|}{\|x\| \vee \|y\|} \right)^2$$

$$\geqslant c K_2 \tilde{\delta}_X \left(\frac{\|x - y\|}{2} \right) \qquad \forall x, y \in B(X) \qquad (2.18)$$

here c is the positive constant such that $\tilde{\delta}_X(t_1) t_1^2 \leqslant c \tilde{\delta}_X(t_2)/t_2^2$ whenever $0 \leqslant t_1 \leqslant t_2$. Since $\tilde{\delta}_X(t)$ is convex, the function $\tilde{\phi}(t) = cK_2 \tilde{\delta}_X(t/2)/t$ is positive and nondecreasing. Thus, (2.18) implies that

$$\langle jx - jy, x - y \rangle \geqslant \phi(\|x - y\|) \|x - y\| \qquad \forall x, y \in B(X),$$

where $\phi(t) = (1 + t)^{-1} t \tilde{\phi}(t)$ is a strictly increasing function from R^+ into R^+ such that $\phi(0) = 0$. As a result, Theorem 1 then says that a Banach space X is uniformly convex if and only if J is strongly monotone in $B(X)$, which provides a uniform version of the result given in [23, Lemma 2.12]. Also, from (2.16) it is seen that a Banach space X has modulus of convexity of power tye 2 if and only if J is uniformly monotone, which is a slight extension of [22, Proposition 2.11]. Corresponding to the well-known fact that the continuity of J characterizes the smoothness of X, we now can conclude that the monotonicity of J characterizes the convexity of X. This presents a link between the theory of monotone operators and geometry of Banach spaces, which is very useful in tackling problems in these two areas (see [22, 27, 28], for instance).

Remark 3. We emphasize that the inequality developed in Lemma 2 also is of practical and theoretical importance although this is not our main concern here. Some interesting applications of this type of inequalities do exist, see, for example, [10, 11, 13, 15, 18, 24]. However, we remark that not only does the inequality in Lemma 2 generalise that given in [18, Lemma 2.1] to general uniformly convex Banach spaces, but also the method used for proving the inequality is completely elementary and also constructive (in particular, it is not necessary to apply Martingle theory like Prus and Smarzewski in [18]).

3. Characteristic Inequalities of Uniformly Smooth Banach Spaces

Let X be a real Banach space with modulus of smoothness $\rho_X(\tau)$ and

$$\mathscr{F} = \{\varphi : R^+ \to R^+ : \varphi(0) = 0, \varphi \text{ convex, nondecreasing and there exists a constant } K > 0 \text{ such that } \varphi(\tau) \leqslant K \rho_X(\tau)\}.$$

In this section we prove the following duality results of Theorem 1:

THEOREM 2. *For any $1 < p < \infty$, the following statements are equivalent:*

(i) *X is uniformly smooth;*

(ii) *J_p is single-valued and there is $\varphi_p \in \mathscr{F}$ such that*

$$\|J_p x - J_p y\| \leqslant (\|x\| \vee \|y\|)^{p-1} \bar{\varphi}_p\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right) \qquad \forall x, y \in X, \tag{3.1}$$

where $\bar{\varphi}_p(t) = \varphi_p(t)/t$;

(iii) *there exists a $\varphi_p \in \mathscr{F}$ such that*

$$\|x+y\|^p \leqslant \|x\|^p + p\langle J_p x, y\rangle + \sigma_p(x, y) \qquad \forall x, y \in X, \tag{3.2}$$

where

$$\sigma_p(x, y) = p \int_0^1 \frac{(\|x+ty\| \vee \|x\|)}{t} \varphi_p\left(\frac{t\|y\|}{\|x+ty\| \vee \|x\|}\right) dt;$$

(iv) *there exists a $j_p \in J_p$ such that*

$$\|x+y\|^p \leqslant \|x\|^p + p\langle j_p x, y\rangle + \sigma_p(x, y) \qquad \forall x, y \in X. \tag{3.3}$$

To prove this theorem, we need the following lemma.

LEMMA 4. *Let X be a uniformly smooth Banach space, $J_p: X \to X^*$ and $J_q^*: X^* \to X$ be the duality mappings with gauge function $\phi(t) = t^{p-1}$ and $\phi(s) = s^{q-1}$, respectively. Then $J_p^{-1} = J_q^*$.*

Proof. The uniform smoothness of X implies that X is reflexive and that X^* is uniformly convex and reflexive [16, Proposition 1.e.3]). Therefore, from the properties (J2) and (J3), J_p is single-valued and surjective. This implies that the inverse $J_p^{-1}: X^* = D(J_p^{-1}) \to X = X^{**}$ exists and is given by

$$J_p^{-1} x^* = \{x \in X: j_p x = x^*\} \qquad \forall x^* \in X^*.$$

On the other hand, let $\Phi(x) = (1/p)\|x\|^p$ for every x in X. It is easy to show that Φ is continuous, convex, and that its conjugate is given by $\Phi^*(x) = (1/q)\|x^*\|^q$ for every $x^* \in X^*$. From the property (J5), it follows that

$$J_p x = \partial\Phi(x), \quad \forall x \in X; \qquad J^* x_q^* = \partial\Phi^*(x^*), \quad \forall x^* \in X^*.$$

By using the fact that $x^* \in \partial\Phi(x)$ if and only if $x \in \partial\Phi^*(x^*)$ [2, p. 203], we conclude that $J_p^{-1}x^* = J_q^* x^*$ for every $x^* \in X^*$. This completes the proof.

Proof of Theorem 2. (i) $\Rightarrow$ (ii). It is known [16] that J_p is single-valued and that X^* is uniformly convex. Let $\delta_X^*(\varepsilon)$ be the modulus of convexity of X^*. Then, by Theorem 1(ii), the inequality

$$\langle J_q^* x^* - j_q^* y^*, x^* - y^*\rangle \geqslant K_q(\|x^*\| \vee \|y^*\|)^q \delta_{X^*}\left(\frac{\|x^*-y^*\|}{2(\|x^*\| \vee \|y^*\|)}\right)$$

holds for every $x^*, y^* \in X^*$, where j_q^* is an arbitrary selection of the duality mapping $J_q^*: X^* \to 2^X$ and K_q is the constant given in (2.13). Hence, we have

$$\|j_q^* x^* - j_q^* y^*\| \geqslant \frac{1}{2} K_q(\|x^*\| \vee \|y^*\|)^{q-1} \bar{\delta}_{X^*}\left(\frac{\|x^* - y^*\|}{2(\|x^*\| \vee \|y^*\|)}\right),$$

where

$$\bar{\delta}_{X^*}(\varepsilon) = \delta_{X^*}(\varepsilon)/\varepsilon \qquad \forall \in \varepsilon(0, 2).$$

In particular, putting $x^* = J_p x$ and $y^* = J_p y$ in this inequality, which is always possible by Lemma 4, implies (by Lemma 4) that

$$\|x - y\| \geqslant \frac{1}{2} K_q(\|x\| \vee \|y\|) \bar{\delta}_{X^*}\left(\frac{\|J_p x - J_p y\|}{2(\|x\| \vee \|y\|)^{p-1}}\right)$$

and in turn

$$\frac{2\|x - y\|}{K_q(\|x\| \vee \|y\|)} \geqslant \bar{\delta}_{X^*}\left(\frac{\|J_p x - J_p y\|}{2(\|x\| \vee \|y\|)^{p-1}}\right).$$

By the property ($\rho 3$) of the modulus of smoothness, we then obtain

$$\bar{\rho}_X\left(\frac{8\|x - y\|}{K_q(\|x\| \vee \|y\|)}\right) \geqslant \bar{\rho}_X\left(4\bar{\delta}_{X^*}\left(\frac{J_p x - J_p y\|}{2(\|x\| \vee \|y\|)^{p-1}}\right)\right),$$

where

$$\bar{\rho}_X(\tau) = \rho_X(\tau)/\tau \qquad \forall \tau > 0.$$

On the other hand, by the property ($\rho 5$), we have

$$\rho_X(\tau) \geqslant \tfrac{1}{2}\tau\varepsilon - \delta_{X^*}(\varepsilon) \qquad \forall \varepsilon \in (0, 2), \tau > 0.$$

This implies in particular that

$$\rho_X\left(\frac{4\delta_{X^*}(\varepsilon)}{\varepsilon}\right) \geqslant \frac{\varepsilon}{2} \frac{4\delta_{X^*}(\varepsilon)}{\varepsilon} - \delta_{X^*}(\varepsilon) = \delta_{X^*}(\varepsilon) \qquad \forall \varepsilon \in [0, 2].$$

Hence

$$\rho_X(4\bar{\delta}_{X^*}(\varepsilon)) \geqslant \tfrac{1}{4}\varepsilon(4\bar{\delta}_{X^*}(\varepsilon))$$

namely, $\bar{\rho}_X(4\delta_{X^*}(\varepsilon)) \geqslant (1/4)\varepsilon$. From (3.4), it follows that

$$\bar{\rho}_X\left(\frac{8\,\|x-y\|}{K_q(\|x\| \vee \|y\|)}\right) \geqslant \bar{\rho}_X\left(4\delta_{X^*}\left(\frac{\|J_p x - J_p y\|}{2(\|x\| \vee \|y\|)^{p-1}}\right)\right) \geqslant \frac{1}{8}\,\frac{\|J_p x - J_p y\|}{(\|x\| \vee \|y\|)^{p-1}}.$$

That is,

$$\begin{aligned}\|J_p x - J_p y\| &\leqslant 8(\|x\| \vee \|y\|)^{p-1}\bar{\rho}_X\left(\frac{8\,\|x-y\|}{K_q(\|x\| \vee \|y\|)}\right)\\ &= K_q\,\frac{(\|x\| \vee \|y\|)^p}{\|x-y\|}\,\rho_X\left(\frac{8\,\|x-y\|}{K_q(\|x\| \vee \|y\|)}\right). \qquad (3.5)\end{aligned}$$

We now consider the following cases:

Case I. $8/K_q \leqslant 1$. Since $\rho_X(\cdot)$ is convex (the property ($\rho 1$)), we have

$$\begin{aligned}\|J_p x - J_p y\| &\leqslant 8\,\frac{(\|x\| \vee \|y\|)^p}{\|x-y\|}\,\rho_X\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right)\\ &= 8(\|x\| \vee \|y\|)^{p-1}\bar{\rho}_X\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right).\end{aligned}$$

Case II. $8/K_q > 1$. Making use of the property ($\rho 6$), we obtain

$$\begin{aligned}\|J_p x - J_p y\| &\leqslant \frac{8^2(\|x\| \vee \|y)^{p-2}\|x-y\|}{K_q}\\ &\quad\times \rho_X\left(\frac{8\,\|x-y\|}{K_q\|x\| \vee \|y\|}\right)\Big/\left(\frac{8\,\|x-y\|}{K_q\|x\| \vee \|y\|}\right)^2\\ &\leqslant c\,\frac{8^2(\|x\| \vee \|y\|)^{p-2}\|x-y\|}{K_q}\,\rho_X\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right)\Big/\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right)^2\\ &= \frac{8^2 c(\|x\| \vee \|y\|)^p}{K_q\|x-y\|}\,\rho_X\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right)\\ &= 8^2 c K_q^{-1}(\|x\| \vee \|y\|)^{p-1}\bar{\rho}_X\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right).\end{aligned}$$

Consequently, (3.1) follows by taking $\varphi_p(t) = \max\{8, 8^2 c K_q^{-1}\}\,\rho_X(t)$.

(ii) $\Rightarrow$ (iii). Since J_p is single-valued, so is J. This implies that X is smooth (that is, the norm of X is Gateaux differentiable) and that J_p is continuous from the norm topology of X into weak-star topology of X^* [7]. Thus, the function $\Phi(t) = \|x + ty\|^p$, $t \in (0, 1]$, is continuously differentiable with its derivative $\Phi'(t) = p\langle J_p(x+ty), y\rangle$. By the Newton–Leibnitz formula and (3.1), it then follows that

$$\begin{aligned}
&\|x+y\|^p - \|x\|^p - p\langle J_p x, y\rangle \\
&\quad = \Phi(1) - \Phi(0) - \Phi'(0) \\
&\quad = p\int_0^1 \langle J_p(x+ty) - J_p x, y\rangle\, dt \\
&\quad \leqslant p\int_0^1 \|J_p(x+ty) - J_p x\|\, \|y\|\, dt \\
&\quad \leqslant p\|y\| \int_0^1 (\|x+ty\| \vee \|x\|)^{p-1}\, \bar{\varphi}_p\left(\frac{t\|y\|}{\|x+ty\| \vee \|x\|}\right) dt \\
&\quad = p\int_0^1 \frac{(\|x+ty\| \vee \|x\|)^p}{t}\, \varphi_p\left(\frac{t\|y\|}{\|x+ty\| \vee \|x\|}\right) dt
\end{aligned}$$

which establishes the inequality (3.2).

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). For any $x \in S(X)$ and $\|y\| \leqslant \tau$, the inequality (3.3) implies

$$\|x+y\|^p \leqslant \|x\|^p + p\langle j_p x, y\rangle + \sigma_p(x, y) = 1 + \sigma_p(x, y) + p\langle j_p x, y\rangle$$

and

$$\|x-y\|^p \leqslant \|x\|^p - p\langle j_p x, y\rangle + \sigma_p(x, -y) = 1 + \sigma_p(x, -y) - p\langle j_p x, y\rangle$$

Let $\alpha = 1 + \max\{\sigma_p(x, y), \sigma_p(x, -y)\}$. It then follows that

$$\begin{aligned}
\|x+y\| + \|x-y\| &\leqslant (\alpha + p\langle j_p x, y\rangle)^{1/p} + (\alpha - p\langle j_p x, y\rangle)^{1/p} \\
&= \alpha^{1/p}\left[\left(1 + \frac{p}{\alpha}\langle j_p x, y\rangle\right)^{1/p} + \left(1 - \frac{p}{\alpha}\langle j_p x, y\rangle\right)^{1/p}\right].
\end{aligned}$$

Since $|(p/\alpha)\langle j_p x, y\rangle| \leqslant [1 + \max\{\sigma_p(x, y),\ \sigma_p(x, -y)\}]^{-1}\ p\tau \to 0$ as $\tau \to 0$, whenever τ is small enough we have

$$\begin{aligned}
\left(1 + \frac{p}{\alpha}\langle j_p x, y\rangle\right)^{1/p} &= \sum_{n=0}^{\infty} \binom{1/p}{n} \left(\frac{p}{\alpha}\langle j_p x, y\rangle\right)^n \\
\left(1 - \frac{p}{\alpha}\langle j_p x, y\rangle\right)^{1/p} &= \sum_{n=0}^{\infty} \binom{1/p}{n} \left(-\frac{p}{\alpha}\langle j_p x, y\rangle\right)^n,
\end{aligned}$$

where

$$\binom{1/p}{n} = \frac{(1/p)(1/p-1)\cdots(1/p-n+1)}{n!}.$$

In consequence, whenever τ is small enough we obtain

$$\|x+y\|+\|x-y\| \leqslant 2\alpha^{1/p}\left[\sum_{n=0}^{\infty}\binom{1/p}{2n}\left(\frac{p}{\alpha}\langle J_p x, y\rangle\right)^{2n}\right]$$
$$\leqslant 2\alpha^{1/p} \leqslant 2\alpha \leqslant 2(1+\sigma(\tau)),$$

where

$$\sigma(\tau)=\sup\{\sigma_p(x, y), \sigma_p(x, -y) : x \in S(X), \|y\| \leqslant \tau\}$$

(notice that $\binom{1/p}{2n} \leqslant 0$ for any $n \geqslant 1$, and $\alpha > 1$). By the definition of modulus of smoothness of X, it then follows that

$$\rho_X(\tau) \leqslant \sigma(\tau) \qquad \forall \tau > 0.$$

From the expression of $\sigma_p(x, y)$, it is easy to see that $\sigma(\tau)/\tau \to 0$ as $\tau \to 0$. Consequently, $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$, namely, X is uniformly smooth. With this, the proof of Theorem 2 is complete.

Remark 4. From the proof of Theorem 2, it is seen that the inequalities (3.1)–(3.2) can be rewritten in the form

$$\|J_p x - J_p y\| \leqslant L(\|x\| \vee \|y\|)^{p-1} \bar{\rho}_x\left(\frac{\|x-y\|}{\|x\| \vee \|y\|}\right) \tag{3.1$'$}$$

$$\|x+y\|^p \leqslant \|x\|^p + p\langle J_p x, y\rangle + \sigma_p(x, y) \tag{3.2$'$}$$

with

$$\sigma_p(x, y) = pL\int_0^1 \frac{(\|x+ty\| \vee \|x\|)^p}{t}\,\rho_X\left(\frac{t\|y\|}{\|x+ty\| \vee \|x\|}\right) dt,$$

where the constant $L=\max\{8, 64cK_q^{-1}\}$, K_q, and c are defined respectively by (2.13) and by

$$c=\frac{4\tau_0}{\rho_H(\tau_0)}\prod_{j=1}^{\infty}\left(1+\frac{15\tau_0}{4\times 2^j}\right) \quad \text{with} \quad \tau_0=\frac{\sqrt{339}-18}{30} \tag{3.6}$$

(cf. the proof of [16, Proposition 1.e.5]).

Remark 5. The inequalities (3.1)$'$ and (3.2)$'$ imply that when X has modulus of smoothness s ($s > 1$) there exists a positive constant L_1 such that

$$\|J_s x - J_s y\| \leqslant L_1\|x-y\|^{s-1} \tag{3.7}$$

$$\|x+y\|^s \leqslant \|x\|^s + s\langle J_s x, y\rangle + L_1\|y\|^s. \tag{3.8}$$

Moreover, from the implication (iv)$\Rightarrow$(i) in the proof Theorem 2, it is easy to see that $\rho_X(\tau) \leqslant (1/2)L_1\tau^s$ when the inequality (3.8) is satisfied. Accordingly, we know that the following statements are equivalent:

(i) X has modulus of smoothness of power type s $(s>1)$;

(ii) J_s is Hölder continuous with order of continuity $s-1$;

(iii) the inequality (3.8) holds for every $x, y \in X$.

Remark 6. Recall that the modulus of continuity of J_p is defined by

$$W_p(t)=\sup\{\|J_p x - J_p y\| : \|x-y\| \leqslant t\} \qquad \forall t>0.$$

Hence (3.7) implies that

(a) $W_s(t) \leqslant L_1 t^{s-1}$, whenever X has modulus of smoothness of power tye s $(s>1)$.

More generally, from (3.1) and by a similar argument to that following (3.5) in the proof of Theorem 2, we see that

(b) $W_2(t) \leqslant L(t)\, \bar{\rho}_X(t)$, where $L(t)=L\max\{2t, c\}$ is bounded on every bounded interval of R^1.

As stated in Remark 5 and the property (J2) of duality mapping, these imply that a Banach space is uniformly smooth (respectively, has modulus of smoothness of power type s $(s>1)$) if and only if the duality mapping J is uniformly continuous on every bounded subset of X (respectively, J_s is Hölder continuous). Therefore, Theorem 2 not only clarifies the quantitative relation between smoothness of X and continuity of duality mapping J_p, but also provides a direct and constructive proof for the property (J2) of duality mappings. In particular, Remark 5(i)–(ii) strengthens the property (J2).

Remark 7. The inequalities (3.1)′ and (3.7) improve that offered by Al'ber and Notik [1, Theorem 1] in the sense that here an estimate on the modulus of continuity of J_p, rather than the semi-inner product $\langle Jx - Jy, x-y\rangle$, is given. Also, the inequalities (3.2) and (3.8) generalise and improve those developed by Reich [19] and Liu [17] in the sense that the term $\sigma_p(x, y)$ here is explicitly specified by means of the modulus of smoothness of X and, what is more, it is shown here that these types of inequalities characterise the uniform smoothness of X. For applications of the inequalities (3.1)′ and (3.2)′, see, for example, [21, 27, 28].

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