POSTPROCESSING FOURIER GALERKIN METHOD FOR THE NAVIER–STOKES EQUATIONS

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Abstract. A full discrete two-level postprocessing Fourier Galerkin scheme for the unsteady Navier–Stokes equations with a periodic boundary condition is proposed in this paper. By defining a new projection, the interaction between the large and small eddies is reflected by the associated space splitting to some extent. Therefore, a weakly coupled system of the large and small eddies is obtained. Stability and error estimates for the weakly coupled two-level scheme are established in the paper. The proposed scheme is an effective algorithm because nonlinearity is treated only in the coarse-level subspace by solving the coarse-level standard Galerkin equation, and the matrix of the linear algebraic equations arising in each fine-level time stepping is a sparse matrix, especially for large scale computations. As a result, the new scheme saves a lot of CPU time and memory in comparison with the standard Fourier Galerkin method for deriving an approximation of prescribed accuracy.

Key words. two-level method, postprocessing, Fourier Galerkin method, Navier–Stokes equations

AMS subject classifications. 65M70, 65M12, 76D05

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1. Introduction. The purpose of this paper is to study the properties of a two-level postprocessing scheme for the standard Fourier Galerkin method (SGM).

We consider the functional form of the incompressible Navier–Stokes equations

\[
\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u(0) = u_0,
\]

defined in \( \Omega = [0, 2\pi]^d, \ d = 2, 3, \) with a periodic boundary condition. Here \( A \) is the Stokes operator, \( B \) the projection of the nonlinearity on the space of divergence-free functions, \( f \) the density of external forces, \( u_0 \) the initial velocity field, and \( \nu > 0 \) the viscosity.

The spatial discretization we investigate is of spectral Fourier type. For any \( m, M \in \mathbb{N} \) satisfying \( m < M \), let \( S_m \) and \( S_M \) denote the spaces of trigonometric polynomials of degree \( \leq m/2 \) and degree \( \leq M/2 \) in each variable. Let \( P_m \) and \( P_M \) denote the spectral projections onto \( S_m \) and \( S_M \). A natural decomposition of \( S_M \)

\[
S_M = S_m + Q_m S_M, \quad \text{where} \quad Q_m = (I - P_m).
\]

The usual two-level numerical schemes look for an approximate solution of the form

\[
u_{\text{app}} = v + w \quad \text{with} \quad v \in S_m, \ w \in Q_m S_M.
\]

For example, two-level schemes based on the approximate inertial manifold (AIM) and inertial manifold with delay (IMD) proposed in [6] and [7] look for such an approximation by approximating the interaction between the large and small eddies via

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AIM and IMD (see [1], [3], [5], [12], [14], [16], and references therein). Generally, such algorithms are obtained by omitting the second order and part of the first order small eddy terms both in the large and small eddy equations. Since the above decomposition has no relation to the nonlinearity, the interaction of the large and small eddies has to be reflected by the coupled system that $v$ and $w$ satisfy. Therefore, they are usually strong coupled schemes. An exception can be found in recent papers [8], [9], [15]. By omitting all the first order and second order terms of $w$ in the nonlinearity, the authors proposed very effective weakly coupled postprocessing (PP) and dynamical postprocessing (DPP) schemes for two-dimensional (2D) Navier–Stokes equations. Their analyses show that

\begin{equation}
\|u - u_{app}\|_{L^2} = O(L_m^{1/2}m^{-1/2} + L_M^{1/2}M^{-1/2}),
\end{equation}

where $L_M$, $L_m$ are certain logarithmic terms of $M$ and $m$ and $l_1$ is a positive integer which takes different values for different algorithms. Under the assumption for deriving the above analysis result, the coarse-level SGM approximation admits

\begin{equation}
\|u - u_{SGM}\|_{L^2} = O(L_m^{1/2}m^{-1/2}).
\end{equation}

It is obvious that PP and DPP schemes improve the convergence rate of the coarse-level SGM approximation for one order (that is, $m^{-1}$). The omitting of all $w$ terms in the nonlinearity leads to a weakly coupled system, which indicates that the small eddies have a slight influence on the large eddies and their self-evolution. In fact, such approximation is a first order linearization of the nonlinear term. We believe this is acceptable only for large viscosity cases, and our later numerical results also agree with our presumption. To improve the convergence rates of the above PP and DPP algorithms and make the algorithms applicable for smaller viscosity cases, a second order linearization of the nonlinear term is needed. But in usual $L^2$-orthogonal projection based space decomposition, a second order linearization of the nonlinear term in both large and small eddy equations will lead to a strong coupled system; that is, when computing $v$ we need information of $w$ and vice versa. Noticing the idea in [13], if the space splitting can reflect the interaction between the large and small eddies to some extent, we can reasonably expect to have a weakly coupled scheme with a second order linearized nonlinear term both in large and small eddy equations. To construct a weakly coupled two-level dynamical postprocessing Galerkin scheme (see (3.1)–(3.2)) with a second order linearized nonlinear term such that it can further improve the convergence rates of PP and DPP is the main motivation of this paper. In fact, by using a similar technique developed in [18], we will show that our scheme admits for appropriate time step length $k > 0$

\begin{equation}
\|u(t_n) - u_{app}^n\|_{L^2} + \left( k \nu \sum_{i=0}^{n} \|u(t_i) - u_{app}^i\|_{H^1}^2 \right)^{1/2} \leq C(k + m^{-6+d} + M^{-1/2}).
\end{equation}

Compared with (1.2) and (1.3), the result to be obtained indicates that the proposed scheme can improve the $L^2$-convergence rates of PP and DPP schemes for one order and can double the $L^2$-convergence rate of the coarse-level SGM approximation in the 2D case.

2. Preliminaries. First, we introduce for $d = 2, 3$

\[ H = \{ v \in (H^0_{\text{per}}(\Omega))^d, \nabla \cdot v = 0 \}, \quad V = \{ v \in (H^1_{\text{per}}(\Omega))^d, \nabla \cdot v = 0 \}. \]
Denoting by $P$ the $L^2$-orthogonal projection from $(H^0_{per}(\Omega))^d$ onto $H$, we introduce the bilinear operator $B(u,v) = P[(u \cdot \nabla)v]$, the Stokes operator $A = -P\Delta$, and its power $A^\alpha$ for any $\alpha \in \mathbb{R}$ with domain $D(A^\alpha)$. And $|A^\alpha \cdot |$ is equivalent to the standard Sobolev norm $\|\cdot\|_{H^{2\alpha}}$ in the periodic function case. In the rest of this paper, we use $(\cdot, \cdot)$ and $|\cdot|$ to denote the $L^2$-inner product and its related norm. For more details, we refer readers to [17].

We also introduce

$$a(u, v) = (Au, v)_V = (A^\frac{1}{2}u, A^\frac{1}{2}v) \quad \forall u, v \in V,$$

$$b(u, v, w) = (B(u, v), w)_V = \int_{\Omega} (u \cdot \nabla)v \cdot w dx \quad \forall u, v, w \in V.$$ 

It is classical (see [17]) that $a(\cdot, \cdot)$ is $V$-coercive and

\begin{align}
(2.1) \quad & b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in V, \\
& b(u, v, w) \leq c|A^{s_1}u||A^{\frac{1}{2}+s_2}v||A^{s_3}w| \\
(2.2) \quad & \forall u \in D(A^{s_1}), v \in D(A^{\frac{1}{2}+s_2}), w \in D(A^{s_3}),
\end{align}

where $s_1+s_2+s_3 \geq \varphi, (s_1, s_2, s_3) \neq (\frac{d}{2}, 0, 0), (0, \frac{d}{2}, 0), (0, 0, \frac{d}{2})$, and $c > 0$ is a constant.

In this paper, we always use $c$ to denote a generic positive constant. Furthermore,

\begin{align}
(2.3) \quad & b(u, v, w) \leq c|u|_{L^\infty}|A^\frac{3}{2}v||w|, \quad c|u||A^\frac{3}{2}v|_{L^\infty}|w|, \quad c|u||A^\frac{3}{2}v||w|_{L^\infty}.
\end{align}

For given $M \in \mathbb{N}$ and $S_M, P_M, Q_M$ defined previously in the introduction, let us recall (see [4]) that for $s_1 \geq s_2$ and $v \in D(A^{s_1})$

\begin{align}
(2.4) \quad & |A^{s_1}P_Mv| \leq cM^{(s_1-s_2)}|A^{s_2}v|, \quad |A^{s_2}Q_Mv| \leq cM^{(s_2-s_1)}|A^{s_1}v|.
\end{align}

If we denote

\begin{align}
(2.5) \quad & \sigma_d(M) = \left\{ \begin{array}{ll}
1 + 2\ln\sqrt{2M} & \text{when } d = 2, \\
M & \text{when } d = 3,
\end{array} \right.
\end{align}

by using the finite dimensional case of the Brezis–Gallouet inequality [2] when $d = 2$ and Agmon’s inequality and (2.4) when $d = 3$, we have

\begin{align}
(2.6) \quad & |u_M|_{L^\infty} \leq c\sigma_d^\frac{1}{d}(M)|A^\frac{1}{2}u_M| \quad \forall u_M \in S_M.
\end{align}

For the convenience of later analysis, we denote by $T$ a positive constant satisfying $0 \leq T \leq +\infty$ for $d = 2$ and $0 \leq T < T^*$ for $d = 3$, where $T^*$ denotes the length of the time interval on which the strong solution exists. We make the following assumption (A).

(A) \quad There exist positive constants $M_0, M_1, M_2$ such that $|u(t)| \leq M_0, |A^{\frac{1}{2}}u(t)| \leq M_1, |Au(t)| \leq M_2 \quad \forall t \in [0, T].$

By assumption (A), if $D^k f \in L^\infty([0, T]; H)$ for $k = 0, 1, 2$, there exist constants $t_0 > 0$ and $\kappa > 0$ such that (see [10])

\begin{align}
(2.7) \quad & |u_{tt}(t)| \leq \kappa \quad \forall t \in [t_0, T].
\end{align}
To conclude this section, for given time step length $k > 0$, we give the full discrete SGM for (1.1) in functional form: for $\bar{u}_M^0 = P_M u_0$

\begin{equation}
\bar{u}_M^{n+1} + k\nu A \bar{u}_M^{n+1} + kP_M B(\bar{u}_M^{n+1}, \bar{u}_M^{n+1}) = kP_M f + \bar{u}_M^n,
\end{equation}

or, in weak form, $\forall v \in S_M$ (equivalently, $\forall v \in S_M \cap H$)

\begin{equation}
(\bar{u}_M^{n+1}, v) + k\nu a(\bar{u}_M^{n+1}, v) + kb(\bar{u}_M^{n+1}, \bar{u}_M^{n+1}, v) = k(f, v) + (\bar{u}_M^n, v).
\end{equation}

3. **Two-level postprocessing scheme.** For given positive integers $m < M$, we use $S_m$ and $S_M$ to denote the divergence-free subspaces of trigonometric polynomials of degree $\leq m/2$ and $\leq M/2$ in each variable. We consider the full discrete approximation of (1.1) by defining the following sequences:

\begin{equation}
\bar{u}_m^n \in S_m \quad \text{and} \quad \bar{u}_M^n \in S_M \quad \forall n \geq 0.
\end{equation}

They are given by, for $\bar{u}_M^0 = P_M u_0$, $\bar{u}_m^0 = P_m u_0$, and time step length $k > 0$,

\begin{equation}
(\bar{u}_m^{n+1}, v) + k\nu a(\bar{u}_m^{n+1}, v) + kb(\bar{u}_m^{n+1}, \bar{u}_m^{n+1}, v) = k(f, v) + (\bar{u}_m^n, v) \quad \forall v \in S_m,
\end{equation}

\begin{equation}
(\bar{u}_M^{n+1}, v) + k\nu a(\bar{u}_M^{n+1}, v) + kb(\bar{u}_M^{n+1}, \bar{u}_M^{n+1}, v) = k(f, v) + (\bar{u}_M^n, v) + kb(\bar{u}_m^{n+1}, \bar{u}_m^{n+1}, v) \quad \forall v \in S_M.
\end{equation}

The idea of constructing the scheme (3.1)–(3.2) is borrowed from [13], in which we considered a scalar semilinear parabolic equation in the finite element case. In short, to simplify the computation of the SGM scheme (2.8) (or (2.9)), we rewrite the nonlinear term as

\begin{equation}
B(\bar{u}_M^{n+1}, \bar{u}_M^{n+1}) = B(u_m^{n+1}, \bar{u}_M^{n+1}) + B(\bar{u}_M^{n+1} - u_m^{n+1}, u_m^{n+1}) + B(\bar{u}_M^{n+1} - u_m^{n+1}, u_m^{n+1} - u_m^{n+1}),
\end{equation}

where $u_m^{n+1} \in S_m$ is a rough approximation of $\bar{u}_m^{n+1}$. Then we get (3.2) by omitting the second order small term on the right-hand side of the above identity. Regarding $u_m^{n+1} \in S_m$ as an initial guess of the final approximation $u_m^{n+1} \in S_M$, (3.2) is a one-step Newton iteration, which provides a postprocessed approximation of $u_m^{n+1}$ in $S_M$. Thus we call this scheme the dynamical postprocessing scheme of Newton type (DPPN). A peculiar property of the scheme compared with the SGM is that the coefficient matrix of the linear algebraic equation arising in solving (3.2) is a relative sparse matrix. The smaller the ratio $\frac{n}{2}$ is, the sparser the matrix is. Therefore, the scheme can save a lot of memory and CPU time when the bilinear form is small. In the scheme, the rough approximation $u_m^{n+1} \in S_m$ is obtained by solving (3.1), which is nothing but an SGM equation in $S_m$ except for a successive updating of the large eddy approximation in the previous time step $(P_m u_M^0 = u_m^0 + P_m (u_M^n - u_m^n)$, generally, and $P_m (u_M^n - u_m^n) \neq 0$ in our scheme). Subsequently, we will show that (3.1) is the restriction of (3.2) in $S_m$.

We assume that $u_m^{n+1} \in S_m$ is a certain approximation of the time discrete true solution $u \in V$ and the SGM approximation $\bar{u}_M^{n+1} \in S_M$ at $t_{n+1} = (n + 1)k$ within $S_m$, where $1 \leq m < M$. Then we introduce the following bilinear form on $V \times V$: $\forall w, v \in V$

\begin{equation}
\mathcal{L}_m^{n+1}(w, v) = (w, v) + k\nu a(w, v) + kb(u_m^{n+1}, w, v) + kb(w, u_m^{n+1}, v).
\end{equation}

This bilinear form is actually the Frechet derivative of the time discrete Navier–Stokes operator at $u_m^{n+1}$. For given $g \in V'$, the linear equation

\begin{equation}
\mathcal{L}_m^{n+1}(w, v) = (g, v)_{V'} \quad \forall v \in V
\end{equation}
is well-posed in $V$ if $c_{\sigma_d}(m)k|A^{\hat{w}}u^{n+1}_m|^2 \leq \nu$. Then we define a new projection $P^{n+1}_m$ from $V$ (or $S_M$) onto $S_m$: for any given $w \in V$, find $P^{n+1}_m w \in S_m$ such that

$$L^{n+1}_m (w - P^{n+1}_m w, v) = 0 \quad \forall v \in S_m. \tag{3.3}$$

The superscript $n + 1$ means that $L^{n+1}_m (\cdot, \cdot)$ and $P^{n+1}_m$ depend on the evolution of the nonlinear system (1.1) (or SGM system (2.8)) via the large eddy approximation $u^{n+1}_m$. It is easy to obtain that $P_m P^{n+1}_m = P^{n+1}_m, P^{n+1}_m P_m = P_m$. With this new projection, we can easily verify that (3.1) is the restriction of (3.2) in $S_m$.

Denoting $Q^{n+1}_m = I - P^{n+1}_m$, we split $S_M$ and $V$ into

$$S_M = S_m + \hat{S}^{n+1}_M, \quad V = S_m + \hat{V}^{n+1},$$

where $S_m = P_m S_M = P^{n+1}_m S_M, \hat{S}^{n+1}_M = Q^{n+1}_m S_M$, and $\hat{V}^{n+1} = Q^{n+1}_m V$. In the rest of this paper, we always use the following decomposition:

$$w_M = w_m + \hat{w}_M, \quad w = w_m + \hat{w} \quad \forall w_M \in S_M, w \in V.$$

The following lemma shows that $\hat{V}^{n+1}$ (and $\hat{S}^{n+1}_M$) is a small eddy subspace, which carries only a little part of the total energy.

**Lemma 3.1.** For any $w \in V$ and $\hat{w} = (I - P^{n+1}_m) w \in \hat{V}^{n+1}$, we have

$$|P_m \hat{w}| \leq |Q_m \hat{w}|, \quad |\hat{w}| \leq \sqrt{2}|Q_m \hat{w}|,$

provided that $k$ and $m$ satisfy

$$c_{\sigma_d}(m)k|A^{\hat{w}}u^{n+1}_m|^2 \leq \frac{1}{2}. \tag{3.4}$$

**Proof.** Thanks to the property (2.1) and $L^{n+1}_m (\hat{w}, P_m \hat{w}) = 0$, we have

$$|P_m \hat{w}|^2 + k\nu|A^{\hat{w}}P_m \hat{w}|^2 = -kb(u^{n+1}_m, Q_m \hat{w}, P_m \hat{w}) - kb(P_m \hat{w} + Q_m \hat{w}, u^{n+1}_m, P_m \hat{w}).$$

Due to (2.1)–(2.3), (2.6), and the Sobolev interpolation inequality, we have

$$kb(u^{n+1}_m, Q_m \hat{w}, P_m \hat{w}) = -kb(u^{n+1}_m, P_m \hat{w}, Q_m \hat{w}) \leq c \nu|u^{n+1}_m|_{L^\infty} |A^{\hat{w}}P_m \hat{w}| |Q_m \hat{w}|$$

$$\leq c k \sigma_d(m)|A^{\hat{w}}u^{n+1}_m| |A^{\hat{w}}P_m \hat{w}| |Q_m \hat{w}| \leq \frac{k^2 \nu}{3} |A^{\hat{w}}P_m \hat{w}|^2 + \frac{c_{\sigma_d}(m)k|A^{\hat{w}}u^{n+1}_m|^2}{\nu} |Q_m \hat{w}|^2,$$

$kb(P_m \hat{w}, u^{n+1}_m, P_m \hat{w}) = -kb(P_m \hat{w}, P_m \hat{w}, u^{n+1}_m) \leq c \nu|u^{n+1}_m|_{L^\infty} |A^{\hat{w}}P_m \hat{w}| |P_m \hat{w}|$

$$\leq c k \sigma_d(m)|A^{\hat{w}}u^{n+1}_m| |A^{\hat{w}}P_m \hat{w}| |P_m \hat{w}| \leq \frac{k^2 \nu}{3} |A^{\hat{w}}P_m \hat{w}|^2 + \frac{c_{\sigma_d}(m)k|A^{\hat{w}}u^{n+1}_m|^2}{\nu} |P_m \hat{w}|^2,$$

$kb(Q_m \hat{w}, u^{n+1}_m, P_m \hat{w}) = -kb(Q_m \hat{w}, P_m \hat{w}, u^{n+1}_m) \leq c \nu|u^{n+1}_m|_{L^\infty} |A^{\hat{w}}P_m \hat{w}| |Q_m \hat{w}|$

$$\leq c k \sigma_d(m)|A^{\hat{w}}u^{n+1}_m| |Q_m \hat{w}| |A^{\hat{w}}P_m \hat{w}| \leq \frac{k^2 \nu}{3} |A^{\hat{w}}P_m \hat{w}|^2 + \frac{c_{\sigma_d}(m)k|A^{\hat{w}}u^{n+1}_m|^2}{\nu} |Q_m \hat{w}|^2.$$
Noticing that $Q_m \hat{w} = Q_m w$ for any $w \in V$ and Lemma 3.1, it is easy to derive that $P_m^{n+1}$ is a bounded projection in $V$.

**Corollary 3.2.** For any given $m \in \mathbb{N}$, if $k$ and $m$ satisfy (3.4), we have

$$|P_m^{n+1}w| \leq c|w|, \quad |(I - P_m^{n+1})w| \leq c|w| \quad \forall w \in V.$$ 

**Remark 1.** From (3.5), we get

$$|P_m \hat{w}|^2 \leq c\sigma_d(m)k|Q_m \hat{w}|^2.$$ 

That is, $|P_m \hat{w}| \to 0$ when $k \to 0$, which means $|(P_m - P_m^{n+1})w| \to 0$ when $k \to 0$ for any $w \in V$. So the usual $L^2$-orthogonal projection $P_m$ is the limit case of our new projection $P_m^{n+1}$ when $k$ tends to zero.

Thanks to [9], in the 2D case, the approximation error of SGM in the sense of usual $L^2$-orthogonal projection is

$$|P_m \hat{u}(t_n) - u^{n+1}_M| = O(L^4_m m^{-3}).$$

The usual $L^2$-orthogonal projection $P_m$-based two-level schemes, for example, the PP and DPP schemes and the schemes in [8], [9], and [15], correct only the large eddies $\bar{u}^{n+1}_m$ in $Q_m S_M$. Thus the entire error is dominated by the large eddy approximation error $O(L^4_m/m^3)$. In the case of the new projection $P_m^{n+1}$, the correction $\hat{w}^{n+1}_m = v^{n+1}_M - \bar{u}^{n+1}_m$ will not only provide a suitable approximation of the truncated small eddies but will also correct the large eddies in the sense of usual $L^2$-orthogonal projection. Therefore, it is reasonable to expect that

$$|P_m \hat{u}(t_n) - u^{n+1}_M| = o(L^4_m m^{-3}).$$

**4. Numerical experiments.** Before further investigation, we do some numerical experiments first. Considering the computing resources, we do computation only in the 2D case.

In the following numerical implementations, we take the initial divergence-free velocity field and external force as follows:

$$u_0(x) = \sum_{|k| \neq 0} \frac{\text{sgn}(k_1)}{10|k|^4} \left( \frac{k_2}{-k_1} \right) e^{ikx}, \quad f(x) = \sum_{k \in \mathbb{Z}^2, |k| \neq 0} \frac{\sin(k_1 + k_2)}{|k|^2} \left( \frac{k_2}{-k_1} \right) e^{ikx}.$$ 

Such a configuration ensures $u_0 \in V/D(A)$ and $f \in H$. For different $\nu$, we compare the performance of the DPPN scheme (3.1)–(3.2) with the SGM scheme (2.8) and the PP and DPP schemes. For the PP and DPP schemes we use the same backward Euler time discretization as DPPN and SGM with the same time step length $k$. Here we use the following full discrete DPP scheme (4.1), (4.2) and PP scheme (4.1), (4.3):

\begin{align*}
(4.1) \quad &v^{n+1}_m + k\nu A v^{n+1}_m + kP_m B(v^{n+1}_m, v^{n+1}_m) = v^n_m + kP_m f, \\
(4.2) \quad &\hat{w}^{n+1}_M + k\nu A \hat{w}^{n+1}_M + kP_M B(\hat{w}^{n+1}_M, \hat{w}^{n+1}_M) = \hat{w}^n_M + kP_M f, \\
(4.3) \quad &\nu A \hat{w}^N_M + P_M B(\hat{w}^N_m, \hat{w}^N_m) = P_M f, \quad N = \lfloor T/k \rfloor.
\end{align*}

For PP, DPP, and DPPN, we set $M = 61$ and let $m$ vary from 3 to $M$ while we take different $m$ ($M$ in (2.8)) ranging from 3 to $M$ for SGM. The time stepping in the algorithms is achieved by a Newton iteration with tolerance $10^{-9}$, and Gauss–Seidel iteration with tolerance $10^{-9}$ is used in solving the linear algebraic equations. Here we
set the time step length $k$ to be as large as such that any further decreasing of it will not improve the $L^2$-accuracy of the SGM approximation with $m = M$ significantly. Therefore, we take it for granted that the errors of all approximate solutions are dominated by spatial discretization error. By means of precomputations, we find that $k = 0.0001$ is a suitable choice for such a time step length. Finally, the errors of all approximations are computed by comparing them with the larger SGM approximation with $M = 71$.

In Figures 1–4, the vertical axes represent the total $L^2$-relative errors at $T = 2$, and the horizontal axes represent the number of modes in each variable. The error curves of PP for $\nu < 0.01$ are not plotted because the errors are very large.

According to the error estimates of PP and DPP mentioned in the introduction, the total error should decrease rapidly and reach the error level of SGM-$M$ (with $m = M$) at certain $m_0$. After that, the error curve should remain flat. For PP and DPP, such an error curve pattern is well preserved for $\nu \geq 0.1$ and badly deformed for $\nu \leq 0.01$. Additionally, the errors of PP and DPP are greater than the errors of SGM for all $m$ when $\nu < 0.1$. That means the total error of them is dominated by the small eddy error, which indicates that updating the small eddy part by PP and DPP is impractical for small viscosity cases—even worse than treating the truncation part as zero. The numerical results also verify our assertion that PP and DPP are feasible only for large viscosity cases in the introduction because the contribution of the small eddies to large eddies and to small eddies themselves cannot be ignored in such cases. On the contrary, such an error curve pattern is preserved in DPPN for smaller $\nu$. For further small viscosity cases, for example, $\nu \leq 0.001$, the energy decay of the solution with respect to the modes becomes much slower than that of larger viscosity cases. Unless we choose a larger $M$ and a relatively larger $m_0$ to ensure that the small eddies really carry a small part of the entire energy, ignoring small eddy terms in nonlinearity, even a second order small eddy term in DPPN, will make $m_0$ shift to the right rapidly and deform the error curve pattern rapidly. It is interesting that there are some oscillations in the DPPN error curves for $m < m_0$ in the case of $\nu \leq 0.01$. Indeed, the SGM error curves also present similar oscillations. The oscillations of DPPN error curves are merely a reflection of such SGM oscillations on a smaller error level.

At last, Table 1 gives the CPU time comparison, especially for DPPN. Here

$$\text{CPU} = \frac{\text{CPU time used by PP, DPP, or DPPN}}{\text{CPU time used by SGM for } m = M}.$$
Fig. 3. \( \nu = 0.01, M = 61, k = 0.001 \).

Fig. 4. \( \nu = 0.001, M = 61, k = 0.0001 \).

Table 1

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>SGM</th>
<th>PP((m_0, \text{CPU}))</th>
<th>DPP((m_0, \text{CPU}))</th>
<th>DPPN((m_0, \text{CPU}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-1} )</td>
<td>2.47E-4</td>
<td>2.57E-4</td>
<td>2.57E-4</td>
<td>2.52E-4</td>
</tr>
<tr>
<td>( 10^{-2} )</td>
<td>6.02E-3</td>
<td>6.00E-3</td>
<td>5.91E-3</td>
<td>5.99E-3</td>
</tr>
<tr>
<td>( 10^{-3} )</td>
<td>1.13E-1</td>
<td>1.12E-1</td>
<td>1.10E-1</td>
<td>1.07E-1</td>
</tr>
</tbody>
</table>

We conclude that PP and DPP are the most effective schemes for large viscosity cases, while DPPN is more effective for small viscosity cases.

5. Analysis of the numerical results. In this section, we provide an analytic explanation of the numerical experiments presented in the previous section.

5.1. Stability. In this subsection, we will use the induction argument to establish the stability theorem. First, we prove the following lemma, which guarantees the usage of Lemma 3.1 in the \((n + 1)\)th step if \( |A^2 u^n_M| \) is bounded. For convenience, we denote \( t_n = nk \) in the remaining part.

**Lemma 5.1.** For given positive integer \( n \), suppose that there exists a constant \( M_1 > 0 \) independent of \( m \) and \( M \) such that

\[
\max_{0 \leq i \leq n} |A^2 u^i_M| \leq M_1.
\]

Then

\[
|A^2 u^{n+1}_M| \leq 3 \max_{0 \leq i \leq n} |A^2 u^i_M| \leq 3M_1,
\]

provided that \( cM_1^2 \sigma_d(m)k \leq \nu \) and \( k|f|^2 \leq \frac{1}{4}\nu M_1^2 \).

**Proof.** Taking \( v = 2\delta = 2(u^{n+1}_m - P_m u^n_M) \) in (3.1) and noticing the identity

\[
(a - b, 2a) = |a|^2 - |b|^2 + |a - b|^2
\]

(5.1) and (2.1), we have

\[
2|\delta|^2 + k\nu|A^2 u^{n+1}_M|^2 + k\nu|A^2 \delta|^2 \leq k\nu|A^2 P_m u^n_M|^2 + 2k|b(u^{n+1}_m, u^n_m, \delta)| + 2k(|f|, \delta)|
\]

\[
= k\nu|A^2 P_m u^n_M|^2 + 2k|b(u^{n+1}_m, P_m u^n_M, \delta)| + 2k(|f|, \delta)|.
\]
Since
\[ 2k|b(u_{m+}^{n+1}, P_m u_M^n, \delta)| \leq ck|u_{m+}^{n+1}|L_\infty|A^\frac{1}{2} u_M^n||\delta| \leq cM_1 \sigma^2_d(m)k|A^\frac{1}{2} u_m^{n+1}||\delta| \leq |\delta|^2 + cM^2_1 \sigma_d(m)k^2|A^\frac{1}{2} u_m^{n+1}|^2, \]
\[ 2k|(f, \delta)| \leq 2k|f||\delta| \leq |\delta|^2 + k^2|f|^2, \]
we can finally derive
\[ \frac{k\nu}{2}|A^\frac{1}{2} u_{m+}^{n+1}|^2 \leq k\nu|A^\frac{1}{2} P_m u_M^n|^2 + k^2|f|^2 \]
as long as \( k \) is sufficiently small such that \( cM^2_1 \sigma_d(m)k \leq \frac{\nu}{2} \). This leads to the result of this lemma by demanding \( k|f|^2 \leq \frac{1}{2}\nu M^2_1 \).

**Theorem 5.2.** There exist \( N^* \in \mathbb{N} \) and \( M_1 > 0 \) such that
\[ |A^\frac{1}{2} u_M^i|^2 \leq M_1 \quad \forall 0 \leq n \leq N^* \]
if \( k \) and \( m \) satisfy
\[ \frac{ck}{2} \sigma_d(m)M_1^2 \leq \nu, \quad k|f|^2 \leq \frac{1}{2}\nu M^2_1, \quad m^2 - \nu \geq \frac{cM_1}{\nu}. \]

Here \( N^* = +\infty \) and \( M_1 = (|A^\frac{1}{2} u_0|^2 + \frac{c\nu^2}{M_1^2})^\frac{1}{2} \) when \( d = 2 \). For \( d = 3 \), if \( |A^\frac{1}{2} u_0| \) and \( |f| \) are sufficiently small, we have \( N^* = +\infty \).

Here the condition \( m^2 - \nu \geq \frac{cM_1}{\nu} \) can be removed if we consider the local boundedness of the approximate solution rather than global boundedness.

**Proof.** Suppose that
\[ |A^\frac{1}{2} u_M^i|^2 \leq M_1 \quad \forall 0 \leq i \leq n < N^*. \]
Thanks to Lemma 5.1, the projection \( P_m^{n+1} \) can be well defined. Being aware of (3.3), we rewrite the scheme (3.1)-(3.2) as
\[ (u_M^{i+1}, v) + k\nu(u_M^{i+1}, v) + kb(u_M^{i+1}, u_M^{i+1}, v) - kb(\hat{u}_M^{i+1}, \hat{u}_M^{i+1}, v) = k(f, v) + (n^{i+1}_M, v) \quad \forall v \in S_M. \]

Now let us consider the 2D case first. By taking \( v = 2Au_M^{i+1} \) in (5.4) and using (2.1), (5.1), we get
\[ |A^\frac{1}{2} u_M^{i+1}|^2 + 2k|Au_M^{i+1}|^2 \leq |A^\frac{1}{2} u_M^{i+1}|^2 + 2k|(f, Au_M^{i+1})| + 2k|b(u_M^{i+1}, \hat{u}_M^{i+1}, Au_m^{i+1})|. \]
Here we use the property \( b(u, u, Au) = 0 \ \forall u \in D(A) \), which is valid only in the 2D case (see [17]). For the right-hand side terms, we have (for \( i \leq n \))
\[ 2k|(f, Au_M^{i+1})| \leq 2k|f||Au_M^{i+1}| \leq \frac{k\nu}{2}|Au_M^{i+1}|^2 + \frac{2k}{\nu}|f|^2, \]
\[ 2k|b(\hat{u}_M^{i+1}, \hat{u}_M^{i+1}, Au_m^{i+1})| \leq c|A^\frac{1}{2} \hat{u}_M^{i+1}||A^\frac{1}{2} \hat{u}_M^{i+1}||Au_m^{i+1}| \leq ckm^{-2}|Au_m^{i+1}||Au_m^{i+1}| \leq ckm^{-1}|Au_m^{i+1}||Au_m^{i+1}| \leq ckm^{-1}M_1|Au_m^{i+1}|^2. \]
If \( m \) is large enough such that \( m \geq \frac{cM_1}{\nu} \), we obtain
\[ (1 + k\nu)|A^\frac{1}{2} u_M^{i+1}|^2 \leq |A^\frac{1}{2} u_M^{i}|^2 + \frac{2k}{\nu}|f|^2 \quad \forall 0 \leq i \leq n. \]
Then we have $|\mathbf{A}^+ u_{M}^i| \leq M_1 = (|\mathbf{A}^+ u_{M}^i|^2 + \frac{c f^2}{\nu} )^{\frac{1}{2}}$ for $0 \leq i \leq n + 1$ using the discrete Gronwall inequality (see [3]). Finally, by the induction argument we get that $|\mathbf{A}^+ u_{M}^i| \leq M_1$ for $0 \leq i \leq N^* = +\infty$. This completes the proof for the 2D case.

For the three-dimensional (3D) case, we have

\begin{align}
|\mathbf{A}^+ u_{M}^{i+1}|^2 + 2k \nu |\mathbf{A}u_{M}^{i+1}|^2 + |\mathbf{A}^+ (u_{M}^i - u_{M}^i)|^2 \\
\leq |\mathbf{A}^+ u_{M}^i| + 2k|f,\mathbf{A}u_{M}^{i+1}| + 2k|b(u_{M}^{i+1}, u_{M}^{i+1}, \mathbf{A}u_{M}^{i+1})| \\
+ 2k|b(u_{M}^i, u_{M}^i, \mathbf{A}u_{M}^{i+1})| + 2k|b(u_{M}^{i+1}, u_{M}^{i+1}, \mathbf{A}u_{M}^{i+1})|.
\end{align}

For convenience, we denote

\[ y_i = \max_{0 \leq i \leq n} |\mathbf{A}^+ u_{M}^i|^2. \]

Noticing Lemmas 5.1 and 3.1, simple estimates of the right-hand side of the inequality (5.5) admit

\begin{align}
y_{i+1} \leq y_i + \frac{ck}{\nu^3} y_i^3 + \frac{ck}{\nu} |f|^2 \quad \forall i \geq 0.
\end{align}

Now let us re-estimate the right-hand side terms of (5.5) based on the assumption $\max_{0 \leq i \leq M} |\mathbf{A}^+ u_{M}^i| \leq M_1$ and the result of Lemma 5.1, $|\mathbf{A}^+ u_{M}^{i+1}|^2 \leq 3y_i \leq 3M_1^2$:

\[ 2k|f,\mathbf{A}u_{M}^{i+1}| \leq 2k|f| |\mathbf{A}u_{M}^{i+1}| \leq \frac{k \nu}{4} |\mathbf{A}u_{M}^{i+1}|^2 + \frac{ck}{\nu} |f|^2, \]

\[ 2k|b(u_{M}^{i+1}, u_{M}^{i+1}, \mathbf{A}u_{M}^{i+1})| \leq 2k|b(u_{M}^{i+1}, u_{M}^{i+1}, \mathbf{A}u_{M}^{i+1})| + 2k|b(u_{M}^{i+1}, u_{M}^{i+1}, \mathbf{A}u_{M}^{i+1})| \]

\[ \leq c k \sigma_2^2(m) |\mathbf{A}^+ u_{M}^{i+1}| |\mathbf{A}^+ (u_{M}^i - u_{M}^i)| |\mathbf{A}u_{M}^{i+1}| + ck |\mathbf{A}^+ u_{M}^{i+1}| |\mathbf{A}^+ (u_{M}^i - u_{M}^i)| |\mathbf{A}u_{M}^{i+1}| \]

\[ \leq \frac{k \nu}{4} |\mathbf{A}u_{M}^{i+1}|^2 + c k \sigma_2(m) M_1^2 |\mathbf{A}^+ u_{M}^{i+1}| |\mathbf{A}^+ (u_{M}^i - u_{M}^i)| |\mathbf{A}u_{M}^{i+1}| \]

\[ \leq \frac{k \nu}{4} |\mathbf{A}u_{M}^{i+1}|^2 + c k \sigma_2(m) M_1^2 |\mathbf{A}^+ u_{M}^{i+1}| |\mathbf{A}^+ (u_{M}^i - u_{M}^i)| |\mathbf{A}u_{M}^{i+1}| \]

\[ \leq \frac{k \nu}{4} |\mathbf{A}u_{M}^{i+1}|^2 + \frac{ck}{\nu^3} y_i^3 + \frac{ck}{\nu} |f|^2 \quad \forall i \leq n. \]

Then we have

\begin{align}
y_{i+1} \leq y_i + \frac{ck}{\nu^3} y_i^3 + \frac{ck}{\nu} |f|^2 \quad \forall i \leq n.
\end{align}
If we denote \( \tilde{y}_i = 1 + y_i \) and \( \tilde{y}(t) = [(t - t_i)\tilde{y}_{i+1} + (t_{i+1} - t)\tilde{y}_i]/k \) for \( t_i < t < t_{i+1} \), the combination of (5.6) and (5.7) admits

\[
\tilde{y}'' \leq c'y^3 \quad \forall 0 \leq t \leq t_{n+1},
\]

where \( c' = \max(\frac{|f|}{\nu}, \frac{|\tilde{y}|}{\nu}) \). Integration of this inequality admits

\[
y_i \leq \frac{y_0}{\sqrt{1 - c'y_0^2t_i}} \quad \forall 0 \leq t_i \leq t_{n+1} \leq \frac{1}{c'(1 + |A\hat{u}_0|^2)^2}.
\]

Then we have

\[
|A\hat{u}_M|^2 \leq 2(1 + |A\hat{u}_0|^2) \triangleq M_1^2 \quad \forall 0 \leq i \leq \frac{3}{4c'(1 + |A\hat{u}_0|^2)k} \triangleq N^*.
\]

To get the global boundedness of the approximate solution, taking \( v = 2u_{i+1}^\delta \) in (5.4) and using (2.1), (5.1), we get

\[
|u_{i+1}^\delta|^2 + 2kv|A\hat{u}_{i+1}^\delta|^2 \leq |u_M|^2 + 2k|b(\hat{u}_{i+1}^\delta, \hat{u}_{i+1}^\delta) + |2k|f(\hat{u}_{i+1}^\delta)|.
\]

Under the conditions (5.2), using (2.3), (2.6), and Lemmas 5.1 and 3.1, we can easily derive

\[
|u_{i+1}^\delta|^2 + kv|A\hat{u}_{i+1}^\delta|^2 \leq |u_M|^2 + \frac{|f|^2}{\nu} \quad \forall 0 \leq i \leq n.
\]

Therefore, we have (for any fixed integer \( i_0 > 0 \))

\[
\sup_{0 \leq i \leq n+1} |u_{iM}| < \infty \quad \text{and} \quad \sup_{0 < i_0 < i \leq n+1} k \sum_{j=i-i_0}^i |A\hat{u}_{jM}|^2 < \infty.
\]

With such estimates, if \( |A\hat{u}_0| \) and \( |f| \) are sufficiently small, similar arguments in [10] will prove the global boundedness of \( |A\hat{u}_M| \) for \( 0 \leq i \leq +\infty \). \( \square \)

5.2. Error estimation. We will use the energy method and the idea in [18] to establish the error estimate in the \( L^\infty(0, T; H) \cap L^2(0, T; V) \)-norm. The gist of the idea in [18] is that the distance between the Galerkin approximation of a nonlinear parabolic equation and the elliptic projection of the exact solution is superconvergent in the \( H^1 \)-norm; i.e., it is of the same order as the distance in the \( L^2 \)-norm.

In this subsection, we always assume that (A) is valid for \( 0 \leq T \leq +\infty \) in both the 2D and 3D cases. Furthermore, we also assume that

\[
|Au_{M}| \leq M_2 \quad \forall n \geq 0.
\]

Here we use the same symbol \( M_2 \) that appeared in (A) to denote the bound of the approximate solution, and this will not cause any significant difference. In fact, under the conditions of Theorem 5.2, we can prove (5.8) employing a method similar to that used in [10].

From now on, we denote

\[
\begin{align*}
u^n &= u(t_n) = p^n + q^n = P_Mu_n + Q_Mu^n, \quad p^n = U_n + \hat{U} = P^{n}M_p^n + Q^{n}mp^n, \\
\epsilon^n_M = p^n - u_M^n = e^n_m + \epsilon^n_M, \quad \epsilon^n_m = U_m^n - u_m^n, \quad \epsilon^n_M = \hat{U} - \hat{u}_M^n.
\end{align*}
\]
Now we can rewrite the Navier–Stokes equation (1.1) in $S_M$ at $t = t_{n+1}$: $\forall v \in S_M$

$$(p^{n+1}, v) + k\nu(p^{n+1}, v) + kb(p^{n+1}, p^{n+1}, v) + kQ^{n+1}(v) = (p^n, v) + k(f, v) + k(h^{n+1}, v),$$

where

$$Q^{n+1}(v) = b(p^{n+1}, q^{n+1}, v) + b(q^{n+1}, p^{n+1}, v) + b(q^{n+1}, q^{n+1}, v),$$

$$h^{n+1} = \frac{1}{k} \int_{t_n}^{t_{n+1}} P_M(u_t(s) - u_t(t_{n+1})) ds.$$

Noticing $p^{n+1} = U^{n+1} + \hat{U}^{n+1}$ and

$$b(U_m^{n+1}, p^{n+1}, v) = b(e^{n+1}_m, p^{n+1}, v) + b(u^{n+1}_m, e^{n+1}_m, v),$$

$$b(p^{n+1}, U_m^{n+1}, v) = b(e^{n+1}_m, U_m^{n+1}, v) + b(u_m^{n+1}, e^{n+1}_m, v)$$

we obtain that for all $v \in S_M$

$$(e^{n+1}_m + 1, v) + k\nu(a(e^{n+1}_m, v) + k(b(e^{n+1}_m, \hat{U}^{n+1}, v) + b(u^{n+1}_m, e^{n+1}_m, v) + k(b(e^{n+1}_m, U^{n+1}_m, v) + k(b(u^{n+1}_m, e^{n+1}_m, v) + k(b(U^{n+1}_m, \hat{U}^{n+1}, v) + kQ^{n+1}(v) = (e^0_M, v) + k(h^{n+1}, v).$$

Taking $v = 2e^{n+1}_M$ in this equation admits

$$|e^{n+1}_M|^2 + |e^{n+1}_M| - 2k\nu|A \dddot{e}^{n+1}_M|^2 = |e^0_M|^2 - 2k\nu(e^{n+1}_m, \hat{U}^{n+1}, e^{n+1}_m)$$

$$- 2k\nu(e^{n+1}_m, U^{n+1}_m, e^{n+1}_m) - 2k\nu(h^{n+1}, e^{n+1}_m, e^{n+1}_m)$$

Using (2.1), (2.2), (2.4), (2.7), Corollary 3.2, the Sobolev interpolation inequality, and standard estimations on the trilinear forms, we get

$$2k\nu(e^{n+1}_m, \hat{U}^{n+1}, e^{n+1}_m) \leq cM_2 k|e^{n+1}_M||A \dddot{e}^{n+1}_M| \leq \frac{k\nu}{3}|A \dddot{e}^{n+1}_M|^2 + \frac{cM_2^2}{\nu}|e^{n+1}_M|^2,$$

$$2k\nu(e^{n+1}_m, U^{n+1}_m, e^{n+1}_m) \leq cM_2 k|e^{n+1}_M||A \dddot{U}^{n+1}| \leq \frac{k\nu}{3}|A \dddot{U}^{n+1}|^2 + \frac{cM_2^2}{\nu}|e^{n+1}_M|^2,$$

$$kQ^{n+1}(e^{n+1}_M) \leq cM_2^2 k|A \dddot{e}^{n+1}_M| \leq \frac{k\nu}{3}|A \dddot{e}^{n+1}_M|^2 + \frac{cM_2^2}{\nu}|e^{n+1}_M|^2,$$

$$2k(h^{n+1}, e^{n+1}_m) \leq cM_2^2 |u_t||e^{n+1}_M| \leq \frac{cM_2^2}{\nu}|e^{n+1}_M|^2 + cvk^2 \frac{\kappa^2}{M_2^2}.$$

Being aware of $e^0_M = 0$, we have

$$|e^{n+1}_M|^2 + \sum_{i=1}^{n+1} k\nu|A \dddot{e}^{n+1}_M|^2 \leq \sum_{i=1}^{n+1} \frac{cM_2^2}{\nu}|e^{n+1}_M|^2 + \frac{cM_2^2}{\nu^2m^2-4} k\nu \sum_{i=1}^{n+1} |A \dddot{U}^{n+1}|^2$$

$$+ \frac{cM_2^2}{\nu^4} + cvk^2 \frac{\kappa^2}{M_2^2}.$$
Estimating the equation of $Q_m u_i$ similarly, we can easily get

$$
k^2 v \sum_{i=1}^{n+1} |A_M^T Q_m u_i|^2 \leq c \left( M^2_2 + \frac{M^2_1 + |A_M^T f|^2}{\nu} \right) m^{-4} + c k^2 T k^2 M^2_2.
$$

Thanks to Lemma 3.1, we have $|A_M^T \hat{U}| \leq c |A_M^T Q_m \hat{U}| = c |A_M^T Q_m u_i|$. Therefore, if $ck^2 M^2_2 \leq \nu$, combining the above two inequalities and using the discrete Gronwall inequality in [11] leads to

$$
(5.10) \quad |e_M^{n+1}|^2 + \sum_{i=1}^{n+1} k^n |A_M^T e_M^{i}|^2 \leq \exp \left( \frac{c M^2_2 t_{n+1}}{\nu} \right) \left( C_1 m^2(d-6) + C_2 M^{-4} + C_3 k^2 \right),
$$

where $C_1 = C_1(T, \nu, M_1, M_2, |A_M^T f|)$, $C_2 = C_2(T, \nu, M_2)$, and $C_3 = C_3(T, \nu, M_2, \kappa)$.

Noticing $u^{n+1} - u_M^{n+1} = e_M^{n+1} + Q_M u^{n+1}$ and the above inequality, we eventually derive the following theorem.

**Theorem 5.3.** With the conditions of Theorem 5.2, we assume that (A), (5.8) are valid and (2.7) is valid for $t_0 = 0$. If the time step length is small such that $ck^2 M^2_2 \leq \nu$, we have

$$
|u^n - u_M^n| + \left( k^2 \sum_{i=0}^{n} |A_M^T (u^n - u_M^n)|^2 \right)^{\frac{1}{2}} \leq C_1^* m^{d-6} + C_2^* M^{-2} + C_3^* k \quad \forall 0 \leq nk \leq T,
$$

where $C_1^*$ and $C_2^*$ are positive constants depending on $T, \nu, M_1, M_2$, and $|A_M^T f|$ and $C_3^*$ is a positive constant depending on $T, \nu, M_2$, and $\kappa$.

**Remark 2.** In the proof of Theorem 5.3, if we restrict ourselves to the 2D case, only the $H^1$-bound, that is, $M_1$, for the true solution and the approximate solution will appear in the estimates of the following trilinear forms:

$$
2k |e_M^{n+1}, \hat{U}^{n+1}, e_M^{n+1}| + 2k |e_M^{n+1}, \hat{U}_M^{n+1}, e_M^{n+1}| + 2k |\hat{u}_M^{n+1}, e_M^{n+1}, e_M^{n+1}|.
$$

Thus the assumption (5.8) is not necessary. And these estimates ensure that the exponential term in the error constant will involve only $M_1$ rather than $M_2$ in the 3D case in (5.10). Furthermore, we can derive the error estimate without the restriction $ck^2 M^2_2 \leq \nu$.

**6. Conclusion.** First, it is clear that the DPPN scheme can greatly improve the convergence rates of the coarse-level SGM and the DPP approximation from the numerical results and the analytic analysis. For large viscosity, the DPP proposed in [15] is the most effective scheme since the neglect of the contribution of small eddies to large eddies is acceptable. But for smaller viscosities when such neglect is no longer acceptable, DPPN is more effective and accurate.

Second, the idea of reflecting the interaction between the large and the small eddies by system dependent space splitting, which makes the numerical system weakly coupled, comes from the tangent space correction proposed in [13] and IMD. Such an idea may be extended to general nonlinear parabolic PDEs, the high order time discrete scheme, and the finite element case. Although we considered only the periodic boundary condition case in the present paper, the results can be proven for the Dirichlet boundary case with a little modification.

Third, although the fully implicit Fourier Galerkin method is unconditionally stable, it is not as commonly used as the explicit one because the iterative matrix
is almost a full matrix. This makes the computing costs, both of CPU time and of memory, very high. Our two-level DPPN scheme shows that the coefficient matrix arising from solving the fine-level equation can be a very sparse one, which makes the implicit Fourier Galerkin scheme more practical.

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