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PULLBACK $\mathcal{D}$-ATTRACTORS FOR THE NON-AUTONOMOUS NEWTON-BOUSSINESQ EQUATION IN TWO-DIMENSIONAL BOUNDED DOMAIN

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Abstract. We investigate the asymptotic behavior of solutions of a class of non-autonomous Newton-Boussinesq equation in two-dimensional bounded domain. The existence of pullback global attractors is proved in $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$, respectively.

1. **Introduction.** In this paper, we investigate the asymptotic behavior of solutions of the non-autonomous Newton-Boussinesq equation. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Consider the system of equations defined in $\Omega \times R^+$:

$$\partial_t \zeta + u\partial_x \zeta + v\partial_y \zeta = \Delta\zeta - \frac{R_a}{P_r}\partial_x \eta + f(x,y,t), \tag{1.1}$$

$$\Delta\Psi = \zeta,\ \ u = \Psi_y, v = -\Psi_x, \tag{1.2}$$

$$\partial_t \eta + u\partial_x \eta + v\partial_y \eta = \frac{1}{P_r}\Delta\eta + g(x,y,t), \tag{1.3}$$

where $\vec{u} = (u,v)$ is the velocity field of the fluid, η is the flow temperature, Ψ is the flow function and ζ is the vortex. The positive constants P_r and R_a are the Prandtl number and the Rayleigh number, respectively. f and g are external terms.

Notice that system (1.1)-(1.3) can be rewritten as follows: for every $(x,y) \in \Omega$ and $t > 0$,

$$\frac{\partial\zeta}{\partial t} - \Delta\zeta + J(\Psi,\zeta) + \frac{R_a}{P_r}\frac{\partial\eta}{\partial x} = f(x,y,t), \tag{1.4}$$

$$\Delta\Psi = \zeta, \tag{1.5}$$

$$\frac{\partial\eta}{\partial t} - \frac{1}{P_r}\Delta\eta + J(\Psi,\eta) = g(x,y,t), \tag{1.6}$$

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with the boundary conditions

$$\zeta|_{\partial\Omega} = 0, \eta|_{\partial\Omega} = 0, \Psi|_{\partial\Omega} = 0, \tag{1.7}$$

and the initial conditions

$$\zeta(x, y, \tau) = \zeta_\tau(x, y), \eta(x, y, \tau) = \eta_\tau(x, y), \tag{1.8}$$

where the function J is given by

$$J(u, v) = u_y v_x - u_x v_y. \tag{1.9}$$

The Newton-Boussinesq equation describes many physical phenomena such as Benard flow. If the domain is bounded, the existence, uniqueness and the asymptotic behavior of solutions of system (1.1)-(1.3) have been studied by several authors(see [1]-[4]). And in [5], the authors have proved the existence of a global attractor for the autonomous Newton-Boussinesq equation defined in a two-dimensional channel. As far as we know, there is no results on uniform or pullback attractors for non-autonomous Newton-Boussinesq equation.

In this paper, we will investigate the asymptotic behavior of solutions of non-autonomous Newton-Boussinesq equation when the system is defined in a two-dimensional bounded domain. In particular, we will prove that the system has pullback attractors in $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$, respectively. And the pullback asymptotic compactness of the system will be proved by using uniform a priori estimates of solutions.

The existence and uniqueness of solutions for problem (1.4)-(1.8) can be proved by a standard method as in [1] for the autonomous case. More precisely, for every $(\zeta_\tau, \eta_\tau) \in L^2(\Omega) \times L^2(\Omega)$, problem (1.4)-(1.8) has a unique solution (ζ, η) such that for every $T > 0$,

$$\zeta \in C^0([\tau, \infty), L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)), \eta \in C^0([\tau, \infty), L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)).$$

Furthermore, the solution is continuous with respect to initial data (ζ_τ, η_τ) in $L^2(\Omega) \times L^2(\Omega)$.

It is easy to verify that J satisfies:

$$\int_\Omega J(u, v) v \mathrm{d}x\mathrm{d}y = 0, \ \textit{for all } u \in H^1(\Omega), \ v \in H^2(\Omega) \cap H_0^1(\Omega), \tag{1.10}$$

$$\| J(u, v) \| \leq C \| u \|_{H^2} \| v \|_{H^2}, \ \textit{for all } u \in H^2(\Omega), v \in H^2(\Omega), \tag{1.11}$$

$$\| J(u, v) \| \leq C \| u \|_{H^3} \| \nabla v \|, \ \textit{for all } u \in H^3(\Omega), v \in H^1(\Omega). \tag{1.12}$$

Throughout this paper, we frequently use the Poincaré inequality

$$\| u \| \leq \lambda \| \nabla u \|, \forall u \in H_0^1(\Omega),$$

where λ is a positive constant.

The following notations will be used throughout this paper. We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and inner product in $L^2(\Omega)$ and use $\| \cdot \|_p$ to denote the norm in $L^p(\Omega)$ $(p \neq 2)$. The norm of any Banach space X is written as $\| \cdot \|_X$. The letter C is a generic positive constant which may change its values from line to line or even in the same line.

2. **Preliminaries and abstract results.** In this section, we recall some definitions and results concerning the pullback attractor. The reader is referred to [6]-[9] for details. Let Θ be a non-empty set, X be a metric space. And we denote by $\mathcal{D}$ the collection of families of subsets of X:

$$\mathcal{D} = \{D = \{D(\omega)\}_{\omega\in\Theta} : D(\omega) \subseteq X \ for\ every\ \omega \in \Theta\}.$$

Definition 2.1. A family of mapping $\{\theta_t\}_{t\in\mathbb{R}}$ from Θ to itself is called a family of shift operators on Θ if $\{\theta_t\}_{t\in\mathbb{R}}$ satisfies the group properties:

(1) $\theta_0\omega = \omega,\ \forall\omega \in \Theta$;
(2) $\theta_t(\theta_\tau\omega) = \theta_{t+\tau}\omega, \forall\omega \in \Theta\ and\ t, \tau \in \mathbb{R}$.

Definition 2.2. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a family of shift operators on Θ. Then a continuous θ-cocycle ϕ on X is a mapping

$$\phi : \mathbb{R}^+ \times \Theta \times X \to X, (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which satisfies, for all $\omega \in \Theta$ and $t, \tau \in \mathbb{R}^+$,

(1) $\phi(0, \omega, \cdot)$ is the identity on X;
(2) $\phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau\omega, \cdot) \circ \phi(\tau, \omega, \cdot)$;
(3) $\phi(t, \omega, \cdot) : X \to X$ is continuous.

Definition 2.3. Let $\mathcal{D}$ be a collection of families of subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subseteq X : \omega \in \Theta\}$ with $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Theta$ imply that $\tilde{D} \in \mathcal{D}$.

Definition 2.4. Let $\mathcal{D}$ be a collection of families of subsets of X and $\{K(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Theta}$ is called a pullback absorbing set for ϕ in $\mathcal{D}$ if, for every $B \in \mathcal{D}$ and $\omega \in \Theta$, there exists $T(\omega, B) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega)\ for\ all\ t \geq T(\omega, B).$$

Definition 2.5. Let $\mathcal{D}$ be a collection of families of subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if, for every $\omega \in \Theta$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$.

Definition 2.6. Let $\mathcal{D}$ be a collection of families of subsets of X and $\{\mathcal{A}(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$. Then $\{\mathcal{A}(\omega)\}_{\omega\in\Theta}$ is called a $\mathcal{D}$-pullback global attractor for ϕ if the following conditions are satisfied, for every $\omega \in \Theta$,
(1) $\mathcal{A}(\omega)$ is compact;
(2) $\{\mathcal{A}(\omega)\}_{\omega\in\Theta}$ is invariant, that is,

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \forall t \geq 0;$$

(3) $\{\mathcal{A}(\omega)\}_{\omega\in\Theta}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$,

$$\lim_{t\to\infty} \mathrm{dist}(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where dist is the Hausdorff semi-metric given by $\mathrm{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \| a - b \|_X$ for any $A \subset X$ and $B \subset X$.

Proposition 1. *Let $\mathcal{D}$ be an inclusion-closed collection of families of subsets of X and ϕ a continuous θ-cocycle on X. Suppose that $\{K(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$ is a closed absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-pullback global attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

To construct a cocycle ϕ for problem (1.4)-(1.8), we denote by $\Theta = \mathbb{R}$, and define a shift operator θ_t on Θ for every $t \in \mathbb{R}$ by:

$$\theta_t(\tau) = t + \tau, \forall \tau \in \mathbb{R}.$$

Let ϕ be a mapping from $\mathbb{R}^+ \times \Theta \times (L^2(\Omega) \times L^2(\Omega))$ to $L^2(\Omega) \times L^2(\Omega)$ given by

$$\phi(t, \tau, (\zeta_\tau, \eta_\tau)) = (\zeta(t + \tau, \tau, \zeta_\tau), \eta(t + \tau, \tau, \eta_\tau)),$$

where $t \geq 0, \tau \in \mathbb{R}, (\zeta_\tau, \eta_\tau) \in L^2(\Omega) \times L^2(\Omega)$, and (ζ, η) is the solution of problem (1.4)-(1.8). By the uniqueness of solutions, we find that for every $t, s \geq 0, \tau \in \mathbb{R}$ and $(\zeta_\tau, \eta_\tau) \in L^2(\Omega) \times L^2(\Omega)$,

$$\phi(t + s, \tau, (\zeta_\tau, \eta_\tau)) = \phi(t, s + \tau, (\phi(s, \tau, (\zeta_\tau, \eta_\tau)))).$$

Then we see that ϕ is a continuous θ-cocycle on $L^2(\Omega) \times L^2(\Omega)$.

For convenience, if $E \subseteq L^2(\Omega) \times L^2(\Omega)$, we denote

$$\| E \| = \sup_{x \in E} \| x \|_{L^2(\Omega) \times L^2(\Omega)} .$$

Let $D = \{D(t)\}_{t \in \mathbb{R}}$ be a family of subsets of $L^2(\Omega) \times L^2(\Omega)$, i.e., $D(t) \subseteq L^2(\Omega) \times L^2(\Omega)$ for every $t \in \mathbb{R}$. In this paper, we are interested in a family $D = \{D(t)\}_{t \in \mathbb{R}}$ satisfying

$$\lim_{t \to -\infty} e^{\sigma t} \| D(t) \|^2 = 0, \tag{2.1}$$

where σ is a positive constant satisfying $0 < \sigma < \min\{\frac{1}{2\lambda^2 P_r}, \frac{1}{2\lambda^2}\}$. We write the collection of all families satisfying (2.1) as $\mathcal{D}_\sigma$, that is,

$$\mathcal{D}_\sigma = \{D = \{D(t)\}_{t \in \mathbb{R}} : D \ satisfies \ (2.1)\}. \tag{2.2}$$

As we will see later, for deriving the uniform estimates of solutions, we need the following conditions for the external force terms

$$\int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] \mathrm{d}\xi < K < \infty, \forall \tau \in \mathbb{R}, \tag{2.3}$$

where K is a positive constant.

3. Uniform estimates of solutions.

In this section, we derive uniform estimates of solutions of problem (1.4)-(1.8) when $t \to \infty$. These estimates are necessary for proving the existence of a bounded pullback absorbing set and the pullback asymptotic compactness of the θ-cocycle ϕ associated with the system.

Lemma 3.1. *For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,*

$$\begin{aligned} &\| \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 \\ &\qquad \leq C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] \mathrm{d}\xi, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\int_{\tau - t}^{\tau} e^{\sigma \xi} \Big(\| \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \Big) \mathrm{d}\xi \\ &\qquad \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] \mathrm{d}\xi, \end{aligned} \tag{3.2}$$

$$\int_{\tau-t}^{\tau} e^{\sigma\xi}\Big(\parallel \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 + \parallel \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \Big)\mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi, \tag{3.3}$$

where $(\zeta_0(\tau-t),\eta_0(\tau-t)) \in D(\tau-t)$*, and* C *is a positive constant independent of* τ *or* D*.*

Proof. Taking the inner product of (1.6) with η in $L^2(\Omega)$ and using (1.10) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel \eta \parallel^2 + \frac{1}{P_r} \parallel \nabla\eta \parallel^2 = \int_\Omega g\eta \mathrm{d}x\mathrm{d}y \leq \parallel g \parallel \parallel \eta \parallel \\ \leq \frac{1}{4\lambda^2 P_r} \parallel \eta \parallel^2 + C \parallel g(t) \parallel^2 . \tag{3.4}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel \eta \parallel^2 + \frac{1}{P_r} \parallel \nabla\eta \parallel^2 + \frac{1}{2\lambda^2 P_r} \parallel \eta \parallel^2 \leq C \parallel g(t) \parallel^2 . \tag{3.5}$$

Multiplying (3.5) by $e^{\sigma t}$ and then integrating it between $\tau - t$ and τ with $t \geq 0$, we get,

$$e^{\sigma\tau} \parallel \eta(\tau,\tau-t,\eta_0(\tau-t)) \parallel^2 + \frac{1}{P_r}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ \leq e^{\sigma(\tau-t)} \parallel \eta_0(\tau-t) \parallel^2 + (\sigma - \frac{1}{2\lambda^2 P_r})\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ + C\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel g(\xi) \parallel^2 \mathrm{d}\xi. \tag{3.6}$$

Noticing that $(\zeta_0(\tau-t),\eta_0(\tau-t)) \in D(\tau-t)$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, and we find that for every $\tau \in \mathbb{R}$, there exists $T_1 = T_1(\tau, D) > 0$ such that for all $t \geq T_1$,

$$e^{\sigma(\tau-t)} \parallel \eta_0(\tau-t) \parallel^2 \leq \int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi,$$

which along with (3.6) shows that, for all $t \geq T_1$,

$$\|\eta(\tau,\tau-t,\eta_0(\tau-t)) \|^2 + \frac{1}{P_r}e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ + (\frac{1}{2\lambda^2 P_r} - \sigma)e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi. \tag{3.7}$$

From (3.7) we have

$$\parallel \eta(\tau,\tau-t,\eta_0(\tau-t)) \parallel^2 \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi; \tag{3.8}$$

$$\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi; \tag{3.9}$$

$$\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi. \tag{3.10}$$

Taking the inner product of (1.4) with ζ in $L^2(\Omega)$ and using (1.10) we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\parallel\zeta\parallel^2+\parallel\nabla\zeta\parallel^2+\frac{R_a}{P_r}\int_\Omega\eta_x\zeta\mathrm{d}x\mathrm{d}y=\int_\Omega f(t)\zeta\mathrm{d}x\mathrm{d}y, \tag{3.11}$$

because

$$\begin{aligned}\Big|\frac{R_a}{P_r}\int_\Omega\eta_x\zeta\mathrm{d}x\mathrm{d}y\Big|=\Big|\frac{R_a}{P_r}\int_\Omega\zeta_x\eta\mathrm{d}x\mathrm{d}y\Big|\leq&\frac{R_a}{P_r}\parallel\nabla\zeta\parallel\parallel\eta\parallel\\ \leq&\frac{\parallel\nabla\zeta\parallel^2}{4}+C\parallel\eta\parallel^2,\end{aligned} \tag{3.12}$$

$$\int_\Omega f(t)\zeta\mathrm{d}x\mathrm{d}y\leq\parallel f(t)\parallel\parallel\zeta\parallel\leq\lambda\parallel f(t)\parallel\parallel\nabla\zeta\parallel\leq\frac{\parallel\nabla\zeta\parallel^2}{4}+C\parallel f(t)\parallel^2, \tag{3.13}$$

so taking (3.11)-(3.13) into account, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\parallel\zeta\parallel^2+\frac{1}{2}\parallel\nabla\zeta\parallel^2+\frac{1}{2\lambda^2}\parallel\zeta\parallel^2\leq C\parallel\eta\parallel^2+C\parallel f(t)\parallel^2. \tag{3.14}$$

Multiplying (3.14) by $e^{\sigma t}$ and then integrating it between $\tau-t$ and τ, we obtain

$$\begin{aligned}&e^{\sigma\tau}\parallel\zeta(\tau,\tau-t,\zeta_0(\tau-t))\parallel^2+\frac{1}{2}\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t))\parallel^2\mathrm{d}\xi\\ \leq&e^{\sigma(\tau-t)}\parallel\zeta_0(\tau-t)\parallel^2+(\sigma-\frac{1}{2\lambda^2})\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\zeta(\xi,\tau-t,\zeta_0(\tau-t))\parallel^2\mathrm{d}\xi\\ &+C\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\eta(\xi,\tau-t,\eta_0(\tau-t))\parallel^2\mathrm{d}\xi+C\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel f(\xi)\parallel^2\mathrm{d}\xi.\end{aligned} \tag{3.15}$$

Noticing that $(\zeta_0(\tau-t),\eta_0(\tau-t))\in D(\tau-t)$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, we find that for every $\tau\in\mathbb{R}$, there exists $T_2=T_2(\tau,D)>0$ such that for all $t\geq T_2$,

$$e^{\sigma(\tau-t)}\parallel\zeta_0(\tau-t)\parallel^2\leq\int_{-\infty}^{\tau}e^{\sigma\xi}[\parallel f(\xi)\parallel^2+\parallel g(\xi)\parallel^2]\mathrm{d}\xi,$$

which along with (3.15) and (3.9) shows that, for all $t\geq T=\max\{T_1,T_2\}>0$,

$$\begin{aligned}&\parallel\zeta(\tau,\tau-t,\zeta_0(\tau-t))\parallel^2+\frac{1}{2}e^{-\sigma\tau}\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t))\parallel^2\mathrm{d}\xi\\ &+(\frac{1}{2\lambda^2}-\sigma)e^{-\sigma\tau}\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\zeta(\xi,\tau-t,\zeta_0(\tau-t))\parallel^2\mathrm{d}\xi\\ \leq&Ce^{-\sigma\tau}\int_{-\infty}^{\tau}e^{\sigma\xi}[\parallel f(\xi)\parallel^2+\parallel g(\xi)\parallel^2]\mathrm{d}\xi.\end{aligned} \tag{3.16}$$

From (3.16) we have

$$\parallel\zeta(\tau,\tau-t,\zeta_0(\tau-t))\parallel^2\leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau}e^{\sigma\xi}[\parallel f(\xi)\parallel^2+\parallel g(\xi)\parallel^2]\mathrm{d}\xi; \tag{3.17}$$

$$\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\zeta(\xi,\tau-t,\zeta_0(\tau-t))\parallel^2\mathrm{d}\xi\leq C\int_{-\infty}^{\tau}e^{\sigma\xi}[\parallel f(\xi)\parallel^2+\parallel g(\xi)\parallel^2]\mathrm{d}\xi. \tag{3.18}$$

$$\int_{\tau-t}^{\tau}e^{\sigma\xi}\parallel\nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t))\parallel^2\mathrm{d}\xi\leq C\int_{-\infty}^{\tau}e^{\sigma\xi}[\parallel f(\xi)\parallel^2+\parallel g(\xi)\parallel^2]\mathrm{d}\xi. \tag{3.19}$$

Thus, the proof is completed. □

Lemma 3.2. *For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,*

$$\begin{aligned}\int_{\tau-2}^{\tau} e^{\sigma\xi}(\| \zeta(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 + \| \eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2)\mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi;\end{aligned} \tag{3.20}$$

$$\begin{aligned}\int_{\tau-2}^{\tau} e^{\sigma\xi}(\| \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 + \| \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2)\mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi,\end{aligned} \tag{3.21}$$

where $(\zeta_0(\tau-t), \eta_0(\tau-t)) \in D(\tau-t)$, and C is a positive constant independent of τ or D.

Proof. Note that (3.5) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \eta \|^2 + \frac{1}{2\lambda^2 P_r} \| \eta \|^2 \leq C \| g(t) \|^2 . \tag{3.22}$$

Let $s \in [\tau-2, \tau]$ and $t \geq 2$. Multiplying (3.22) by $e^{\sigma t}$, then relabeling t as ξ and integrating it with respect to ξ over $(\tau-t, s)$, we have

$$\begin{aligned}e^{\sigma s} \| \eta(s,\tau-t,\eta_0(\tau-t)) \|^2 + (\frac{1}{2\lambda^2 P_r} - \sigma)\int_{\tau-t}^{s} e^{\sigma\xi} \| \eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ \leq e^{\sigma(\tau-t)} \| \eta_0(\tau-t) \|^2 + C\int_{-\infty}^{\tau} e^{\sigma\xi} \| g(\xi) \|^2 \,\mathrm{d}\xi.\end{aligned} \tag{3.23}$$

Noticing that $(\zeta_0(\tau-t), \eta_0(\tau-t)) \in D(\tau-t)$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, and we find that for every $\tau \in \mathbb{R}$, there exists $T_3 = T_3(\tau, D) > 2$ such that for all $t \geq T_3$,

$$e^{\sigma(\tau-t)} \| \eta_0(\tau-t) \|^2 \leq \int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi.$$

So

$$\begin{aligned}e^{\sigma s} \| \eta(s,\tau-t,\eta_0(\tau-t)) \|^2 + \int_{\tau-t}^{s} e^{\sigma\xi} \| \eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi,\end{aligned} \tag{3.24}$$

for any $s \in [\tau-2, \tau]$.

Multiplying (3.5) by $e^{\sigma t}$ and integrating it over $(\tau-2, \tau)$, by (3.24) we get that, for all $t \geq T_3$,

$$\begin{aligned}e^{\sigma\tau} \|\eta(\tau,\tau-t,\eta_0(\tau-t)) \|^2 + \frac{1}{P_r}\int_{\tau-2}^{\tau} e^{\sigma\xi} \| \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ + (\frac{1}{2\lambda^2 P_r} - \sigma)\int_{\tau-2}^{\tau} e^{\sigma\xi} \| \eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ \leq e^{\sigma(\tau-2)} \| \eta(\tau-2,\tau-t,\eta_0(\tau-t)) \|^2 + C\int_{\tau-2}^{\tau} e^{\sigma\xi} \| g(\xi) \|^2 \,\mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi.\end{aligned} \tag{3.25}$$

From (3.25) we have

$$\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi; \quad (3.26)$$

$$\begin{aligned}\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi.\end{aligned} \quad (3.27)$$

Note that (3.14) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel \zeta \parallel^2 + \frac{1}{2\lambda^2} \parallel \zeta \parallel^2 \leq C \parallel \eta \parallel^2 + C \parallel f(t) \parallel^2 . \quad (3.28)$$

Let $s \in [\tau-2,\tau]$ and $t \geq 2$, multiplying (3.28) by $e^{\sigma t}$, then relabeling t as ξ and integrating it with respect to ξ over $(\tau-t,s)$, we have

$$\begin{aligned} & e^{\sigma s} \parallel \zeta(s,\tau-t,\zeta_0(\tau-t)) \parallel^2 + (\frac{1}{2\lambda^2}-\sigma)\int_{\tau-t}^{s} e^{\sigma\xi} \parallel \zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ & \leq e^{\sigma(\tau-t)} \parallel \zeta_0(\tau-t) \parallel^2 + C\int_{\tau-t}^{s} e^{\sigma\xi} \parallel \eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ & \quad + C\int_{\tau-t}^{s} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \end{aligned} \quad (3.29)$$

Noticing that $(\zeta_0(\tau-t),\eta_0(\tau-t)) \in D(\tau-t)$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, and we find that for every $\tau \in \mathbb{R}$, there exists $T_4 = T_4(\tau,D) > 2$ such that for all $t \geq T_4$,

$$e^{\sigma(\tau-t)} \parallel \zeta_0(\tau-t) \parallel^2 \leq \int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi,$$

which along with (3.29) and (3.24) implies that: there exists $T_5 = \max\{T_3,T_4\} > 2$, such that for all $t \geq T_5$ and for any $s \in [\tau-2,\tau]$,

$$e^{\sigma s} \parallel \zeta(s,\tau-t,\zeta_0(\tau-t)) \parallel^2 \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi. \quad (3.30)$$

Multiplying (3.14) by $e^{\sigma t}$ and integrating it over $(\tau-2,\tau)$, using (3.26) and (3.30) we get

$$\begin{aligned} & e^{\sigma\tau} \parallel \zeta(\tau,\tau-t,\zeta_0(\tau-t)) \parallel^2 + \frac{1}{2}\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ & \quad + (\frac{1}{2\lambda^2}-\sigma)\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\ & \leq e^{\sigma(\tau-2)} \parallel \zeta(\tau-2,\tau-t,\zeta_0(\tau-t)) \parallel^2 \\ & \quad + C\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi + C\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \\ & \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi. \end{aligned} \quad (3.31)$$

Thanks to (3.31), we have

$$\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi; \quad (3.32)$$

$$\begin{aligned}\int_{\tau-2}^{\tau} e^{\sigma\xi} \|\nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t))\|^2 \,\mathrm{d}\xi \\ \leq C\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi)\|^2 + \| g(\xi)\|^2]\mathrm{d}\xi,\end{aligned} \tag{3.33}$$

which completes the proof. □

Corollary 1. *For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,*

$$\begin{aligned}\int_{\tau-2}^{\tau}[\| \zeta(\xi,\tau-t,\zeta_0(\tau-t))\|^2 + \| \eta(\xi,\tau-t,\eta_0(\tau-t))\|^2]\mathrm{d}\xi \\ \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi)\|^2 + \| g(\xi)\|^2]\mathrm{d}\xi,\end{aligned} \tag{3.34}$$

$$\begin{aligned}\int_{\tau-2}^{\tau}[\| \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t))\|^2 + \| \nabla\eta(\xi,\tau-t,\eta_0(\tau-t))\|^2]\mathrm{d}\xi \\ \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi)\|^2 + \| g(\xi)\|^2]\mathrm{d}\xi,\end{aligned} \tag{3.35}$$

where $(\zeta_0(\tau-t), \eta_0(\tau-t)) \in D(\tau-t)$, and C is a positive constant independent of τ or D.

Proof. It is straightforward from Lemma 3.2. □

Lemma 3.3. *For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,*

$$\begin{aligned}\| \nabla\zeta(\tau,\tau-t,\zeta_0(\tau-t))\|^2 + \| \nabla\eta(\tau,\tau-t,\eta_0(\tau-t))\|^2 \\ \leq C[e^{-\sigma\tau} + e^{-3\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi)\|^2 + \| g(\xi)\|^2]\mathrm{d}\xi,\end{aligned} \tag{3.36}$$

where $(\zeta_0(\tau-t), \eta_0(\tau-t)) \in D(\tau-t)$, and C is a positive constant independent of τ or D.

Proof. Taking the inner product of (1.4) with $-\Delta\zeta$ in $L^2(\Omega)$ we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \nabla\zeta\|^2 + \| \Delta\zeta\|^2 = \int_\Omega J(\Psi,\zeta)\Delta\zeta\mathrm{d}x\mathrm{d}y + \frac{R_a}{P_r}\int_\Omega \eta_x\Delta\zeta\mathrm{d}x\mathrm{d}y \\ - \int_\Omega f\Delta\zeta\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.37}$$

Notice that the first term on the right-hand side of (3.37) is given by

$$\int_\Omega J(\Psi,\zeta)\Delta\zeta\mathrm{d}x\mathrm{d}y = \int_\Omega \Psi_y\zeta_x\Delta\zeta\mathrm{d}x\mathrm{d}y - \int_\Omega \Psi_x\zeta_y\Delta\zeta\mathrm{d}x\mathrm{d}y. \tag{3.38}$$

We now estimate the first term on the right-hand side of (3.38). By Hölder inequality and Nirenberg-Gagliardo inequality, we have

$$\begin{aligned}&\int_\Omega \Psi_y\zeta_x\Delta\zeta\mathrm{d}x\mathrm{d}y \leq \| \Psi_y\|_4\| \zeta_x\|_4\| \Delta\zeta\| \\ &\leq C\| \Psi_y\|^{1/2}\| \Psi_y\|_{H^1}^{1/2}\| \zeta_x\|^{1/2}\| \zeta_x\|_{H^1}^{1/2}\| \Delta\zeta\| \\ &\leq C\| \Psi\|_{H^2}\| \nabla\zeta\|^{1/2}\| \Delta\zeta\|^{3/2} \leq C\| \zeta\|\| \nabla\zeta\|^{1/2}\| \Delta\zeta\|^{3/2} \\ &\leq \frac{1}{8}\| \Delta\zeta\|^2 + C\| \zeta\|^4\| \nabla\zeta\|^2 .\end{aligned} \tag{3.39}$$

Similarly, we have

$$\int_\Omega \Psi_x \zeta_y \Delta\zeta \mathrm{d}x\mathrm{d}y \leq \frac{1}{8} \parallel \Delta\zeta \parallel^2 + C \parallel \zeta \parallel^4 \parallel \nabla\zeta \parallel^2 . \tag{3.40}$$

It is obvious that the last two terms on the right-hand side of (3.37) are bounded by

$$\Big|\frac{R_a}{P_r}\int_\Omega \eta_x \Delta\zeta \mathrm{d}x\mathrm{d}y\Big| + \Big|\int_\Omega f\Delta\zeta \mathrm{d}x\mathrm{d}y\Big| \leq \frac{1}{4} \parallel \Delta\zeta \parallel^2 + C \parallel \nabla\eta \parallel^2 + C \parallel f \parallel^2 . \tag{3.41}$$

Now taking (3.37)-(3.41) into account, the usage of Lemma 3.1 leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \parallel \nabla\zeta \parallel^2 + \parallel \Delta\zeta \parallel^2 \leq & C \parallel \zeta \parallel^4 \parallel \nabla\zeta \parallel^2 + C \parallel \nabla\eta \parallel^2 + C \parallel f \parallel^2 \\ \leq & C[r_0(t)]^2 \parallel \nabla\zeta \parallel^2 + C \parallel \nabla\eta \parallel^2 + C \parallel f \parallel^2 \\ \leq & Ce^{-2\sigma t} \parallel \nabla\zeta \parallel^2 + C \parallel \nabla\eta \parallel^2 + C \parallel f \parallel^2,\end{aligned} \tag{3.42}$$

where $r_0(t) = Ce^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi < CKe^{-\sigma t} \leq Ce^{-\sigma t}$. Let $s \leq \tau$ and $t \geq 2$. Multiplying (3.42) by $e^{\sigma t}$, then relabeling t as ξ and integrating it with respect to ξ over (s, τ) with $\tau - 1 \leq s \leq \tau$, we find that

$$\begin{aligned}& e^{\sigma\tau} \parallel \nabla\zeta(\tau, \tau - t, \zeta_0(\tau - t)) \parallel^2 \\ & \leq e^{\sigma s} \parallel \nabla\zeta(s, \tau - t, \zeta_0(\tau - t)) \parallel^2 + C\int_s^\tau e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi \\ & \quad + C\int_s^\tau e^{-\sigma\xi} \parallel \nabla\zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 \mathrm{d}\xi \\ & \quad + C\int_s^\tau e^{\sigma\xi} \parallel \nabla\eta(\xi, \tau - t, \eta_0(\tau - t)) \parallel^2 \mathrm{d}\xi \\ & \quad + \sigma\int_s^\tau e^{\sigma\xi} \parallel \nabla\zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 \mathrm{d}\xi \\ & \leq e^{\sigma s} \parallel \nabla\zeta(s, \tau - t, \zeta_0(\tau - t)) \parallel^2 + C\int_{\tau-1}^\tau e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi \\ & \quad + Ce^{-2\sigma(\tau-1)}\int_{\tau-1}^\tau e^{\sigma\xi} \parallel \nabla\zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 \mathrm{d}\xi \\ & \quad + C\int_{\tau-1}^\tau e^{\sigma\xi} \parallel \nabla\eta(\xi, \tau - t, \eta_0(\tau - t)) \parallel^2 \mathrm{d}\xi \\ & \quad + \sigma\int_{\tau-1}^\tau e^{\sigma\xi} \parallel \nabla\zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 \mathrm{d}\xi.\end{aligned} \tag{3.43}$$

We now integrate (3.43) with respect to s over $(\tau-1,\tau)$, and by (3.21) we have

$$\begin{aligned}
&e^{\sigma\tau} \parallel \nabla\zeta(\tau,\tau-t,\zeta_0(\tau-t)) \parallel^2 \\
&\leq \int_{\tau-1}^{\tau} e^{\sigma s} \parallel \nabla\zeta(s,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}s + C\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi \\
&\quad + Ce^{-2\sigma(\tau-1)}\int_{\tau-1}^{\tau} e^{\sigma\xi} \parallel \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\
&\quad + C\int_{\tau-1}^{\tau} e^{\sigma\xi} \parallel \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\
&\quad + \sigma\int_{\tau-1}^{\tau} e^{\sigma\xi} \parallel \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \parallel^2 \mathrm{d}\xi \\
&\leq C[1+e^{-2\sigma(\tau-1)}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi \\
&\leq C[1+e^{-2\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi.
\end{aligned} \tag{3.44}$$

So

$$\begin{aligned}
&\parallel \nabla\zeta(\tau,\tau-t,\zeta_0(\tau-t)) \parallel^2 \\
&\leq C[e^{-\sigma\tau}+e^{-3\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2]\mathrm{d}\xi.
\end{aligned} \tag{3.45}$$

Taking the inner product of (1.6) with $-\Delta\eta$ in $L^2(\Omega)$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel \nabla\eta \parallel^2 + \frac{1}{P_r} \parallel \Delta\eta \parallel^2 = \int_\Omega J(\Psi,\eta)\Delta\eta\mathrm{d}x\mathrm{d}y - \int_\Omega g\Delta\eta\mathrm{d}x\mathrm{d}y. \tag{3.46}$$

By arguments similar to getting (3.39) and (3.40), we obtain that

$$\begin{aligned}
\Big|\int_\Omega J(\Psi,\eta)\Delta\eta\mathrm{d}x\mathrm{d}y\Big| &= \Big|\int_\Omega \Psi_y\eta_x\Delta\eta\mathrm{d}x\mathrm{d}y - \int_\Omega \Psi_x\eta_y\Delta\eta\mathrm{d}x\mathrm{d}y\Big| \\
&\leq \frac{1}{4P_r} \parallel \Delta\eta \parallel^2 + C \parallel \zeta \parallel^4 \parallel \nabla\eta \parallel^2,
\end{aligned} \tag{3.47}$$

and

$$\Big|\int_\Omega g\Delta\eta\mathrm{d}x\mathrm{d}y\Big| \leq \frac{1}{4P_r} \parallel \Delta\eta \parallel^2 + C \parallel g(t) \parallel^2 . \tag{3.48}$$

Taking (3.46)-(3.48) into account, the usage of Lemma 3.1 leads to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \parallel \nabla\eta \parallel^2 + \frac{1}{P_r} \parallel \Delta\eta \parallel^2 &\leq C \parallel \zeta \parallel^4 \parallel \nabla\eta \parallel^2 + C \parallel g(t) \parallel^2 \\
&\leq Ce^{-2\sigma t} \parallel \nabla\eta \parallel^2 + C \parallel g(t) \parallel^2 .
\end{aligned} \tag{3.49}$$

Let $s \leq \tau$ and $t \geq 2$. Multiplying (3.49) by $e^{\sigma t}$, then relabeling t as ξ and integrating it with respect to ξ over (s, τ) with $\tau - 1 \leq s \leq \tau$, we find that

$$\begin{aligned}
&e^{\sigma\tau} \|\nabla\eta(\tau, \tau - t, \eta_0(\tau - t))\|^2 \\
&\leq e^{\sigma s} \|\nabla\eta(s, \tau - t, \eta_0(\tau - t))\|^2 + C\int_s^\tau e^{\sigma\xi} \|g(\xi)\|^2 \,\mathrm{d}\xi \\
&\quad + C\int_s^\tau e^{-\sigma\xi} \|\nabla\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}\xi \\
&\quad + \sigma\int_s^\tau e^{\sigma\xi} \|\nabla\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}\xi \\
&\leq e^{\sigma s} \|\nabla\eta(s, \tau - t, \eta_0(\tau - t))\|^2 + C\int_{\tau-1}^\tau e^{\sigma\xi} \|g(\xi)\|^2 \,\mathrm{d}\xi \\
&\quad + Ce^{-2\sigma(\tau-1)}\int_{\tau-1}^\tau e^{\sigma\xi} \|\nabla\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}\xi \\
&\quad + \sigma\int_{\tau-1}^\tau e^{\sigma\xi} \|\nabla\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}\xi.
\end{aligned} \tag{3.50}$$

We now integrate (3.50) with respect to s over $(\tau - 1, \tau)$, and according to (3.21), we have

$$\begin{aligned}
&e^{\sigma\tau} \|\nabla\eta(\tau, \tau - t, \eta_0(\tau - t))\|^2 \\
&\leq \int_{\tau-1}^\tau e^{\sigma s} \|\nabla\eta(s, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}s + C\int_{-\infty}^\tau e^{\sigma\xi} \|g(\xi)\|^2 \,\mathrm{d}\xi \\
&\quad + Ce^{-2\sigma(\tau-1)}\int_{\tau-1}^\tau e^{\sigma\xi} \|\nabla\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}\xi \\
&\quad + \sigma\int_{\tau-1}^\tau e^{\sigma\xi} \|\nabla\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2 \,\mathrm{d}\xi \\
&\leq C[1 + e^{-2\sigma(\tau-1)}]\int_{-\infty}^\tau e^{\sigma\xi}[\|f(\xi)\|^2 + \|g(\xi)\|^2]\mathrm{d}\xi \\
&\leq C[1 + e^{-2\sigma\tau}]\int_{-\infty}^\tau e^{\sigma\xi}[\|f(\xi)\|^2 + \|g(\xi)\|^2]\mathrm{d}\xi.
\end{aligned} \tag{3.51}$$

So

$$\begin{aligned}
&\|\nabla\eta(\tau, \tau - t, \eta_0(\tau - t))\|^2 \\
&\qquad \leq C[e^{-\sigma\tau} + e^{-3\sigma\tau}]\int_{-\infty}^\tau e^{\sigma\xi}[\|f(\xi)\|^2 + \|g(\xi)\|^2]\mathrm{d}\xi,
\end{aligned} \tag{3.52}$$

which completes the proof. □

Lemma 3.4. *For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,*

$$\begin{aligned}
&\int_{\tau-1}^\tau e^{\sigma\xi}[\|\Delta\zeta(\xi, \tau - t, \zeta_0(\tau - t))\|^2 + \|\Delta\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2]\mathrm{d}\xi \\
&\qquad \leq C[1 + e^{-2\sigma\tau}]\int_{-\infty}^\tau e^{\sigma\xi}[\|f(\xi)\|^2 + \|g(\xi)\|^2]\mathrm{d}\xi,
\end{aligned} \tag{3.53}$$

$$\begin{aligned}\int_{\tau-1}^{\tau}[\| \Delta\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 + \| \Delta\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2]\mathrm{d}\xi \\ \leq C[e^{-\sigma\tau}+e^{-3\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi,\end{aligned} \tag{3.54}$$

where $(\zeta_0(\tau-t),\eta_0(\tau-t)) \in D(\tau-t)$*, and* C *is a positive constant independent of* τ *or* D*.*

Proof. Like the proof of Lemma 3.3, we can also prove that: for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,

$$\begin{aligned}\| \nabla\zeta(\tau-1,\tau-t,\zeta_0(\tau-t)) \|^2 + \| \nabla\eta(\tau-1,\tau-t,\eta_0(\tau-t)) \|^2 \\ \leq C[e^{-\sigma\tau}+e^{-3\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi,\end{aligned} \tag{3.55}$$

where $(\zeta_0(\tau-t),\eta_0(\tau-t)) \in D(\tau-t)$, and C is a positive constant independent of τ or D.

Now, multiplying (3.42) by $e^{\sigma t}$ and then integrating it over $(\tau-1,\tau)$, according to (3.55) and (3.21), we have

$$\begin{aligned}&\int_{\tau-1}^{\tau} e^{\sigma\xi} \| \Delta\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ &\quad\leq e^{\sigma(\tau-1)} \| \nabla\zeta(\tau-1,\tau-t,\zeta_0(\tau-t)) \|^2 + C\int_{\tau-1}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \,\mathrm{d}\xi \\ &\qquad+\sigma\int_{\tau-1}^{\tau} e^{\sigma\xi} \| \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ &\qquad+C\int_{\tau-1}^{\tau} e^{-\sigma\xi} \| \nabla\zeta(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ &\qquad+C\int_{\tau-1}^{\tau} e^{\sigma\xi} \| \nabla\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ &\quad\leq C[1+e^{-2\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi.\end{aligned} \tag{3.56}$$

Similarly, multiplying (3.49) by $e^{\sigma t}$ and then integrating it over $(\tau-1,\tau)$, according to (3.55) and (3.21), we can get

$$\begin{aligned}\int_{\tau-1}^{\tau} e^{\sigma\xi} \| \Delta\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ \leq C[1+e^{-2\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi.\end{aligned} \tag{3.57}$$

From (3.56) and (3.57) we can get (3.53) immediately. And (3.54) is straightforward from (3.53). □

Lemma 3.5. *For every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$*, there exists* $T = T(\tau, D) > 2$ *such that for all* $t \geq T$*,*

$$\begin{aligned}\int_{\tau-1}^{\tau}[\| \zeta_\xi(\xi,\tau-t,\zeta_0(\tau-t)) \|^2 + \| \eta_\xi(\xi,\tau-t,\eta_0(\tau-t)) \|^2]\mathrm{d}\xi \\ \leq C[e^{-\sigma\tau}+e^{-2\sigma\tau}+e^{-3\sigma\tau}+e^{-4\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi,\end{aligned} \tag{3.58}$$

where $(\zeta_0(\tau-t),\eta_0(\tau-t))\in D(\tau-t)$*, and* C *is a positive constant independent of* τ *or* D*.*

Proof. From (1.6) and (1.11), we have

$$\begin{aligned}\| \eta_t \| &\leq \frac{1}{P_r} \| \Delta\eta \| + \| J(\Psi,\eta) \| + \| g(t) \| \\ &\leq \frac{1}{P_r} \| \Delta\eta \| + C \| \zeta \| \| \Delta\eta \| + \| g(t) \| .\end{aligned}$$

So

$$\| \eta_t \|^2 \leq C[\| \Delta\eta \|^2 + \| \zeta \|^2 \| \Delta\eta \|^2 + \| g(t) \|^2]. \tag{3.59}$$

Relabeling t as ξ and integrating (3.59) with respect to ξ over $(\tau-1,\tau)$, by Lemma 3.1 and Lemma 3.4, we have

$$\begin{aligned}&\int_{\tau-1}^{\tau} \|\eta_\xi(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ &\quad\leq C\int_{\tau-1}^{\tau} \| \Delta\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi + C\int_{\tau-1}^{\tau} \| g(\xi) \|^2 \,\mathrm{d}\xi \\ &\qquad + C\int_{\tau-1}^{\tau} e^{-\sigma\xi} \| \Delta\eta(\xi,\tau-t,\eta_0(\tau-t)) \|^2 \,\mathrm{d}\xi \\ &\quad\leq C[e^{-\sigma\tau}+e^{-3\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi \\ &\qquad + Ce^{-\sigma(\tau-1)}[e^{-\sigma\tau}+e^{-3\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi \\ &\qquad + Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \| g(\xi) \|^2 \,\mathrm{d}\xi \\ &\quad\leq C[e^{-\sigma\tau}+e^{-2\sigma\tau}+e^{-3\sigma\tau}+e^{-4\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2]\mathrm{d}\xi.\end{aligned}$$

From (1.4) and (1.11), we have

$$\begin{aligned}\| \zeta_t \| &\leq \| \Delta\zeta \| + \| J(\Psi,\zeta) \| + \frac{R_a}{P_r} \| \nabla\eta \| + \| f(t) \| \\ &\leq \| \Delta\zeta \| + C \| \zeta \| \| \Delta\zeta \| + \frac{R_a}{P_r} \| \nabla\eta \| + \| f(t) \| .\end{aligned}$$

So

$$\| \zeta_t \|^2 \leq C[\| \Delta\zeta \|^2 + \| \zeta \|^2 \| \Delta\zeta \|^2 + \| \nabla\eta \|^2 + \| f(t) \|^2]. \tag{3.60}$$

Relabeling t as ξ and integrating (3.60) with respect to ξ over $(\tau-1,\tau)$, by Lemma 3.1, Corollary 1 and Lemma 3.4, we have

$$\begin{aligned}
&\int_{\tau-1}^{\tau} \|\zeta_\xi(\xi,\tau-t,\zeta_0(\tau-t))\|^2 \,\mathrm{d}\xi \\
&\quad\le C\int_{\tau-1}^{\tau} \|\Delta\zeta(\xi,\tau-t,\zeta_0(\tau-t))\|^2 \,\mathrm{d}\xi \\
&\qquad + C\int_{\tau-1}^{\tau} \|\nabla\eta(\xi,\tau-t,\eta_0(\tau-t))\|^2 \,\mathrm{d}\xi + C\int_{\tau-1}^{\tau} \|f(\xi)\|^2 \,\mathrm{d}\xi \\
&\qquad + C\int_{\tau-1}^{\tau} e^{-\sigma\xi} \|\Delta\zeta(\xi,\tau-t,\zeta_0(\tau-t))\|^2 \,\mathrm{d}\xi \\
&\quad\le C[e^{-\sigma\tau}+e^{-2\sigma\tau}+e^{-3\sigma\tau}+e^{-4\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\|f(\xi)\|^2+\|g(\xi)\|^2]\mathrm{d}\xi.
\end{aligned}$$

□

Lemma 3.6. *Let $\frac{\mathrm{d}f}{\mathrm{d}t},\frac{\mathrm{d}g}{\mathrm{d}t}\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$. Then for every $\tau\in\mathbb{R}$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, there exists $T=T(\tau,D)>2$ such that for all $t\ge T$,*

$$\begin{aligned}
&\|\zeta_\tau(\tau,\tau-t,\zeta_0(\tau-t))\|^2+\|\eta_\tau(\tau,\tau-t,\eta_0(\tau-t))\|^2 \\
&\le\Bigg(C[e^{-\sigma\tau}+e^{-2\sigma\tau}+e^{-3\sigma\tau}+e^{-4\sigma\tau}]\int_{-\infty}^{\tau} e^{\sigma\xi}[\|f(\xi)\|^2+\|g(\xi)\|^2]\mathrm{d}\xi \\
&\quad + C\int_{\tau-1}^{\tau}[\|f_\xi(\xi)\|^2+\|g_\xi(\xi)\|^2]\mathrm{d}\xi\Bigg)\exp\Bigg\{C[e^{-\sigma\tau}+e^{-3\sigma\tau}]\cdot \\
&\quad \int_{-\infty}^{\tau} e^{\sigma\xi}[\|f(\xi)\|^2+\|g(\xi)\|^2]\mathrm{d}\xi+C\Bigg\},
\end{aligned}\tag{3.61}$$

where $(\zeta_0(\tau-t),\eta_0(\tau-t))\in D(\tau-t)$, and C is a positive constant independent of τ or D.

Proof. Let $\widetilde{\zeta}=\frac{\mathrm{d}\zeta}{\mathrm{d}t}$ and $\widetilde{\eta}=\frac{\mathrm{d}\eta}{\mathrm{d}t}$. Differentiating (1.4) and (1.6) with respect to t admits

$$\frac{\mathrm{d}\widetilde{\zeta}}{\mathrm{d}t}-\Delta\widetilde{\zeta}+J(\Psi_t,\zeta)+J(\Psi,\widetilde{\zeta})+\frac{R_a}{P_r}\frac{\partial\widetilde{\eta}}{\partial x}=f_t,\tag{3.62}$$

and

$$\frac{\mathrm{d}\widetilde{\eta}}{\mathrm{d}t}-\frac{1}{P_r}\Delta\widetilde{\eta}+J(\Psi_t,\eta)+J(\Psi,\widetilde{\eta})=g_t.\tag{3.63}$$

By taking the inner product of (3.63) with $\widetilde{\eta}$ in $L^2(\Omega)$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\eta}\|^2+\frac{1}{P_r}\|\nabla\widetilde{\eta}\|^2=-(J(\Psi_t,\eta),\widetilde{\eta})+(g_t,\widetilde{\eta}).\tag{3.64}$$

Let us estimate the first term on the right-hand side of (3.64). Due to

$$|(J(\Psi_t,\eta),\widetilde{\eta})|=\Big|\int_\Omega \Psi_{ty}\eta_x\widetilde{\eta}-\int_\Omega \Psi_{tx}\eta_y\widetilde{\eta}\Big|,$$

and

$$\begin{aligned}\Big|\int_\Omega \Psi_{ty}\eta_x\widetilde{\eta}\Big| \leq &\parallel \Psi_{ty} \parallel_4 \parallel \eta_x \parallel_4 \parallel \widetilde{\eta} \parallel \\ \leq &C \parallel \Psi_{ty} \parallel^{1/2} \parallel \Psi_{ty} \parallel_{H^1}^{1/2} \parallel \eta_x \parallel^{1/2} \parallel \eta_x \parallel_{H^1}^{1/2} \parallel \widetilde{\eta} \parallel \\ \leq &C \parallel \widetilde{\zeta} \parallel \parallel \Delta\eta \parallel \parallel \widetilde{\eta} \parallel \\ \leq &C \parallel \widetilde{\eta} \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2 \parallel \Delta\eta \parallel^2 .\end{aligned}$$

Similarly, we have

$$\Big|\int_\Omega \Psi_{tx}\eta_y\widetilde{\eta}\Big| \leq C \parallel \widetilde{\eta} \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2 \parallel \Delta\eta \parallel^2 .$$

So

$$|(J(\Psi_t,\eta),\widetilde{\eta})| \leq C \parallel \widetilde{\eta} \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2 \parallel \Delta\eta \parallel^2 . \tag{3.65}$$

Thanks to

$$|(g_t,\widetilde{\eta})| \leq C \parallel g_t \parallel^2 + C \parallel \widetilde{\eta} \parallel^2, \tag{3.66}$$

by taking (3.64)-(3.66) into account, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel \widetilde{\eta} \parallel^2 + \frac{2}{P_r} \parallel \nabla\widetilde{\eta} \parallel^2 \leq C \parallel \widetilde{\zeta} \parallel^2 \parallel \Delta\eta \parallel^2 + C \parallel \widetilde{\eta} \parallel^2 + C \parallel g_t \parallel^2 . \tag{3.67}$$

Taking the inner product of (3.62) with $\widetilde{\zeta}$ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel \widetilde{\zeta} \parallel^2 + \parallel \nabla\widetilde{\zeta} \parallel^2 = -(J(\Psi_t,\zeta),\widetilde{\zeta}) - \frac{R_a}{P_r}(\frac{\partial\widetilde{\eta}}{\partial x},\widetilde{\zeta}) + (f_t,\widetilde{\zeta}). \tag{3.68}$$

By arguments similar to deriving (3.65), we obtain that

$$|(J(\Psi_t,\zeta),\widetilde{\zeta})| \leq C \parallel \widetilde{\zeta} \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2 \parallel \Delta\zeta \parallel^2 . \tag{3.69}$$

And bacause

$$\Big|\frac{R_a}{P_r}(\frac{\partial\widetilde{\eta}}{\partial x},\widetilde{\zeta})\Big| \leq \frac{R_a}{P_r} \parallel \nabla\widetilde{\eta} \parallel \parallel \widetilde{\zeta} \parallel \leq \frac{1}{P_r} \parallel \nabla\widetilde{\eta} \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2, \tag{3.70}$$

$$|(f_t,\widetilde{\zeta})| \leq C \parallel f_t \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2, \tag{3.71}$$

so taking (3.68)-(3.71) into account, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel \widetilde{\zeta} \parallel^2 \leq \frac{2}{P_r} \parallel \nabla\widetilde{\eta} \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2 \parallel \Delta\zeta \parallel^2 + C \parallel \widetilde{\zeta} \parallel^2 + C \parallel f_t \parallel^2 . \tag{3.72}$$

Adding (3.67) to (3.72), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\parallel \widetilde{\zeta} \parallel^2 + \parallel \widetilde{\eta} \parallel^2) \leq &C \parallel \widetilde{\zeta} \parallel^2 (\parallel \Delta\zeta \parallel^2 + \parallel \Delta\eta \parallel^2) + C(\parallel \widetilde{\zeta} \parallel^2 + \parallel \widetilde{\eta} \parallel^2) \\ &+ C(\parallel f_t \parallel^2 + \parallel g_t \parallel^2) \\ \leq &C(\parallel \widetilde{\zeta} \parallel^2 + \parallel \widetilde{\eta} \parallel^2)(\parallel \Delta\zeta \parallel^2 + \parallel \Delta\eta \parallel^2 + 1) \\ &+ C(\parallel f_t \parallel^2 + \parallel g_t \parallel^2).\end{aligned} \tag{3.73}$$

By the uniform Gronwall lemma, Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}
&\|\widetilde{\zeta}(\tau,\tau-t,\zeta_0(\tau-t))\|^2+\|\widetilde{\eta}(\tau,\tau-t,\eta_0(\tau-t))\|^2\\
\leq&\Bigg(\int_{\tau-1}^{\tau}[\|\widetilde{\zeta}(\xi,\tau-t,\zeta_0(\tau-t))\|^2+\|\widetilde{\eta}(\xi,\tau-t,\eta_0(\tau-t))\|^2]\mathrm{d}\xi\\
&+C\int_{\tau-1}^{\tau}[\|f_\xi(\xi)\|^2+\|g_\xi(\xi)\|^2]\mathrm{d}\xi\Bigg)\cdot\exp\Bigg\{C\int_{\tau-1}^{\tau}[\|\Delta\zeta(\xi,\tau-t,\\
&\zeta_0(\tau-t))\|^2+\|\Delta\eta(\xi,\tau-t,\eta_0(\tau-t))\|^2]\mathrm{d}\xi+C\Bigg\}\\
\leq&\Bigg(C[e^{-\sigma\tau}+e^{-2\sigma\tau}+e^{-3\sigma\tau}+e^{-4\sigma\tau}]\int_{-\infty}^{\tau}e^{\sigma\xi}[\|f(\xi)\|^2+\|g(\xi)\|^2]\mathrm{d}\xi\\
&+C\int_{\tau-1}^{\tau}[\|f_\xi(\xi)\|^2+\|g_\xi(\xi)\|^2]\mathrm{d}\xi\Bigg)\exp\Bigg\{C[e^{-\sigma\tau}+e^{-3\sigma\tau}]\cdot\\
&\int_{-\infty}^{\tau}e^{\sigma\xi}[\|f(\xi)\|^2+\|g(\xi)\|^2]\mathrm{d}\xi+C\Bigg\}
\end{aligned}\tag{3.74}$$

□

4. **Existence of pullback attractors.** In this section, we will prove the existence of $\mathcal{D}_\sigma$-pullback attractors for problem (1.4)-(1.8) in $L^2(\Omega)\times L^2(\Omega)$ and $H^1(\Omega)\times H^1(\Omega)$, respectively.

Theorem 4.1. *Problem (1.4)-(1.8) has a unique $\mathcal{D}_\sigma$ -pullback global attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ in $L^2(\Omega)\times L^2(\Omega)$.*

Proof. For $\tau\in\mathbb{R}$, denote by

$$\begin{aligned}
B_1(\tau)=\Big\{&(\zeta,\eta)\in L^2(\Omega)\times L^2(\Omega):\|\zeta\|^2+\|\eta\|^2\\
&\leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau}e^{\sigma\xi}[\|f(\xi)\|^2+\|g(\xi)\|^2]\mathrm{d}\xi\Big\}.
\end{aligned}$$

Thanks to Lemma 3.1, we know that $B_1=\{B_1(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_\sigma$ is a $\mathcal{D}_\sigma$-pullback absorbing set for ϕ in $L^2(\Omega)\times L^2(\Omega)$.

Now, we will prove that for every $\tau\in\mathbb{R}, D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, and $t_n\to\infty$, $(\zeta_{0,n},\eta_{0,n})\in D(\tau-t_n)$, the sequence $\phi(t_n,\tau-t_n,(\zeta_{0,n},\eta_{0,n}))$ has a convergent subsequence in $L^2(\Omega)\times L^2(\Omega)$, i.e., ϕ is $\mathcal{D}_\sigma$-pullback asymptotically compact in $L^2(\Omega)\times L^2(\Omega)$. By Lemma 3.3, there exist $C=C(\tau)>0$ and $N=N(\tau,D)>0$ such that, for all $n\geq N$,

$$\|\phi(t_n,\tau-t_n,(\zeta_{0,n},\eta_{0,n}))\|_{H^1(\Omega)\times H^1(\Omega)}\leq C.\tag{4.1}$$

By (4.1) and the compactness of embedding $H^1(\Omega)\times H^1(\Omega)\hookrightarrow L^2(\Omega)\times L^2(\Omega)$, the sequence $\phi(t_n,\tau-t_n,(\zeta_{0,n},\eta_{0,n}))$ is precompact in $L^2(\Omega)\times L^2(\Omega)$, which completes the proof. □

Lemma 4.2. *Let $\frac{\mathrm{d}f}{\mathrm{d}t},\frac{\mathrm{d}g}{\mathrm{d}t}\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$. Then ϕ is $\mathcal{D}_\sigma$-pullback asymptotically compact in $H^1(\Omega)\times H^1(\Omega)$, that is, for every $\tau\in\mathbb{R}, D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, and $t_n\to\infty$, $(\zeta_{0,n},\eta_{0,n})\in D(\tau-t_n)$, the sequence $\phi(t_n,\tau-t_n,(\zeta_{0,n},\eta_{0,n}))$ has a convergent subsequence in $H^1(\Omega)\times H^1(\Omega)$.*

Proof. We will show that the sequence $\phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n}))$ has a Cauchy subsequence in $H^1(\Omega) \times H^1(\Omega)$. By Theorem 4.1, $\phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n}))$ is precompact in $L^2(\Omega) \times L^2(\Omega)$. This means that, up to a subsequence,

$$(\zeta(\tau, \tau - t_n, \zeta_{0,n}), \eta(\tau, \tau - t_n, \eta_{0,n})) \text{ is a Cauchy sequence in } L^2(\Omega) \times L^2(\Omega). \tag{4.2}$$

From (1.4) we find that, for any $n, m \geq 1$,

$$\begin{aligned} &-(\Delta\zeta\big(\tau, \tau - t_n, \zeta_{0,n}) - \Delta\zeta(\tau, \tau - t_m, \zeta_{0,m})\big) \\ &\quad = -\big(\zeta_\tau(\tau, \tau - t_n, \zeta_{0,n}) - \zeta_\tau(\tau, \tau - t_m, \zeta_{0,m})\big) \\ &\qquad - \big(J(\Psi, \zeta(\tau, \tau - t_n, \zeta_{0,n})) - J(\Psi, \zeta(\tau, \tau - t_m, \zeta_{0,m}))\big) \\ &\qquad - \frac{R_a}{P_r}\big(\eta_x(\tau, \tau - t_n, \eta_{0,n}) - \eta_x(\tau, \tau - t_m, \eta_{0,m})\big). \end{aligned} \tag{4.3}$$

Multiplying (4.3) by $\zeta(\tau, \tau - t_n, \zeta_{0,n}) - \zeta(\tau, \tau - t_m, \zeta_{0,m})$ and then integrating it on Ω, by (1.10), we get

$$\begin{aligned} &\|\nabla\zeta(\tau, \tau - t_n, \zeta_{0,n}) - \nabla\zeta(\tau, \tau - t_m, \zeta_{0,m})\,\|^2 \\ &\leq \|\,\zeta_\tau(\tau, \tau - t_n, \zeta_{0,n}) - \zeta_\tau(\tau, \tau - t_m, \zeta_{0,m})\,\| \\ &\quad \cdot \|\,\zeta(\tau, \tau - t_n, \zeta_{0,n}) - \zeta(\tau, \tau - t_m, \zeta_{0,m})\,\| \\ &\quad + \frac{R_a}{P_r}\,\|\,\nabla\eta(\tau, \tau - t_n, \eta_{0,n}) - \nabla\eta(\tau, \tau - t_m, \eta_{0,m})\,\| \\ &\quad \cdot \|\,\zeta(\tau, \tau - t_n, \zeta_{0,n}) - \zeta(\tau, \tau - t_m, \zeta_{0,m})\,\|\,. \end{aligned} \tag{4.4}$$

By Lemma 3.3 and Lemma 3.6, there is $N = N(\tau, D) > 0$ such that, for all $n \geq N$,

$$\|\,\nabla\eta(\tau, \tau - t_n, \eta_{0,n}) - \nabla\eta(\tau, \tau - t_m, \eta_{0,m})\,\| \leq C$$

and

$$\|\,\zeta_\tau(\tau, \tau - t_n, \zeta_{0,n}) - \zeta_\tau(\tau, \tau - t_m, \zeta_{0,m})\,\| \leq C,$$

which along with (4.2) and (4.4) implies

$$\nabla\zeta(\tau, \tau - t_n, \zeta_{0,n}) \text{ is a Cauchy sequence in } L^2(\Omega). \tag{4.5}$$

From (1.6) we find that, for any $n, m \geq 1$,

$$\begin{aligned} &-\frac{1}{P_r}\big(\Delta\eta(\tau, \tau - t_n, \eta_{0,n}) - \Delta\eta(\tau, \tau - t_m, \eta_{0,m})\big) \\ &\quad = -\big(\eta_\tau(\tau, \tau - t_n, \eta_{0,n}) - \eta_\tau(\tau, \tau - t_m, \eta_{0,m})\big) \\ &\qquad - \big(J(\Psi, \eta(\tau, \tau - t_n, \eta_{0,n})) - J(\Psi, \eta(\tau, \tau - t_m, \eta_{0,m}))\big). \end{aligned} \tag{4.6}$$

Multiplying (4.6) by $\eta(\tau, \tau - t_n, \eta_{0,n}) - \eta(\tau, \tau - t_m, \eta_{0,m})$ and then integrating it on Ω, by (1.10), we get

$$\begin{aligned} &\frac{1}{P_r}\,\|\,\nabla\eta(\tau, \tau - t_n, \eta_{0,n}) - \nabla\eta(\tau, \tau - t_m, \eta_{0,m})\,\|^2 \\ &\quad \leq \|\,\eta_\tau(\tau, \tau - t_n, \eta_{0,n}) - \eta_\tau(\tau, \tau - t_m, \eta_{0,m})\,\| \\ &\qquad \cdot \|\,\eta(\tau, \tau - t_n, \eta_{0,n}) - \eta(\tau, \tau - t_m, \eta_{0,m})\,\| \end{aligned} \tag{4.7}$$

By Lemma 3.6, there is $N = N(\tau, D) > 0$ such that, for all $n \geq N$,

$$\|\,\eta_\tau(\tau, \tau - t_n, \eta_{0,n}) - \eta_\tau(\tau, \tau - t_m, \eta_{0,m})\,\| \leq C,$$

which along with (4.2) and (4.7) implies

$$\nabla\eta(\tau, \tau - t_n, \eta_{0,n}) \text{ is a Cauchy sequence in } L^2(\Omega). \tag{4.8}$$

Then (4.5) and (4.8) imply that $\big(\zeta(\tau,\tau-t_n,\zeta_{0,n}),\eta(\tau,\tau-t_n,\eta_{0,n})\big)$ is a Cauchy sequence in $H^1(\Omega)\times H^1(\Omega)$. □

Theorem 4.3. *Let $\frac{\mathrm{d}f}{\mathrm{d}t},\frac{\mathrm{d}g}{\mathrm{d}t}\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$. Then problem (1.4)-(1.8) has a $\mathcal{D}_\sigma$-pullback global attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ in $H^1(\Omega)\times H^1(\Omega)$, that is, for all $\tau\in\mathbb{R}$,*
(1) $\mathcal{A}(\tau)$ is compact in $H^1(\Omega)\times H^1(\Omega)$;
(2) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is invariant, that is,

$$\phi(t,\tau,\mathcal{A}(\tau))=\mathcal{A}(t+\tau),\forall t\geq 0;$$

(3) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ attracts every set in $\mathcal{D}_\sigma$ with respect to the $H^1(\Omega)\times H^1(\Omega)$ norm; that is, for all $B=\{B(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_\sigma$,

$$\lim_{t\to\infty}\mathrm{dist}_{H^1(\Omega)\times H^1(\Omega)}(\phi(t,\tau-t,B(\tau-t)),\mathcal{A}(\tau))=0.$$

Proof. We will prove that the attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ in $L^2(\Omega)\times L^2(\Omega)$ obtained in Theorem 4.1 is actually a $\mathcal{D}_\sigma$-pullback global attractor for ϕ in $H^1(\Omega)\times H^1(\Omega)$. And the proof is similar with the proof of Theorem 6.4 in [9], so we omit it. □

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REFERENCES

[1] B. Guo, *Spectral method for solving the two-dimensional Newton-Boussinesq equations*, Acta. Math. Appl. Sinica (English Ser.), **5** (1989), 208–218.
[2] B. Guo, *Nonlinear Galerkin methods for solving two-dimensional Newton-Boussinesq equations*, Chinese Ann. Math. Ser. B, **16** (1995), 379–390.
[3] B. Guo and B. Wang, *Gevrey class regularity and approximate inertial manifolds for the Newton-Boussinesq equations*, Chinese Ann. Math. Ser. B, **19** (1998), 179–188.
[4] B. Guo and B. Wang, *Approximate inertial manifolds to the Newton-Boussinesq equations*, J. Partial Differential Equations, **9** (1996), 237–250.
[5] G. Fucci, B. Wang and P. Singh, *Asymptotic behavior of the Newton-Boussinesq equation in a two-dimensional channel*, Nonlinear Anal., **70** (2009), 2000–2013.
[6] T. Caraballo, G. Łukaszewicz and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Anal., **64** (2006), 484–498.
[7] T. Caraballo, G. Łukaszewicz and J. Real, *Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains*, C. R. Acad. Sci. Paris, **342** (2006), 263–268.
[8] B. Wang, *Pullback attractors for the non-autonomous FitzHugh-Nagumo system on unbounded domains*, Nonlinear Anal., **70** (2009), 3799–3815.
[9] B. Wang and R. Jones, *Asymptotic behavior of a class of non-autonomous degenerate parabolic equations*, Nonlinear Anal., **72** (2010), 3887–3902.

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