PULLBACK $\mathcal{D}$-ATTRACTORS FOR THE NON-AUTONOMOUS NEWTON-BOUSSINESQ EQUATION IN TWO-DIMENSIONAL BOUNDED DOMAIN

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Abstract. We investigate the asymptotic behavior of solutions of a class of non-autonomous Newton-Boussinesq equation in two-dimensional bounded domain. The existence of pullback global attractors is proved in $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$, respectively.

1. Introduction. In this paper, we investigate the asymptotic behavior of solutions of the non-autonomous Newton-Boussinesq equation. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Consider the system of equations defined in $\Omega \times \mathbb{R}^+$:

$$
\begin{align*}
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} &= \Delta \zeta - \frac{R_a}{P_r} \frac{\partial \eta}{\partial x} + f(x, y, t), \\
\Delta \Psi &= \zeta, \ u = \Psi_y, v = -\Psi_x, \\
\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} &= \frac{1}{P_r} \Delta \eta + g(x, y, t),
\end{align*}
$$

where $\vec{u} = (u, v)$ is the velocity field of the fluid, $\eta$ is the flow temperature, $\Psi$ is the flow function and $\zeta$ is the vortex. The positive constants $P_r$ and $R_a$ are the Prandtl number and the Rayleigh number, respectively. $f$ and $g$ are external terms.

Notice that system (1.1)-(1.3) can be rewritten as follows: for every $(x, y) \in \Omega$ and $t > 0$,

$$
\begin{align*}
\frac{\partial \zeta}{\partial t} - \Delta \zeta + J(\Psi, \zeta) + \frac{R_a}{P_r} \frac{\partial \eta}{\partial x} &= f(x, y, t), \\
\Delta \Psi &= \zeta, \\
\frac{\partial \eta}{\partial t} - \frac{1}{P_r} \Delta \eta + J(\Psi, \eta) &= g(x, y, t),
\end{align*}
$$

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with the boundary conditions
\[ \zeta|_{\partial\Omega} = 0, \eta|_{\partial\Omega} = 0, \Psi|_{\partial\Omega} = 0, \tag{1.7} \]
and the initial conditions
\[ \zeta(x, y, \tau) = \zeta_\tau(x, y), \eta(x, y, \tau) = \eta_\tau(x, y), \tag{1.8} \]
where the function \( J \) is given by
\[ J(u, v) = u_y v_x - u_x v_y. \tag{1.9} \]

The Newton-Boussinesq equation describes many physical phenomena such as Benard flow. If the domain is bounded, the existence, uniqueness and the asymptotic behavior of solutions of system (1.1)-(1.3) have been studied by several authors (see [1]-[4]). And in [5], the authors have proved the existence of a global attractor for the autonomous Newton-Boussinesq equation defined in a two-dimensional channel. As far as we know, there is no results on uniform or pullback attractors for non-autonomous Newton-Boussinesq equation.

In this paper, we will investigate the asymptotic behavior of solutions of non-autonomous Newton-Boussinesq equation when the system is defined in a two-dimensional bounded domain. In particular, we will prove that the system has pullback attractors in \( L^2(\Omega) \times L^2(\Omega) \) and \( H^1(\Omega) \times H^1(\Omega) \), respectively. And the pullback asymptotic compactness of the system will be proved by using uniform a priori estimates of solutions.

The existence and uniqueness of solutions for problem (1.4)-(1.8) can be proved by a standard method as in [1] for the autonomous case. More precisely, for every \((\zeta_\tau, \eta_\tau) \in L^2(\Omega) \times L^2(\Omega)\), problem (1.4)-(1.8) has a unique solution \((\zeta, \eta)\) such that for every \(T > 0\),
\[ \zeta \in C^0([\tau, \infty), L^2(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)), \eta \in C^0([\tau, \infty), L^2(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)). \]
Furthermore, the solution is continuous with respect to initial data \((\zeta_\tau, \eta_\tau)\) in \( L^2(\Omega) \times L^2(\Omega) \).

It is easy to verify that \( J \) satisfies:
\[ \int_\Omega J(u, v) \, v \, dx \, dy = 0, \quad \text{for all } u \in H^1(\Omega), \quad v \in H^2(\Omega) \cap H^1_0(\Omega), \tag{1.10} \]
\[ \| J(u, v) \| \leq C \| u \|_{H^2} \| v \|_{H^2}, \quad \text{for all } u \in H^2(\Omega), \quad v \in H^2(\Omega), \tag{1.11} \]
\[ \| J(u, v) \| \leq C \| u \|_{H^3} \| \nabla v \|, \quad \text{for all } u \in H^3(\Omega), \quad v \in H^1(\Omega). \tag{1.12} \]
Throughout this paper, we frequently use the Poincaré inequality
\[ \| u \| \leq \lambda \| \nabla u \|, \quad \forall u \in H^1_0(\Omega), \]
where \( \lambda \) is a positive constant.

The following notations will be used throughout this paper. We denote by \( \| \cdot \| \) and \((\cdot, \cdot)\) the norm and inner product in \( L^2(\Omega) \) and use \( \| \cdot \|_p \) to denote the norm in \( L^p(\Omega) \) \( (p \neq 2) \). The norm of any Banach space \( X \) is written as \( \| \cdot \|_X \). The letter \( C \) is a generic positive constant which may change its values from line to line or even in the same line.
2. Preliminaries and abstract results. In this section, we recall some definitions and results concerning the pullback attractor. The reader is referred to [6]-[9] for details. Let \( \Theta \) be a non-empty set, \( X \) be a metric space. And we denote by \( \mathcal{D} \) the collection of families of subsets of \( X \):

\[
\mathcal{D} = \{ D = \{ D(\omega) \}_{\omega \in \Theta} : D(\omega) \subseteq X \text{ for every } \omega \in \Theta \}.
\]

**Definition 2.1.** A family of mapping \( \{ \theta_t \}_{t \in \mathbb{R}} \) from \( \Theta \) to itself is called a family of shift operators on \( \Theta \) if \( \{ \theta_t \}_{t \in \mathbb{R}} \) satisfies the group properties:

1. \( \theta_0 \omega = \omega, \ \forall \omega \in \Theta; \)
2. \( \theta_t(\theta_s \omega) = \theta_{t+s} \omega, \forall \omega \in \Theta \text{ and } t, s \in \mathbb{R}. \)

**Definition 2.2.** Let \( \{ \theta_t \}_{t \in \mathbb{R}} \) be a family of shift operators on \( \Theta \). Then a continuous \( \theta \)-cocycle \( \phi \) on \( X \) is a mapping

\[
\phi : \mathbb{R}^+ \times \Theta \times X \to X, (t, \omega, x) \mapsto \phi(t, \omega, x),
\]

which satisfies, for all \( \omega \in \Theta \) and \( t, \tau \in \mathbb{R}^+ \),

1. \( \phi(0, \omega, \cdot) \) is the identity on \( X \);
2. \( \phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau \omega, \cdot) \circ \phi(\tau, \omega, \cdot) \);
3. \( \phi(t, \cdot, \cdot) : X \to X \) is continuous.

**Definition 2.3.** Let \( \mathcal{D} \) be a collection of families of subsets of \( X \). Then \( \mathcal{D} \) is called inclusion-closed if \( D = \{ D(\omega) \}_{\omega \in \Theta} \in \mathcal{D} \) and \( \mathcal{D}_\omega = \{ D(\omega) \subseteq X : \omega \in \Theta \} \) with \( D(\omega) \subseteq D(\omega) \) for all \( \omega \in \Theta \) imply that \( D \in \mathcal{D} \).

**Definition 2.4.** Let \( \mathcal{D} \) be a collection of families of subsets of \( X \) and \( \{ K(\omega) \}_{\omega \in \Theta} \in \mathcal{D} \). Then \( \{ K(\omega) \}_{\omega \in \Theta} \) is called a pullback absorbing set for \( \phi \) in \( \mathcal{D} \) if, for every \( B \in \mathcal{D} \) and \( \omega \in \Theta \), there exists \( T(\omega, B) > 0 \) such that

\[
\phi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)) \subseteq K(\omega) \text{ for all } t \geq T(\omega, B).
\]

**Definition 2.5.** Let \( \mathcal{D} \) be a collection of families of subsets of \( X \). Then \( \phi \) is said to be \( \mathcal{D} \)-pullback asymptotically compact in \( X \) if, for every \( \omega \in \Theta \),

\[
\{ \phi(t_n, \theta_{-t_n} \omega, x_n) \}_{n=1}^\infty \text{ has a convergent subsequence in } X \text{ whenever } t_n \to \infty, \text{ and } x_n \in B(\theta_{-t_n} \omega) \text{ with } \{ B(\omega) \}_{\omega \in \Theta} \in \mathcal{D}.
\]

**Definition 2.6.** Let \( \mathcal{D} \) be a collection of families of subsets of \( X \) and \( \{ A(\omega) \}_{\omega \in \Theta} \in \mathcal{D} \). Then \( \{ A(\omega) \}_{\omega \in \Theta} \) is called a \( \mathcal{D} \)-pullback global attractor for \( \phi \) if the following conditions are satisfied, for every \( \omega \in \Theta \),

1. \( A(\omega) \) is compact;
2. \( \{ A(\omega) \}_{\omega \in \Theta} \) is invariant, that is,
3. \( A(\omega) \) attracts every set in \( \mathcal{D} \), that is, for every \( B = \{ B(\omega) \}_{\omega \in \Theta} \in \mathcal{D} \),

\[
\lim_{t \to \infty} \text{dist}(\phi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)), A(\omega)) = 0,
\]

where \( \text{dist} \) is the Hausdorff semi-metric given by \( \text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|_X \) for any \( A \subseteq X \) and \( B \subseteq X \).

**Proposition 1.** Let \( \mathcal{D} \) be an inclusion-closed collection of families of subsets of \( X \) and \( \phi \) a continuous \( \theta \)-cocycle on \( X \). Suppose that \( \{ K(\omega) \}_{\omega \in \Theta} \in \mathcal{D} \) is a closed absorbing set for \( \phi \) in \( \mathcal{D} \) and \( \phi \) is \( \mathcal{D} \)-pullback asymptotically compact in \( X \). Then \( \phi \) has a unique \( \mathcal{D} \)-pullback global attractor \( \{ A(\omega) \}_{\omega \in \Theta} \in \mathcal{D} \) which is given by
\[ A(\omega) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \phi(t, \theta_{-t\omega}, K(\theta_{-t\omega})). \]

To construct a cocycle \( \phi \) for problem (1.4)-(1.8), we denote by \( \Theta = \mathbb{R} \), and define a shift operator \( \theta_t \) on \( \Theta \) for every \( t \in \mathbb{R} \) by:

\[ \theta_t(\tau) = t + \tau, \forall \tau \in \mathbb{R}. \]

Let \( \phi \) be a mapping from \( \mathbb{R}^+ \times \Theta \times (L^2(\Omega) \times L^2(\Omega)) \) to \( L^2(\Omega) \times L^2(\Omega) \) given by

\[ \phi(t, \tau, (\zeta, \eta)) = (\zeta(t + \tau, \tau, \zeta), \eta(t + \tau, \tau, \eta)), \]

where \( t \geq 0, \tau \in \mathbb{R}, (\zeta, \eta) \in L^2(\Omega) \times L^2(\Omega) \), and \( (\zeta, \eta) \) is the solution of problem (1.4)-(1.8). By the uniqueness of solutions, we find that for every \( t, s \geq 0, \tau \in \mathbb{R} \) and \( (\zeta, \eta) \in L^2(\Omega) \times L^2(\Omega) \),

\[ \phi(t + s, \tau, (\zeta, \eta)) = \phi(t, s + \tau, (\phi(s, \tau, (\zeta, \eta)))) \]

Then we see that \( \phi \) is a continuous \( \theta \)-cocycle on \( L^2(\Omega) \times L^2(\Omega) \).

For convenience, if \( E \subseteq L^2(\Omega) \times L^2(\Omega) \), we denote

\[ \| E \| = \sup_{x \in E} \|x\|_{L^2(\Omega) \times L^2(\Omega)}. \]

Let \( D = \{D(t)\}_{t \in \mathbb{R}} \) be a family of subsets of \( L^2(\Omega) \times L^2(\Omega) \), i.e., \( D(t) \subseteq L^2(\Omega) \times L^2(\Omega) \) for every \( t \in \mathbb{R} \). In this paper, we are interested in a family \( D = \{D(t)\}_{t \in \mathbb{R}} \) satisfying

\[ \lim_{t \to -\infty} e^{\sigma t} \| D(t) \|^2 = 0, \]

where \( \sigma \) is a positive constant satisfying \( 0 < \sigma < \min\{\frac{1}{2\sqrt{\tau}}, \frac{1}{2\tau}\} \). We write the collection of all families satisfying (2.1) as \( D_\sigma \), that is,

\[ D_\sigma = \{D = \{D(t)\}_{t \in \mathbb{R}} : D \text{ satisfies (2.1)}\}. \]

As we will see later, for deriving the uniform estimates of solutions, we need the following conditions for the external force terms

\[ \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi < K < \infty, \forall \tau \in \mathbb{R}, \]

where \( K \) is a positive constant.

3. Uniform estimates of solutions. In this section, we derive uniform estimates of solutions of problem (1.4)-(1.8) when \( t \to \infty \). These estimates are necessary for proving the existence of a bounded pullback absorbing set and the pullback asymptotic compactness of the \( \theta \)-cocycle \( \phi \) associated with the system.

**Lemma 3.1.** For every \( \tau \in \mathbb{R} \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma \), there exists \( T = T(\tau, D) > 0 \) such that for all \( t \geq T \),

\[ \| \zeta(\tau - t, \zeta_0(\tau - t)) \|^2 + \| \eta(\tau - t, \eta_0(\tau - t)) \|^2 \leq C e^{-\sigma T} \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi, \]

and

\[ \int_{\tau}^{\tau + t} e^{\sigma \xi} \left( \| \zeta(\xi - t, \zeta_0(\tau - t)) \|^2 + \| \eta(\xi - t, \eta_0(\tau - t)) \|^2 \right) d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi. \]
\[
\int_{\tau-t}^{\tau} e^{\sigma \xi} \left( \| \nabla \xi(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \right) d\xi \\
\leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi,
\]
where \((\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t),\) and \(C\) is a positive constant independent of \(\tau\) or \(D\).

**Proof.** Taking the inner product of (1.6) with \(\eta\) in \(L^2(\Omega)\) and using (1.10) we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \eta \|^2 + \frac{1}{P_r} \| \nabla \eta \|^2 = \int_{\Omega} g \eta dx dy \leq \| g \| \| \eta \| \\
\leq \frac{1}{4\lambda^2 P_r} \| \eta \|^2 + C \| g(t) \|^2.
\]
So
\[
\frac{d}{dt} \| \eta \|^2 + \frac{1}{P_r} \| \nabla \eta \|^2 + \frac{1}{2\lambda^2 P_r} \| \eta \|^2 \leq C \| g(t) \|^2.
\]
Multiplying (3.5) by \(e^{\sigma t}\) and then integrating it between \(\tau - t\) and \(\tau\) with \(t \geq 0\), we get,
\[
e^{\sigma \tau} \| \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 + \frac{1}{P_r} \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi \\
\leq e^{\sigma(\tau-t)} \| \eta_0(\tau - t) \|^2 + (\sigma - \frac{1}{2\lambda^2 P_r}) \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi \\
+ C \int_{\tau-t}^{\tau} e^{\sigma \xi} \| g(\xi) \|^2 d\xi.
\]
Noticing that \((\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t)\) and \(D = \{ D(t) \}_{t \in \mathbb{R}} \in D_\sigma\), and we find that for every \(\tau \in \mathbb{R}\), there exists \(T_1 = T_1(\tau, D) > 0\) such that for all \(t \geq T_1\),
\[
e^{\sigma(\tau-t)} \| \eta_0(\tau - t) \|^2 \leq \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi,
\]
which along with (3.6) shows that, for all \(t \geq T_1\),
\[
\| \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 + \frac{1}{P_r} e^{-\sigma \tau} \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi
\]
\[
+ \left( \frac{1}{2\lambda^2 P_r} - \sigma \right) e^{-\sigma \tau} \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi
\]
\[
\leq C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi.
\]
From (3.7) we have
\[
\| \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 \leq C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi;
\]
\[
\int_{\tau-t}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi;
\]
\[
\int_{\tau-t}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] d\xi.
\]
\[
(3.10)
\]
Taking the inner product of (1.4) with \( \zeta \) in \( L^2(\Omega) \) and using (1.10) we obtain:
\[
\frac{1}{2} \frac{d}{dt} \parallel \zeta \parallel^2 + \parallel \nabla \zeta \parallel^2 + \frac{R_a}{F_r} \int_{\Omega} \eta_x \zeta dx dy = \int_{\Omega} f(t) \zeta dx dy,
\]
(3.11)
because
\[
\left| \frac{R_a}{F_r} \int_{\Omega} \eta_x \zeta dx dy \right| = \left| \frac{R_a}{F_r} \int_{\Omega} \zeta_x \eta dx dy \right| \leq \frac{R_a}{F_r} \parallel \nabla \zeta \parallel \parallel \eta \parallel \leq \frac{\parallel \nabla \zeta \parallel^2}{4} + C \parallel \eta \parallel^2,
\]
(3.12)
\[
\int_{\Omega} f(t) \zeta dx dy \leq \parallel f(t) \parallel \parallel \zeta \parallel \leq \lambda \parallel f(t) \parallel \parallel \nabla \zeta \parallel \leq \frac{\parallel \nabla \zeta \parallel^2}{4} + C \parallel f(t) \parallel^2,
\]
(3.13)
so taking (3.11)-(3.13) into account, we get
\[
\frac{d}{dt} \parallel \zeta \parallel^2 + \frac{1}{2} \parallel \nabla \zeta \parallel^2 + \frac{1}{2 \lambda^2} \parallel \zeta \parallel^2 \leq C \parallel \eta \parallel^2 + C \parallel f(t) \parallel^2.
\]
(3.14)
Multiplying (3.14) by \( e^{\sigma t} \) and then integrating it between \( \tau - t \) and \( \tau \), we obtain
\[
e^{\sigma \tau} \parallel \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \parallel^2 + \frac{1}{2} \int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 d\xi
\leq e^{\sigma(\tau - t)} \parallel \zeta_0(\tau - t) \parallel^2 + (\sigma - \frac{1}{2 \lambda^2}) \int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 d\xi
\]
(3.15)
\[
+ C \int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel \eta(\xi, \tau - t, \eta_0(\tau - t)) \parallel^2 d\xi + C \int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel f(\xi) \parallel^2 d\xi.
\]
Noticing that \( (\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t) \) and \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), we find that for every \( \tau \in \mathbb{R} \), there exists \( T_2 = T_2(\tau, D) > 0 \) such that for all \( t \geq T_2 \),
\[
e^{\sigma(\tau - t)} \parallel \zeta_0(\tau - t) \parallel^2 \leq \int_{-\infty}^{\tau} e^{\sigma \xi} [\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2] d\xi,
\]
which along with (3.15) and (3.9) shows that, for all \( t \geq T = \max\{T_1, T_2\} > 0 \),
\[
\parallel \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \parallel^2 + \frac{1}{2} \int_{\tau - t}^{\tau} e^{-\sigma \xi} \parallel \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 d\xi
\]
(3.16)
\[
+ \left( \frac{1}{2 \lambda^2} - \sigma \right) e^{-\sigma \tau} \int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 d\xi \leq C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} [\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2] d\xi.
\]
From (3.16) we have
\[
\parallel \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \parallel^2 \leq C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} [\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2] d\xi;
\]
(3.17)
\[
\int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2] d\xi.
\]
(3.18)
\[
\int_{\tau - t}^{\tau} e^{\sigma \xi} \parallel \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \parallel^2 d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} [\parallel f(\xi) \parallel^2 + \parallel g(\xi) \parallel^2] d\xi.
\]
(3.19)
Thus, the proof is completed.
Lemma 3.2. For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,

$$
\int_{\tau-2}^{\tau} e^{\xi}(\| \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2) d\xi 
\leq C \int_{-\infty}^{\tau} e^{\sigma \xi}(\| f(\xi) \|^2 + \| g(\xi) \|^2) d\xi;
$$

(3.20)

$$
\int_{\tau-2}^{\tau} e^{\xi}(\| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2) d\xi 
\leq C \int_{-\infty}^{\tau} e^{\sigma \xi}(\| f(\xi) \|^2 + \| g(\xi) \|^2) d\xi,
$$

(3.21)

where $(\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t)$, and $C$ is a positive constant independent of $\tau$ or $D$.

Proof. Note that (3.5) implies

$$
\frac{d}{dt} \| \eta \|^2 + \frac{1}{2\lambda^2 P_r} \| \eta \|^2 \leq C \| g(t) \|^2.
$$

(3.22)

Let $s \in [\tau - 2, \tau]$ and $t \geq 2$. Multiplying (3.22) by $e^{\sigma t}$, then relabeling $t$ as $\xi$ and integrating it with respect to $\xi$ over $(\tau - t, s)$, we have

$$
e^{\sigma s} \| \eta(s, \tau - t, \eta_0(\tau - t)) \|^2 + (\frac{1}{2\lambda^2 P_r} - \sigma) \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi 
\leq e^{\sigma(\tau-t)} \| \eta_0(\tau - t) \|^2 + C \int_{-\infty}^{\tau} e^{\sigma \xi} \| g(\xi) \|^2 d\xi.
$$

(3.23)

Noticing that $(\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t)$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in D_\sigma$, and we find that for every $\tau \in \mathbb{R}$, there exists $T_3 = T_3(\tau, D) > 2$ such that for all $t \geq T_3$,

$$
e^{\sigma(\tau-t)} \| \eta_0(\tau - t) \|^2 \leq \int_{-\infty}^{\tau} e^{\sigma \xi}(\| f(\xi) \|^2 + \| g(\xi) \|^2) d\xi.
$$

So

$$
e^{\sigma s} \| \eta(s, \tau - t, \eta_0(\tau - t)) \|^2 + \int_{\tau-t}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi 
\leq C \int_{-\infty}^{\tau} e^{\sigma \xi}(\| f(\xi) \|^2 + \| g(\xi) \|^2) d\xi,
$$

(3.24)

for any $s \in [\tau - 2, \tau]$.

Multiplying (3.5) by $e^{\sigma t}$ and integrating it over $(\tau - 2, \tau)$, by (3.24) we get that, for all $t \geq T_3$,

$$
e^{\sigma \tau} \| \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 + \frac{1}{P_r} \int_{\tau-2}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi 
+ (\frac{1}{2\lambda^2 P_r} - \sigma) \int_{\tau-2}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi 
\leq e^{\sigma(\tau-2)} \| \eta(\tau - 2, \tau - t, \eta_0(\tau - t)) \|^2 + C \int_{\tau-2}^{\tau} e^{\sigma \xi} \| g(\xi) \|^2 d\xi
$$

(3.25)

$$
\leq C \int_{-\infty}^{\tau} e^{\sigma \xi}(\| f(\xi) \|^2 + \| g(\xi) \|^2) d\xi.
$$
From (3.25) we have
\[
\int_{\tau-2}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \| ^2 \, d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} (\| f(\xi) \| ^2 + \| g(\xi) \| ^2) \, d\xi; \quad (3.26)
\]
\[
\int_{\tau-2}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \| ^2 \, d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} (\| f(\xi) \| ^2 + \| g(\xi) \| ^2) \, d\xi. \quad (3.27)
\]
Note that (3.14) implies
\[
\frac{d}{dt} \| \xi \| ^2 + \frac{1}{2\lambda^2} \| \xi \| ^2 \leq C \| \eta \| ^2 + C \| f(t) \| ^2. \quad (3.28)
\]
Let \( s \in [\tau - 2, \tau] \) and \( t \geq 2 \), multiplying (3.28) by \( e^{\sigma t} \), then relabeling \( t \) as \( \xi \) and integrating it with respect to \( \xi \) over \( (\tau - t, s) \), we have
\[
e^{\sigma s} \| \zeta(s, \tau - t, \zeta_0(\tau - t)) \| ^2 + \frac{1}{2\lambda^2} - \sigma \int_{\tau-t}^{s} e^{\sigma \xi} \| \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \| ^2 \, d\xi \leq e^{\sigma(\tau - t)} \| \zeta_0(\tau - t) \| ^2 + C \int_{\tau-t}^{s} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \| ^2 \, d\xi \] 
\[
+ C \int_{\tau-t}^{s} e^{\sigma \xi} \| f(\xi) \| ^2 \, d\xi. \quad (3.29)
\]
Noticing that \((\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t) \) and \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), and we find that for every \( \tau \in \mathbb{R} \), there exists \( T_4 = T_4(\tau, D) > 2 \) such that for all \( t \geq T_4 \),
\[
e^{\sigma(\tau - t)} \| \zeta_0(\tau - t) \| ^2 \leq \int_{-\infty}^{\tau} e^{\sigma \xi} (\| f(\xi) \| ^2 + \| g(\xi) \| ^2) \, d\xi,
\]
which along with (3.29) and (3.24) implies that: there exists \( T_5 = \max\{T_3, T_4\} > 2 \), such that for all \( t \geq T_5 \) and for any \( s \in [\tau - 2, \tau] \),
\[
e^{\sigma s} \| \zeta(s, \tau - t, \zeta_0(\tau - t)) \| ^2 \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} (\| f(\xi) \| ^2 + \| g(\xi) \| ^2) \, d\xi. \quad (3.30)
\]
Multiplying (3.14) by \( e^{\sigma t} \) and integrating it over \( (\tau - 2, \tau) \), using (3.26) and (3.30) we get
\[
e^{\sigma \tau} \| \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \| ^2 + \frac{1}{2} \int_{\tau-2}^{\tau} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \| ^2 \, d\xi 
\]
\[
+ \frac{1}{2\lambda^2} - \sigma \int_{\tau-2}^{\tau} e^{\sigma \xi} \| \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \| ^2 \, d\xi \leq e^{\sigma(\tau - 2)} \| \zeta(\tau - 2, \tau - t, \zeta_0(\tau - t)) \| ^2 
\]
\[
+ C \int_{\tau-2}^{\tau} e^{\sigma \xi} \| \eta(\xi, \tau - t, \eta_0(\tau - t)) \| ^2 \, d\xi + C \int_{\tau-2}^{\tau} e^{\sigma \xi} \| f(\xi) \| ^2 \, d\xi. \quad (3.31)
\]
Thanks to (3.31), we have
\[
\int_{\tau-2}^{\tau} e^{\sigma \xi} \| \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \| ^2 \, d\xi \leq C \int_{-\infty}^{\tau} e^{\sigma \xi} (\| f(\xi) \| ^2 + \| g(\xi) \| ^2) \, d\xi; \quad (3.32)
\]
Corollary 1. For every \( \tau \in \mathbb{R} \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), there exists \( T = T(\tau, D) > 2 \) such that for all \( t \geq T \),
\[
\int_{\tau-2}^{\tau} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau-t, \zeta_0(\tau-t)) \| ^2 \, d\xi \\
\leq C \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \| ^2 + \| g(\xi) \| ^2 \, d\xi,
\]
which completes the proof. \( \square \)

**Lemma 3.3.** For every \( \tau \in \mathbb{R} \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), there exists \( T = T(\tau, D) > 2 \) such that for all \( t \geq T \),
\[
\| \nabla \zeta(\tau-t, \zeta_0(\tau-t)) \| ^2 + \| \nabla \eta(\tau-t, \eta_0(\tau-t)) \| ^2 \\
\leq C[e^{-\sigma \xi} + e^{-3\sigma \xi}] \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \| ^2 + \| g(\xi) \| ^2 \, d\xi,
\]
where \( (\zeta_0(\tau-t), \eta_0(\tau-t)) \in D(\tau-t) \), and \( C \) is a positive constant independent of \( \tau \) or \( D \).

**Proof.** Taking the inner product of (1.4) with \( -\Delta \zeta \) in \( L^2(\Omega) \) we get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \zeta \| ^2 + \| \Delta \zeta \| ^2 = \int_{\Omega} J(\Psi, \zeta) \Delta \zeta \, dx dy + \frac{R_a}{T} \int_{\Omega} \eta \Delta \zeta \, dx dy \\
- \int_{\Omega} f \Delta \zeta \, dx dy.
\]
Notice that the first term on the right-hand side of (3.37) is given by
\[
\int_{\Omega} J(\Psi, \zeta) \Delta \zeta \, dx dy = \int_{\Omega} \Psi_y \zeta_x \Delta \zeta \, dx dy - \int_{\Omega} \Psi_x \zeta_y \Delta \zeta \, dx dy.
\]
We now estimate the first term on the right-hand side of (3.38). By Hölder inequality and Nirenberg-Gagliardo inequality, we have
\[
\int_{\Omega} \Psi_y \zeta_x \Delta \zeta \, dx dy \leq \| \Psi_y \|_4 \| \zeta_x \|_4 \| \Delta \zeta \| \\
\leq C \| \Psi_y \|^{1/2} \| \Psi_y \|^{1/2} \| \zeta_x \|^{1/2} \| \zeta_x \|^{1/2} \| \Delta \zeta \| \\
\leq C \| \Psi \|_{H^2} \| \nabla \zeta \|^{1/2} \| \Delta \zeta \|^{3/2} \leq C \| \zeta \| \| \nabla \zeta \|^{1/2} \| \Delta \zeta \|^{3/2} \\
\leq \frac{1}{8} \| \Delta \zeta \| ^2 + C \| \zeta \| ^4 \| \nabla \zeta \| ^2.
\]
Similarly, we have

\[
\int_{\Omega} \Psi x \zeta y \Delta \zeta \, dx \, dy \leq \frac{1}{8} \| \Delta \zeta \|^2 + C \| \zeta \|^4 \| \nabla \zeta \|^2. \tag{3.40}
\]

It is obvious that the last two terms on the right-hand side of (3.37) are bounded by

\[
\left| \frac{R_a}{P_r} \int_{\Omega} \eta \Delta \zeta \, dx \, dy \right| + \left| \int_{\Omega} f \Delta \zeta \, dx \, dy \right| \leq \frac{1}{4} \| \Delta \zeta \|^2 + C \| \nabla \eta \|^2 + C \| f \|^2. \tag{3.41}
\]

Now taking (3.37)-(3.41) into account, the usage of Lemma 3.1 leads to

\[
\frac{d}{dt} \| \nabla \zeta \|^2 + \| \Delta \zeta \|^2 \leq C \| \zeta \|^4 \| \nabla \zeta \|^2 + C \| \nabla \eta \|^2 + C \| f \|^2
\]

\[
\leq C \left[ r_0(t)^2 \| \nabla \zeta \|^2 + C \| \nabla \eta \|^2 + C \| f \|^2 \right]
\]

\[
\leq Ce^{-2\sigma t} \| \nabla \zeta \|^2 + C \| \nabla \eta \|^2 + C \| f \|^2,
\]

where \( r_0(t) = Ce^{-\sigma t} \int_{-\infty}^t e^{\sigma \xi} [\| f(\xi) \|^2 + \| g(\xi) \|^2] \, d\xi < CK e^{-\sigma t} \leq Ce^{-\sigma t} \). Let \( s \leq \tau \) and \( t \geq 2 \). Multiplying (3.42) by \( e^{\sigma t} \), then relabeling \( t \) as \( \xi \) and integrating it with respect to \( \xi \) over \((s, \tau)\) with \( \tau - 1 \leq s \leq \tau \), we find that

\[
e^{\sigma \tau} \| \nabla \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \|^2
\]

\[
\leq e^{\sigma s} \| \nabla \zeta(s, \tau - t, \zeta_0(\tau - t)) \|^2 + C \int_s^\tau e^{\sigma \xi} \| f(\xi) \|^2 \, d\xi
\]

\[
+ C \int_s^\tau e^{-\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 \, d\xi
\]

\[
+ C \int_s^\tau e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi
\]

\[
+ \sigma \int_s^\tau e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 \, d\xi
\]

\[
\leq e^{\sigma s} \| \nabla \zeta(s, \tau - t, \zeta_0(\tau - t)) \|^2 + C \int_{\tau - 1}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 \, d\xi
\]

\[
+ Ce^{-2\sigma(\tau - 1)} \int_{\tau - 1}^{\tau} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 \, d\xi
\]

\[
+ C \int_{\tau - 1}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi
\]

\[
+ \sigma \int_{\tau - 1}^{\tau} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 \, d\xi.
\]

\[
(3.43)
\]
We now integrate (3.43) with respect to $s$ over $(\tau - 1, \tau)$, and by (3.21) we have

$$
\begin{align*}
&\leq e^{\sigma \tau} \| \nabla \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \|^2 \\
&\leq \int_{\tau-1}^{\tau} e^{\sigma s} \| \nabla \zeta(s, \tau - t, \zeta_0(\tau - t)) \|^2 \, ds + C \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 \, d\xi \\
&+ Ce^{-2\sigma(\tau-1)} \int_{\tau-1}^{\tau} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 \, d\xi \\
&+ C \int_{\tau-1}^{\tau} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi \\
&+ \sigma \int_{\tau-1}^{\tau} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 \, d\xi \\
&\leq C[1 + e^{-2\sigma(\tau-1)}] \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 \, d\xi \\
&\leq C[1 + e^{-2\sigma}] \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi.
\end{align*}
$$

So

$$
\begin{align*}
&\| \nabla \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \|^2 \\
&\leq C[e^{-\sigma \tau} + e^{-3\sigma \tau}] \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi.
\end{align*}
$$

Taking the inner product of (1.6) with $-\Delta \eta$ in $L^2(\Omega)$ we get

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \nabla \eta \|^2 + \frac{1}{P_r} \| \Delta \eta \|^2 &= \int_{\Omega} J(\Psi, \eta) \Delta \eta \, dx dy - \int_{\Omega} g \Delta \eta \, dx dy. \\
\end{align*}
$$

By arguments similar to getting (3.39) and (3.40), we obtain that

$$
\begin{align*}
\left| \int_{\Omega} J(\Psi, \eta) \Delta \eta \, dx dy \right| &= \left| \int_{\Omega} \Psi_y \eta_x \Delta \eta \, dx dy - \int_{\Omega} \Psi_x \eta_y \Delta \eta \, dx dy \right| \\
&\leq \frac{1}{4P_r} \| \Delta \eta \|^2 + C \| \zeta \|^4 \| \nabla \eta \|^2,
\end{align*}
$$

and

$$
\begin{align*}
\left| \int_{\Omega} g \Delta \eta \, dx dy \right| &\leq \frac{1}{4P_r} \| \Delta \eta \|^2 + C \| g(t) \|^2.
\end{align*}
$$

Taking (3.46)-(3.48) into account, the usage of Lemma 3.1 leads to

$$
\begin{align*}
\frac{d}{dt} \| \nabla \eta \|^2 + \frac{1}{P_r} \| \Delta \eta \|^2 &\leq C \| \zeta \|^4 \| \nabla \eta \|^2 + C \| g(t) \|^2 \\
&\leq Ce^{-2\sigma t} \| \nabla \eta \|^2 + C \| g(t) \|^2.
\end{align*}
$$
Let \( s \leq t \) and \( t \geq 2 \). Multiplying (3.49) by \( e^{\sigma t} \), then relabeling \( t \) as \( \xi \) and integrating it with respect to \( \xi \) over \( (s, \tau) \) with \( \tau - 1 \leq s \leq \tau \), we find that
\[
e^{\sigma \tau} \| \nabla \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 \leq e^{\sigma s} \| \nabla \eta(s, \tau - t, \eta_0(\tau - t)) \|^2 + C \int_s^\tau e^{\sigma \xi} \| g(\xi) \|^2 \, d\xi \\
+ C \int_s^\tau e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi \\
+ e^{\sigma s} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 + C \int_{\tau - 1}^\tau e^{\sigma \xi} \| g(\xi) \|^2 \, d\xi \\
+ C e^{-2\sigma(\tau - 1)} \int_{\tau - 1}^\tau e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi \\
+ \sigma \int_{\tau - 1}^\tau e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi.
\]
(3.50)

We now integrate (3.50) with respect to \( s \) over \( (\tau - 1, \tau) \), and according to (3.21), we have
\[
e^{\sigma \tau} \| \nabla \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 \leq \int_{\tau - 1}^\tau e^{\sigma s} \| \nabla \eta(s, \tau - t, \eta_0(\tau - t)) \|^2 \, ds \\
+ C \int_{-\infty}^-\infty e^{\sigma \xi} \| g(\xi) \|^2 \, d\xi \\
+ C e^{-2\sigma(\tau - 1)} \int_{\tau - 1}^\tau e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi \\
+ \sigma \int_{\tau - 1}^\tau e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi.
\]
(3.51)

So
\[
\| \nabla \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 \leq C[1 + e^{-2\sigma(\tau - 1)}] \int_{-\infty}^-\infty e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \| d\xi \\
\leq C[1 + e^{-2\sigma\tau}] \int_{-\infty}^-\infty e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \| d\xi.
\]
(3.52)

which completes the proof. \(\square\)

**Lemma 3.4.** For every \( \tau \in \mathbb{R} \) and \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma \), there exists \( T = T(\tau, D) > 2 \) such that for all \( t \geq T \),
\[
\int_{\tau - 1}^\tau e^{\sigma \xi} \| \Delta \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \Delta \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \| d\xi \\
\leq C[1 + e^{-2\sigma\tau}] \int_{-\infty}^-\infty e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \| d\xi.
\]
(3.53)
Lemma 3.5. For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,
\[
\int_{\tau}^{t} \| \Delta \zeta(\xi, \tau - t, \zeta(\tau - t)) \|^2 + \| \Delta \eta(\xi, \tau - t, \eta(\tau - t)) \|^2 \, d\xi 
\leq C[e^{-\sigma \tau} + e^{-3\sigma \tau}] \int_{-\infty}^{T} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi,
\]
where $(\zeta(\tau - t), \eta(\tau - t)) \in D(\tau - t)$, and $C$ is a positive constant independent of $\tau$ or $D$.

Proof. Like the proof of Lemma 3.3, we can also prove that: for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,
\[
\int_{\tau}^{t} \| \nabla \zeta(\xi, \tau - t, \zeta(\tau - t)) \|^2 + \| \nabla \eta(\xi, \tau - t, \eta(\tau - t)) \|^2 
\leq C[e^{-\sigma \tau} + e^{-3\sigma \tau}] \int_{-\infty}^{T} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi,
\]
where $(\zeta(\tau - t), \eta(\tau - t)) \in D(\tau - t)$, and $C$ is a positive constant independent of $\tau$ or $D$.

Now, multiplying (3.42) by $e^{\sigma t}$ and then integrating it over $(\tau - 1, \tau)$, according to (3.55) and (3.21), we have
\[
\int_{\tau}^{t} e^{\sigma \xi} \| \Delta \zeta(\xi, \tau - t, \zeta(\tau - t)) \|^2 \, d\xi 
\leq e^{\sigma(\tau - 1)} \| \nabla \zeta(\xi, \tau - 1, \zeta(\tau - t)) \|^2 + C \int_{\tau}^{T} e^{\sigma \xi} \| f(\xi) \|^2 \, d\xi
\]
\[
+ \sigma \int_{\tau}^{T} e^{\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta(\tau - t)) \|^2 \, d\xi
\]
\[
+ C \int_{\tau}^{T} e^{-\sigma \xi} \| \nabla \zeta(\xi, \tau - t, \zeta(\tau - t)) \|^2 \, d\xi
\]
\[
+ C \int_{\tau}^{T} e^{\sigma \xi} \| \nabla \eta(\xi, \tau - t, \eta(\tau - t)) \|^2 \, d\xi
\]
\[
\leq C[1 + e^{-2\sigma \tau}] \int_{-\infty}^{T} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi.
\]

Similarly, multiplying (3.49) by $e^{\sigma t}$ and then integrating it over $(\tau - 1, \tau)$, according to (3.55) and (3.21), we can get
\[
\int_{\tau}^{t} e^{\sigma \xi} \| \Delta \eta(\xi, \tau - t, \eta(\tau - t)) \|^2 \, d\xi 
\leq C[1 + e^{-2\sigma \tau}] \int_{-\infty}^{T} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi.
\]

From (3.56) and (3.57) we can get (3.53) immediately. And (3.54) is straightforward from (3.53). \qed

Lemma 3.5. For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,
\[
\int_{\tau}^{t} \| \xi(\xi, \tau - t, \zeta(\tau - t)) \|^2 + \| \eta(\xi, \tau - t, \eta(\tau - t)) \|^2 \, d\xi
\leq C[e^{-\sigma \tau} + e^{-2\sigma \tau} + e^{-3\sigma \tau} + e^{-4\sigma \tau}] \int_{-\infty}^{T} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 \, d\xi,
\]
where \((\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t)\), and \(C\) is a positive constant independent of \(\tau\) or \(D\).

**Proof.** From (1.6) and (1.11), we have

\[
\| \eta_t \| \leq \frac{1}{P_r} \| \Delta \eta \| + \| J(\Psi, \eta) \| + \| g(t) \| \\
\leq \frac{1}{P_r} \| \Delta \eta \| + C \| \zeta \| \| \Delta \eta \| + \| g(t) \| .
\]

So

\[
\| \eta_t \|^2 \leq C[\| \Delta \eta \|^2 + \| \zeta \|^2 \| \Delta \eta \|^2 + \| g(t) \|^2].
\]

(3.59)

Relabeling \(t\) as \(\xi\) and integrating (3.59) with respect to \(\xi\) over \((\tau - 1, \tau)\), by Lemma 3.1 and Lemma 3.4, we have

\[
\int_{\tau-1}^{\tau} \| \eta_\xi(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi \\
\leq C \int_{\tau-1}^{\tau} \| \Delta \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi + C \int_{\tau-1}^{\tau} \| g(\xi) \|^2 \, d\xi \\
+ C \int_{\tau-1}^{\tau} e^{-\sigma \xi} \| \Delta \eta(\xi, \tau - t, \eta_0(\tau - t)) \|^2 \, d\xi \\
\leq C[e^{-\sigma \tau} + e^{-3\sigma \tau}] \int_{-\infty}^{\tau} e^{\sigma \xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2] \, d\xi \\
+ C e^{-\sigma(\tau-1)}[e^{-\sigma \tau} + e^{-3\sigma \tau}] \int_{-\infty}^{\tau} e^{\sigma \xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2] \, d\xi \\
+ C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} \| g(\xi) \|^2 \, d\xi \\
\leq C[e^{-\sigma \tau} + e^{-2\sigma \tau} + e^{-3\sigma \tau} + e^{-4\sigma \tau}] \int_{-\infty}^{\tau} e^{\sigma \xi}[\| f(\xi) \|^2 + \| g(\xi) \|^2] \, d\xi.
\]

From (1.4) and (1.11), we have

\[
\| \zeta_t \| \leq \| \Delta \zeta \| + \| J(\Psi, \zeta) \| + \frac{Ra}{Pr} \| \nabla \eta \| + \| f(t) \| \\
\leq \| \Delta \zeta \| + C \| \zeta \| \| \Delta \zeta \| + \frac{Ra}{Pr} \| \nabla \eta \| + \| f(t) \| .
\]

So

\[
\| \zeta_t \|^2 \leq C[\| \Delta \zeta \|^2 + \| \zeta \|^2 \| \Delta \zeta \|^2 + \| \nabla \eta \|^2 + \| f(t) \|^2].
\]

(3.60)
Relabeling $t$ as $\xi$ and integrating (3.60) with respect to $\xi$ over $(\tau - 1, \tau)$, by Lemma 3.1, Corollary 1 and Lemma 3.4, we have

\[
\int_{\tau - 1}^{\tau} \| \zeta_\xi(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 d\xi \\
\leq C \int_{\tau - 1}^{\tau} \| \Delta \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 d\xi \\
+ C \int_{\tau - 1}^{\tau} \| \eta_\xi(\xi, \tau - t, \eta_0(\tau - t)) \|^2 d\xi + C \int_{\tau - 1}^{\tau} \| f(\xi) \|^2 d\xi \\
+ C \int_{\tau - 1}^{\tau} e^{-\sigma \xi} \| \Delta \zeta(\xi, \tau - t, \zeta_0(\tau - t)) \|^2 d\xi \\
\leq C[e^{-\sigma t} + e^{-2\sigma t} + e^{-3\sigma t} + e^{-4\sigma t}] \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 d\xi.
\]

Lemma 3.6. Let \( \frac{df}{dt}, \frac{dg}{dx} \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \). Then for every $\tau \in \mathbb{R}$ and $D = \{ D(t) \}_{t \in \mathbb{R}} \in D_{\sigma}$, there exists $T = T(\tau, D) > 2$ such that for all $t \geq T$,

\[
\| \zeta(\tau, \tau - t, \zeta_0(\tau - t)) \|^2 + \| \eta(\tau, \tau - t, \eta_0(\tau - t)) \|^2 \\
\leq \left( C[e^{-\sigma t} + e^{-2\sigma t} + e^{-3\sigma t} + e^{-4\sigma t}] \int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 d\xi \right) \\
+ C \int_{\tau - 1}^{\tau} \left( \| f(\xi) \|^2 + \| g(\xi) \|^2 \right) d\xi \exp \left( C[e^{-\sigma t} + e^{-3\sigma t}] \right) \\
\int_{-\infty}^{\tau} e^{\sigma \xi} \| f(\xi) \|^2 + \| g(\xi) \|^2 d\xi + C,
\]

where $(\zeta_0(\tau - t), \eta_0(\tau - t)) \in D(\tau - t)$, and $C$ is a positive constant independent of $\tau$ or $D$.

Proof. Let $\tilde{\zeta} = \frac{d}{dt} \zeta$ and $\tilde{\eta} = \frac{d}{dx} \eta$. Differentiating (1.4) and (1.6) with respect to $t$ admits

\[
\frac{d}{dt} \tilde{\zeta} - \Delta \tilde{\zeta} + J(\Psi_t, \zeta) + J(\Psi, \tilde{\zeta}) + \frac{R_{\alpha}}{P_r} \frac{\partial \tilde{\eta}}{\partial x} = f_t, \\
\frac{d}{dt} \tilde{\eta} - \frac{1}{P_r} \Delta \tilde{\eta} + J(\Psi_t, \eta) + J(\Psi, \tilde{\eta}) = g_t.
\]

By taking the inner product of (3.63) with $\tilde{\eta}$ in $L^2(\Omega)$ we get

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{\eta} \|^2 + \frac{1}{P_r} \| \nabla \tilde{\eta} \|^2 = -(J(\Psi_t, \eta), \tilde{\eta}) + (g_t, \tilde{\eta}).
\]

Let us estimate the first term on the right-hand side of (3.64). Due to

\[
|(J(\Psi_t, \eta), \tilde{\eta})| = \left| \int_{\Omega} \Psi_t \eta_x \tilde{\eta} - \int_{\Omega} \Psi_t \eta \tilde{\eta} \right|.
\]
and
\[
\left| \int_{\Omega} \Psi_{tx} \eta_x \tilde{\eta} \right| \leq \| \Psi_{tx} \|_4 \| \eta_x \|_4 \| \tilde{\eta} \|
\]
\[
\leq C \| \Psi_{tx} \|_2 \| \eta_x \|_2 \| \tilde{\eta} \|
\]
\[
\leq C \| \tilde{\zeta} \|_2 \| \Delta \eta \|_2 \| \tilde{\eta} \|
\]
\[
\leq C \| \tilde{\eta} \|^2 + C \| \tilde{\zeta} \|^2 \| \Delta \eta \|^2 .
\]

Similarly, we have
\[
\left| \int_{\Omega} \Psi_{tx} \eta_y \tilde{\eta} \right| \leq C \| \tilde{\eta} \|^2 + C \| \tilde{\zeta} \|^2 \| \Delta \eta \|^2 .
\]

So
\[
|(J(\Psi_t, \eta), \tilde{\eta})| \leq C \| \tilde{\eta} \|^2 + C \| \tilde{\zeta} \|^2 \| \Delta \eta \|^2 .
\]

Thanks to
\[
|(g_t, \tilde{\eta})| \leq C \| g_t \|^2 + C \| \tilde{\eta} \|^2 ,
\]

by taking (3.64)-(3.66) into account, we have
\[
\frac{d}{dt} \| \tilde{\eta} \|^2 + \frac{2}{P_r} \| \nabla \tilde{\eta} \|^2 \leq C \| \tilde{\zeta} \|^2 \| \Delta \eta \|^2 + C \| \tilde{\eta} \|^2 + C \| g_t \|^2 .
\]

Taking the inner product of (3.62) with \( \tilde{\zeta} \) in \( L^2(\Omega) \), we get
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{\zeta} \|^2 + \| \nabla \tilde{\zeta} \|^2 = -(J(\Psi_t, \zeta), \tilde{\zeta}) - \frac{R_a}{P_r} \left( \frac{\partial \tilde{\eta}}{\partial x}, \tilde{\zeta} \right) + (f_t, \tilde{\zeta}).
\]

By arguments similar to deriving (3.65), we obtain that
\[
|(J(\Psi_t, \zeta), \tilde{\zeta})| \leq C \| \tilde{\zeta} \|^2 + C \| \tilde{\eta} \|^2 \| \Delta \zeta \|^2 .
\]

And because
\[
\left| \frac{R_a}{P_r} \left( \frac{\partial \tilde{\eta}}{\partial x}, \tilde{\zeta} \right) \right| \leq \frac{R_a}{P_r} \| \nabla \tilde{\eta} \| \| \tilde{\zeta} \| \leq \frac{1}{P_r} \| \nabla \tilde{\eta} \|^2 + C \| \tilde{\zeta} \|^2 ,
\]

\[
|(f_t, \tilde{\zeta})| \leq C \| f_t \|^2 + C \| \tilde{\zeta} \|^2 ,
\]

so taking (3.68)-(3.71) into account, we have
\[
\frac{d}{dt} \| \tilde{\zeta} \|^2 \leq \frac{2}{P_r} \| \nabla \tilde{\eta} \|^2 + C \| \tilde{\zeta} \|^2 \| \Delta \zeta \|^2 + C \| \tilde{\zeta} \|^2 + C \| f_t \|^2 .
\]

Adding (3.72) to (3.72), we have
\[
\frac{d}{dt} \left( \| \tilde{\zeta} \|^2 + \| \tilde{\eta} \|^2 \right) \leq C \| \tilde{\zeta} \|^2 \| \Delta \zeta \|^2 + \| \Delta \eta \|^2 + C(\| \tilde{\zeta} \|^2 + \| \tilde{\eta} \|^2)
\]
\[
+ C(\| f_t \|^2 + \| g_t \|^2)
\]
\[
\leq C(\| \tilde{\zeta} \|^2 + \| \tilde{\eta} \|^2)(\| \Delta \zeta \|^2 + \| \Delta \eta \|^2 + 1)
\]
\[
+ C(\| f_t \|^2 + \| g_t \|^2).
\]
By the uniform Gronwall lemma, Lemma 3.4 and Lemma 3.5, we have
\[
\begin{align*}
&\|\zeta(\tau, \tau - t, \zeta_0(\tau - t))\|^2 + \|\eta(\tau, \tau - t, \eta_0(\tau - t))\|^2 \\
&\leq \left(\int_{\tau-1}^{\tau} \left[\|\zeta(\xi, \tau - t, \zeta_0(\tau - t))\|^2 + \|\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2\right]d\xi \\
&\quad + C\int_{\tau-1}^{\tau} \left[\|f_\xi(\xi)\|^2 + \|g_\xi(\xi)\|^2\right]d\xi \right) \exp\left\{C\int_{\tau-1}^{\tau} \|\Delta\zeta(\xi, \tau - t, \\
&\quad \zeta_0(\tau - t))\|^2 + \|\Delta\eta(\xi, \tau - t, \eta_0(\tau - t))\|^2\right\}d\xi + C
\end{align*}
\]
(3.74)

4. **Existence of pullback attractors.** In this section, we will prove the existence of \(\mathcal{D}_\sigma\)-pullback attractors for problem (1.4)-(1.8) in \(L^2(\Omega) \times L^2(\Omega)\) and \(H^1(\Omega) \times H^1(\Omega)\), respectively.

**Theorem 4.1.** Problem (1.4)-(1.8) has a unique \(\mathcal{D}_\sigma\)-pullback global attractor \(\{A(\tau)\}_{\tau \in \mathbb{R}}\) in \(L^2(\Omega) \times L^2(\Omega)\).

**Proof.** For \(\tau \in \mathbb{R}\), denote by
\[
B_1(\tau) = \left\{ (\zeta, \eta) \in L^2(\Omega) \times L^2(\Omega) : \|\zeta\|^2 + \|\eta\|^2 \leq C e^{-\sigma \tau} \int_{-\infty}^{\tau} e^{\sigma \xi} \|f(\xi)\|^2 + \|g(\xi)\|^2 \right\}.
\]

Thanks to Lemma 3.1, we know that \(B_1 = \{B_1(\tau)\}_{\tau \in \mathbb{R}} \in \mathcal{D}_\sigma\) is a \(\mathcal{D}_\sigma\)-pullback absorbing set for \(\phi\) in \(L^2(\Omega) \times L^2(\Omega)\).

Now, we will prove that for every \(\tau \in \mathbb{R}\), \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma\), and \(t_n \to \infty\), \((\zeta_{0,n}, \eta_{0,n}) \in D(\tau - t_n)\), the sequence \(\phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n}))\) has a convergent subsequence in \(L^2(\Omega) \times L^2(\Omega)\), i.e., \(\phi\) is \(\mathcal{D}_\sigma\)-pullback asymptotically compact in \(L^2(\Omega) \times L^2(\Omega)\). By Lemma 3.3, there exist \(C = C(\tau) > 0\) and \(N = N(\tau, D) > 0\) such that, for all \(n \geq N\),
\[
\|\phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n}))\|_{H^1(\Omega) \times H^1(\Omega)} \leq C.
\]
(4.1)

By (4.1) and the compactness of embedding \(H^1(\Omega) \times H^1(\Omega) \hookrightarrow L^2(\Omega) \times L^2(\Omega)\), the sequence \(\phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n}))\) is precompact in \(L^2(\Omega) \times L^2(\Omega)\), which completes the proof.

**Lemma 4.2.** Let \(\frac{d}{dt}, \frac{d^2}{dt^2} \in L^2_{loc}(\mathbb{R}; L^2(\Omega))\). Then \(\phi\) is \(\mathcal{D}_\sigma\)-pullback asymptotically compact in \(H^1(\Omega) \times H^1(\Omega)\), that is, for every \(\tau \in \mathbb{R}\), \(D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_\sigma\), and \(t_n \to \infty\), \((\zeta_{0,n}, \eta_{0,n}) \in D(\tau - t_n)\), the sequence \(\phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n}))\) has a convergent subsequence in \(H^1(\Omega) \times H^1(\Omega)\).
Proof. We will show that the sequence \( \phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n})) \) has a Cauchy subsequence in \( H^1(\Omega) \times H^1(\Omega) \). By Theorem 4.1, \( \phi(t_n, \tau - t_n, (\zeta_{0,n}, \eta_{0,n})) \) is precompact in \( L^2(\Omega) \times L^2(\Omega) \). This means that, up to a subsequence,

\[
(\zeta(\tau, \tau - t_n, \zeta_{0,n}), \eta(\tau, \tau - t_n, \eta_{0,n})) \text{ is a Cauchy sequence in } L^2(\Omega) \times L^2(\Omega).
\]

From (1.4) we find that, for any \( n, m \geq 1 \),

\[
-(\Delta \zeta(\tau, \tau - t_n, \zeta_{0,n}) - \Delta \zeta(\tau, \tau - t_m, \zeta_{0,m})) = - (\zeta_r(\tau, \tau - t_n, \zeta_{0,n}) - \zeta_r(\tau, \tau - t_m, \zeta_{0,m}))
- (J(\Psi, \zeta(\tau, \tau - t_n, \zeta_{0,n})) - J(\Psi, \zeta(\tau, \tau - t_m, \zeta_{0,m})))
- \frac{R_n}{P_r} (\eta_r(\tau, \tau - t_n, \eta_{0,n}) - \eta_r(\tau, \tau - t_m, \eta_{0,m})).
\]

Multiplying (4.3) by \( \zeta(\tau, \tau - t_n, \zeta_{0,n}) - \zeta(\tau, \tau - t_m, \zeta_{0,m}) \) and then integrating it on \( \Omega \), by (1.10), we get

\[
\left\| \nabla \zeta(\tau, \tau - t_n, \zeta_{0,n}) - \nabla \zeta(\tau, \tau - t_m, \zeta_{0,m}) \right\|^2 \\
\leq \left\| \zeta_r(\tau, \tau - t_n, \zeta_{0,n}) - \zeta_r(\tau, \tau - t_m, \zeta_{0,m}) \right\|
\cdot \left\| \zeta(\tau, \tau - t_n, \zeta_{0,n}) - \zeta(\tau, \tau - t_m, \zeta_{0,m}) \right\|
\frac{R_n}{P_r} \left\| \nabla \eta_r(\tau, t - t_n, \eta_{0,n}) - \nabla \eta_r(\tau, t - t_m, \eta_{0,m}) \right\|
\cdot \left\| \zeta(\tau, \tau - t_n, \zeta_{0,n}) - \zeta(\tau, \tau - t_m, \zeta_{0,m}) \right\|.
\]

By Lemma 3.3 and Lemma 3.6, there is \( N = N(\tau, D) > 0 \) such that, for all \( n \geq N \),

\[
\left\| \nabla \eta(\tau, \tau - t_n, \eta_{0,n}) - \nabla \eta(\tau, \tau - t_m, \eta_{0,m}) \right\| \leq C
\]

and

\[
\left\| \zeta_r(\tau, \tau - t_n, \zeta_{0,n}) - \zeta_r(\tau, \tau - t_m, \zeta_{0,m}) \right\| \leq C,
\]

which along with (4.2) and (4.4) implies

\[
\nabla \zeta(\tau, \tau - t_n, \zeta_{0,n}) \text{ is a Cauchy sequence in } L^2(\Omega).
\]

From (1.6) we find that, for any \( n, m \geq 1 \),

\[
- \frac{1}{P_r} \left( \Delta \eta(\tau, \tau - t_n, \eta_{0,n}) - \Delta \eta(\tau, \tau - t_m, \eta_{0,m}) \right)
= - \left( \eta_r(\tau, \tau - t_n, \eta_{0,n}) - \eta_r(\tau, \tau - t_m, \eta_{0,m}) \right)
- (J(\Psi, \eta(\tau, \tau - t_n, \eta_{0,n})) - J(\Psi, \eta(\tau, \tau - t_m, \eta_{0,m}))).
\]

Multiplying (4.6) by \( \eta(\tau, \tau - t_n, \eta_{0,n}) - \eta(\tau, \tau - t_m, \eta_{0,m}) \) and then integrating it on \( \Omega \), by (1.10), we get

\[
\frac{1}{P_r} \left\| \nabla \eta(\tau, \tau - t_n, \eta_{0,n}) - \nabla \eta(\tau, \tau - t_m, \eta_{0,m}) \right\|^2 \\
\leq \left\| \eta_r(\tau, \tau - t_n, \eta_{0,n}) - \eta_r(\tau, \tau - t_m, \eta_{0,m}) \right\|
\cdot \left\| \eta(\tau, \tau - t_n, \eta_{0,n}) - \eta(\tau, \tau - t_m, \eta_{0,m}) \right\|
\]

By Lemma 3.6, there is \( N = N(\tau, D) > 0 \) such that, for all \( n \geq N \),

\[
\left\| \eta_r(\tau, \tau - t_n, \eta_{0,n}) - \eta_r(\tau, \tau - t_m, \eta_{0,m}) \right\| \leq C,
\]

which along with (4.2) and (4.7) implies

\[
\nabla \eta(\tau, \tau - t_n, \eta_{0,n}) \text{ is a Cauchy sequence in } L^2(\Omega).
\]
Then (4.5) and (4.8) imply that \( (\zeta(\tau, \tau - t, \zeta_0, n), \eta(\tau, \tau - t, \eta_0, n)) \) is a Cauchy sequence in \( H^1(\Omega) \times H^1(\Omega) \).

**Theorem 4.3.** Let \( \frac{d\zeta}{dt}, \frac{d\eta}{dt} \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \). Then problem (1.4)-(1.8) has a \( D_{\sigma} \)-pullback global attractor \( \{A(\tau)\}_{\tau \in \mathbb{R}} \) in \( H^1(\Omega) \times H^1(\Omega) \), that is, for all \( \tau \in \mathbb{R} \),

1. \( A(\tau) \) is compact in \( H^1(\Omega) \times H^1(\Omega) \);
2. \( \{A(\tau)\}_{\tau \in \mathbb{R}} \) is invariant, that is, \( \phi(t, \tau, A(\tau)) = A(t - \tau), \forall t \geq 0; \)
3. \( \{A(\tau)\}_{\tau \in \mathbb{R}} \) attracts every set in \( D_{\sigma} \) with respect to the \( H^1(\Omega) \times H^1(\Omega) \) norm; that is, for all \( B = \{B(\tau)\}_{\tau \in \mathbb{R}} \in D_{\sigma} \),

\[
\lim_{t \to \infty} \text{dist}_{H^1(\Omega) \times H^1(\Omega)}(\phi(t, \tau - t, B(\tau - t)), A(\tau)) = 0.
\]

**Proof.** We will prove that the attractor \( \{A(\tau)\}_{\tau \in \mathbb{R}} \) in \( L^2(\Omega) \times L^2(\Omega) \) obtained in Theorem 4.1 is actually a \( D_{\sigma} \)-pullback global attractor for \( \phi \) in \( H^1(\Omega) \times H^1(\Omega) \).

And the proof is similar with the proof of Theorem 6.4 in [9], so we omit it.

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