INFINITE-DIMENSIONAL BAYESIAN APPROACH FOR INVERSE SCATTERING PROBLEMS OF A FRACTIONAL HELMHOLTZ EQUATION

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Abstract. This paper focuses on a fractional Helmholtz equation describing wave propagation in the attenuating medium. According to physical interpretations, the fractional Helmholtz equation can be divided into loss- and dispersion-dominated fractional Helmholtz equations. In the first part of this work, we establish the well-posedness of the loss-dominated fractional Helmholtz equation (an integer- and fractional-order mixed elliptic equation) for a general wavenumber and prove the Lipschitz continuity of the scattering field with respect to the scatterer. Meanwhile, we only prove the well-posedness of the dispersion-dominated fractional Helmholtz equation (a high-order fractional elliptic equation) for a sufficiently small wavenumber due to its complexity. In the second part, we generalize infinite-dimensional Bayesian inverse theory to allow a part of the noise depends on the target function (the function that needs to be estimated). We also prove that the estimated function tends to be the true function if both the model reduction error and the white noise vanish. We eventually apply our theory to the loss-dominated model with an absorbing boundary condition.

1. Introduction

Numerous physical models have been proposed [1, 13, 37] to describe the attenuation effect which is an important phenomenon for considering wave propagation in some attenuating medium. When studying scattering problems with attenuating medium, researchers usually focus on the following Helmholtz equation:

$$\Delta u + k^2 n(x) u = 0, \tag{1.1}$$

where u denotes the wavefield, k denotes the wavenumber, and n denotes the refractive index with an imaginary component [14]. In other words, the analysis of scattering problems with absorbing medium can be incorporated into the classical studies on Helmholtz equations [2, 4]. However, the attenuation effect actually incorporates two effects, namely, amplitude loss and velocity dispersion. The aforementioned model (1.1) mixes these two effects together. Hence, the attenuation effect can hardly be compensated when we handle some inverse problems, such as reverse-time migration [47].

We consider the space fractional wave equations proposed in [50] that can separate the two effects incorporated in the attenuation effect. Before revealing the form of this new fractional model, we introduce the time fractional wave equation. Based

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on Caputo's fractional derivative [41], the isotropic stress-strain (σ-ϵ) relation can be deduced in the following form [10]:

$$\sigma = \frac{M_0}{t_0^{-2\gamma}} \frac{\partial^{2\gamma}\epsilon}{\partial t^{2\gamma}},$$

where M_0 is the bulk modulus and t_0 is a reference time. The following wave equation with Caputo's fractional derivative can then be established:

$$\frac{\partial^{2-2\gamma(x)}}{\partial t^{2-2\gamma(x)}} u = c(x)^2 \omega^{-2\gamma(x)} \Delta u, \tag{1.2}$$

where $c^2(x) = c_0^2(x)\cos^2(\pi\gamma(x)/2)$ and c_0 is the sound velocity. Denote $Q(x)$ to be the quality factor that is commonly used to characterize seismic attenuation. In this case, the fractional-order $\gamma(x)$ relates to the quality factor as follows:

$$\gamma(x) = \frac{1}{\pi}\arctan\left(\frac{1}{Q(x)}\right). \tag{1.3}$$

Restricted to the seismic frequency band, the quality factor $Q(x)$ is approximately constant in the frequency domain, while the time fractional wave equation (1.2) describes the constant Q attenuation precisely [47]. Carcione et al. [11, 12] successfully solve the time fractional wave equation (1.2) by using the Grunwald-Letnikow and central-difference approximations for time discretization. However, since the fractional-order in time brought memory effect along the time evolution, producing an accurate solution requires a large amount of computer memory and computational time, thereby limiting the application of the time fractional wave equation in seismic explorations.

Based on the time fractional wave equation (1.2), after performing some intricate calculations in the angular and space frequency domains, Zhu, Carcione, and Harris [49, 50] proposed the following space fractional model:

$$\frac{1}{c(x)^2}\frac{\partial^2}{\partial t^2}u = \Delta u + \Big(-\eta(x)(-\Delta)^{\gamma(x)+1}u - \Delta\Big)u - \tau(x)\frac{\partial}{\partial t}(-\Delta)^{\gamma(x)+1/2}u, \tag{1.4}$$

with coefficients are varying in space as follows:

$$\eta(x) = c_0(x)^{2\gamma(x)}\omega_0^{-2\gamma(x)}\cos(\pi\gamma(x)), \quad \tau(x) = c_0(x)^{2\gamma(x)}\omega_0^{-2\gamma(x)}\sin(\pi\gamma(x)). \tag{1.5}$$

We provide some explanations for the notations used in (1.4) and (1.5). Let ω_0 be a reference frequency ($\omega_0 = 1/t_0$), $c_0(x)$ be the phase velocity, and $c(x)$ be the space acoustic velocity. The fractional power $\gamma(x)$ relates to the quality factor $Q(x)$ according to formula (1.3) and, obviously, takes values between 0 and 1/2. Equation (1.4) exhibits no memory effect in time, thereby circumventing the computational difficulties brought by the time-fractional operator.

The second and third terms on the right hand side of model (1.4) primarily represent the dispersion-effect and amplitude loss-effect, respectively. Therefore, by using this model, the dispersion-effect and loss-effect can be separated to some extent. Obviously, the dispersion-dominated equation [50] takes the following form:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u = -\eta(-\Delta)^{\gamma+1}u, \tag{1.6}$$

and the amplitude loss-dominated equation [50] can be written as follows:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u = \Delta u - \tau\frac{\partial}{\partial t}(-\Delta)^{\gamma+1/2}u. \tag{1.7}$$

The space fractional model (1.4) has two advantages. First, this model can be solved quickly by spectral methods [48] or other numerical methods compared with the time fractional model (1.2). Second, the space fractional model separates the dispersion-effect and amplitude loss-effect from each other, thereby allowing researchers to compensate these two effects separately when handling some inverse problems [47].

Now, let us consider the time-harmonic solution of equation (1.4). As usual, assuming that the solution takes the form $e^{-i\omega t}u(x)$, we derive an equation that can be called the fractional Helmholtz equation as follows:

$$-\eta(-\Delta)^{\gamma+1}u + i\omega\tau(-\Delta)^{\gamma+1/2}u + k^2(1+q(x))u = 0, \tag{1.8}$$

where ω denotes the angular frequency, k represents the wavenumber, and $q(\cdot)$ is a function that is assumed to be larger than -1. Equation (1.8) can also be separated into the loss- and dispersion-dominated models. Specifically, the loss-dominated fractional Helmholtz equation can be derived from equation (1.7) as follows:

$$\Delta u + i\omega\tau(-\Delta)^{\gamma+1/2}u + k^2(1+q(x))u = 0. \tag{1.9}$$

And the dispersion-dominated fractional Helmholtz equation can be derived from equation (1.6) as follows:

$$(-\Delta)^{\gamma+1}u - k^2(1+q)u = 0. \tag{1.10}$$

Given that the background velocity can sometimes be obtained by interpolations of well data and anomalous detection is one of the key problems in geophysical explorations [43, 47], we can assume that the scatterer is compactly supported and that the strong absorbing medium is located in some parts of the scatterer. These hypotheses are formally formulated in Assumption 1 of Section 3. The well-posedness with a general wavenumber $k > 0$ has been constructed for equation (1.9). For equation (1.10), the problem seems to be difficult. Specifically, we can only constuct a unique solution for a sufficiently small wavenumber. The interior regularity of elliptic equations and the Gauss-Green theorem for fractional Laplacian are used to construct appropriate weak formulations of these two fractional Helmholtz equations. The properties of fractional Laplacian is used along with an iterative procedure to obtain our results, which are different to the case of classical Helmholtz equation (1.1).

Although, the forward models studied in this paper are clear, we attempt in the second part of our work to construct a Bayesian inverse theory for inverse scattering problems related to the fractional Helmholtz equation. Before going further, let us recall some basic developments in Bayesian inverse theory, which follows two philosophies. One philosophy involves discretizing the forward problem and using the Bayesian methodology to solve a finite-dimensional problem ("discrete first, inverse second"[DFIS]). Kaipio and Somersalo [29] provide an excellent introduction for the DFIS method, especially the large inverse problems arising in differential equations. The other philosophy involves constructing Bayesian inverse theory in infinite-dimensional space, in which discretization of the continuous problem is postponed to the final step ("inverse first, discrete second"[IFDS]). The IFDS method could be dating back to 1970, Franklin [22] formulate PDE's inverse problems in terms of Bayes' formula on some Hilbert space. Recently, Lasanen

[33, 34, 35, 36] develop a fully nonlinear theory. Cotter et al. [15, 18, 44] establish a mathematical framework for a range of inverse problems for functions, given noisy observations. They also reveal the relationship between regularization techniques and the Bayesian framework. Recently, Trillos and Sanz-Alonso [45] firstly provide a mathematical foundation to the Bayesian learning of the order-and other inputs-of fractional elliptic equations.

In this study, we employ the IFDS method to construct the Bayesian theory for the inverse scattering problem. Let X, Y be separable Hilbert space, equipped with the Borel σ-algebra, and let $\mathcal{G} : X \to Y$ be a measurable mapping. Then, the inverse problem can be sought of as finding x from y, where

$$y = \mathcal{G}(x) + \eta, \tag{1.11}$$

and $\eta \in Y$ denotes noise. An important assumption in the literature [15, 18, 44] is that the noise η is independent of x. However, in previous studies on inverse scattering problems, some model reduction errors may be brought into the forward problem (e.g., absorbing boundary condition [4]). By denoting the model reduction error as ϵ, equation (1.11) can be reformulated as follows:

$$y = \mathcal{G}_a(x) + \epsilon + \eta \tag{1.12}$$

with $\mathcal{G}_a : X \to Y$ is a measurable mapping. The error ϵ usually depends on x; thus, we need to generalize infinite-dimensional Bayesian inverse theory to incorporate this situation.

Following the principles of DFIS, a Bayesian approximation error approach is developed [29, 31, 30] to handle the model approximate errors that are produced by some finite-dimensional approximations. Given that this method can yield acceptable inversion results with only a rough approximate forward solver, the aforementioned Bayesian approximation error approach seems to be a promising method for solving inverse scattering problems. A novel iterative updating algorithm of model error has been recently designed by Calvetti et al. [9], which provides new ideas for learning model errors. However, there seems no special infinite-dimensional Bayesian inverse theory for the model reduction error induced by the hypotheses of constructing the mathematical models (e.g., the error induced by some absorbing boundary conditions).

Based on the aforementioned considerations and the requirements for analyzing inverse scattering problems, we modify the theory presented in [15, 18, 19, 44] to allow a part of the noise to depend on the state variable x. Afterward, we prove that the estimated function tends to be the true function when both the model reduction error ϵ and the white noise η vanish under a simple setting. Finally, we apply the theory to an inverse scattering problem related to equation (1.9). In summary, the contributions of our work are as follows:

- The well-posedness is obtained for a scattering problem related to the loss-dominated fractional Helmholtz equation. Based on the well-posedness result, the Lipschitz continuity of the forward map is obtained, which is useful for analyzing inverse scattering problems.
- Fractional Gauss-Green formula for the regional fractional Laplace operator [23, 46] has been used to formulate appropriate weak forms for the loss- and dispersion-dominated models. Combined with a unique continuation result for the Laplace operator, the fractional Gauss-Green formula has also been used to prove the uniqueness for the loss-dominated equation with a general

wavenumber. The dispersion-dominated model has been reformulated to an integer- and fractional-order mixed elliptic system, and an iterative method has been employed, which are different to the case of classical Helmholtz equations.

- A generalized infinite-dimensional Bayesian inverse method, which can be called infinite-dimensional Bayesian model error method, is developed. In addition, the relationship of this model with some regularization methods is discussed. If both the model reduction error and the white noise vanish, it is proved that the estimated function tends to be the true function.

The contents of this paper are organized as follows. In Section 2, notations are introduced and some basic knowledge on the fractional Laplace operator is presented. In Section 3, we prove well-posedness for the scattering field equation related to the loss-dominated equation firstly. Then, we prove well-posedness for a scattering field equation related to the dispersion-dominated equation with a sufficiently small wavenumber. In Section 4, we derive well-posedenss of the posterior measure when some model reduction errors are considered. Then, we prove that the estimated solution tends to be the true function if both the model error and the white noise vanish. At the end of this section, the general theory is applied to an inverse scattering problem related to the loss-dominated fractional Helmholtz equation. In Section 5, we provide a short summary and propose some further questions.

2. Preliminaries

2.1. **Notations.** In this section, we provide an explanation of the notations used throughout this paper.

- Let $n \in \mathbb{N}$ be an integer, and let $\mathbb{R}^n$ denotes n-dimensional Euclidean space; as usual, $\mathbb{R}$ means $\mathbb{R}^1$.
- Let $\Gamma(\cdot)$ be the usual Gamma function. For an introduction to the Gamma function, the reader may refer to [41].
- For $s \in \mathbb{R}, p \in [1, \infty)$ and a bounded domain $D \subset \mathbb{R}^n$, let $W^{s,p}(D)$ denotes the Sobolev space, which roughly means that the s order weak derivative of a function belongs to the space $L^p(D)$. For brevity, we usually denote $W^{s,p}(D)$ as $H^s(D)$.
- Let $D \subset \mathbb{R}^n$ be a bounded domain, $C(D)$ be continuous functions, and $C_u(D)$ be the uniformly bounded continuous functions.
- Let C be a general constant which may be different from line to line.
- Let $\mathcal{H}$ be a Hilbert space, $L^+(\mathcal{H})$ be the set of all symmetric and positive operators, and $L_1^+(\mathcal{H})$ be the operators of trace class and belong to $L^+(\mathcal{H})$.
- Let $\mathcal{H}$ be a Hilbert space. For an operator $\mathcal{C} \in L_1^+(\mathcal{H})$, let $\mathcal{N}(a, \mathcal{C})$ be a Gaussian measure on $\mathcal{H}$ with mean $a \in \mathcal{H}$ and covariance operator $\mathcal{C}$.
- Let η and ϵ be two random variables, with $\eta \perp \epsilon$ indicating that these two random variables are independent.

2.2. **Fractional Laplace operator.** In this part, we provide an elementary introduction to the fractional Laplace operator that is used throughout this paper. Let $0 < \alpha < 1$, and set

$$\mathcal{L}^1(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable}, \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2\alpha}} dx < \infty \right\}.$$

For $u \in \mathcal{L}^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $\epsilon > 0$, we write

$$(-\Delta)^\alpha_\epsilon u(x) = C_{n,\alpha} \int_{\{y\in\mathbb{R}^n,\, |y-x|>\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}} dy$$

with

$$C_{n,\alpha} = \frac{\alpha 2^{2\alpha}\Gamma(\frac{n+2\alpha}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\alpha)}, \tag{2.1}$$

where Γ denotes the usual Gamma function. The fractional Laplacian $(-\Delta)^\alpha u$ of the function u is defined by the formula

$$(-\Delta)^\alpha u(x) = C_{n,\alpha}\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}} dy = \lim_{\epsilon\downarrow 0} (-\Delta)^\alpha_\epsilon u(x), \quad x \in \mathbb{R}^n \tag{2.2}$$

provided that the limit exists [7]. Apart from this definition, one can also define $(-\Delta)^\alpha$ by using the method of bilinear Dirichlet forms [24], that is, $(-\Delta)^\alpha$ is the closed self-adjoint operator on $L^2(\mathbb{R}^n)$ associated with the bilinear symmetric closed form

$$\mathcal{E}(u,\varphi) = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\alpha}} dxdy, \quad u,\varphi \in H^\alpha(\mathbb{R}^n), \tag{2.3}$$

in the sense that

$$D((-\Delta)^\alpha) = \{u \in H^\alpha(\mathbb{R}^n),\, (-\Delta)^\alpha u \in L^2(\mathbb{R}^n)\}$$

and

$$\mathcal{E}(u,\varphi) = ((-\Delta)^\alpha u,\, \varphi) = \int_{\mathbb{R}^n} \varphi(-\Delta)^\alpha u\, dx, \quad \forall\, u \in D((-\Delta)^\alpha),\, \varphi \in H^\alpha(\mathbb{R}^n).$$

The fractional Laplace operator has at least ten equivalent definitions, and the equivalence has been proven in [32]. Given that we may need to face fractional elliptic equations in a bounded domain in Section 3, we present here the definition of regional fractional Laplacian [23]. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $\mathcal{L}^1(\Omega)$ be the measurable function u on Ω such that $\int_\Omega \frac{|u(x)|}{(1+|x|)^{n+2\alpha}}dx < \infty$. For $u \in \mathcal{L}^1(\Omega)$, $x \in \Omega$, and $\epsilon > 0$, we write

$$A^\alpha_{\Omega,\epsilon} u(x) = C_{n,\alpha} \int_{y\in\Omega, |y-x|>\epsilon} \frac{u(y)-y(x)}{|x-y|^{n+2\alpha}}\, dy, \tag{2.4}$$

where $C_{n,\alpha}$ is defined as in (2.1).

Definition 2.1. Let $u \in \mathcal{L}^1(\Omega)$. The regional fractional Laplacian A^α_Ω is defined by the formula

$$A^\alpha_\Omega u(x) = \lim_{\epsilon\downarrow 0} A^\alpha_{\Omega,\epsilon} u(x), \quad x \in \Omega, \tag{2.5}$$

provided that the limit exists.

When $\Omega = \mathbb{R}^n$, the notation $A^\alpha_{\mathbb{R}^n}$ represents the fractional power of the Laplacian defined in (2.2). In order to give the Gauss-Green formula in the fractional Laplace operator setting, we give the following definition [23].

Definition 2.2. For $0 \le s < 2$, $u \in C^1(\Omega)$, and $z \in \partial\Omega$, we define the operator $\mathcal{N}^s$ on $\partial\Omega$ by

$$\mathcal{N}^s u(z) := -\lim_{t\downarrow 0} \frac{du(z - t\mathbf{n}(z))}{dt} t^s, \tag{2.6}$$

provided that the limit exists. Here, $\mathbf{n}(z)$ denotes the outward normal vector of $\partial\Omega$ at point $z \in \partial\Omega$.

Let $\rho(x) := \mathrm{dist}(x, \partial\Omega) = \inf\{|y-x| : y \in \partial\Omega\}, \quad x \in \Omega$, and for a real number $\delta > 0$, we set $\Omega_\delta := \{x \in \Omega, : 0 < \rho(x) < \delta\}$. Let $\beta > 0$ be a real number, define

$$h_\beta(x) = \begin{cases} \rho(x)^{\beta-1} & \forall\, x \in \Omega_\delta,\, \beta \in (0,1) \cup (1,\infty), \\ \ln(\rho(x)) & \forall\, x \in \Omega_\delta,\, \beta = 1. \end{cases} \tag{2.7}$$

For $1 < \beta \le 2$, we define the space

$$C^2_\beta(\bar{\Omega}) := \{u : u(x) = f(x)h_\beta(x) + g(x), \quad \forall\, x \in \Omega \text{ for some } f, g \in C^2(\bar{\Omega})\}. \tag{2.8}$$

Having these preparations, now, we can state the following fractional Gauss-Green formula [23, 46].

Lemma 2.3. *Let $1/2 < \alpha < 1$ and let A^α_Ω be the nonlocal operator defined in Definition 2.1. Then, for every $u := fh_{2\alpha} + g \in C^2_{2\alpha}(\bar{\Omega})$ and $\varphi \in C^2_{2\alpha}(\bar{\Omega})$,*

$$\begin{aligned} \int_\Omega A^\alpha_\Omega u(x)\varphi(x)dx =& \frac{1}{2}C_{n,\alpha}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\alpha}}\,dxdy \\ &- B_{n,\alpha}\int_{\partial\Omega} \varphi\mathcal{N}^{2-2\alpha}u\,dS, \end{aligned} \tag{2.9}$$

where dS denotes the surface measure, $B_{n,\alpha}$ is a constant related to $C_{n,s}$ which can be found in [23] *or* [46].

Throughout the rest of this paper, A^0_Ω will be understood as the identity operator.

3. Forward Problem

In this section, we will prove the well-posedness for the loss- and dispersion-dominated equations. Before going further, let us make more specific assumptions about these two equations which are valid in what follows.

Assumption 1:

(1) In order to make our presentation more concise, without loss of generality, we assume the space dimension $n = 2$.
(2) Assume $q(\cdot)$ be a bounded function and has compact support. Denote B_R as a ball centered at the origin, then there exists $R > 0$ such that $\mathrm{supp}(q) \subset B_R$. In addition, we assume that there exist two constants $q_{\min}, q_{\max}$ such that $-1 < q_{\min} \le q(\cdot) \le q_{\max} < \infty$.
(3) Let γ be a piecewise constant function, and without loss of generality, in this paper we assume $\gamma(x) = \tilde{\gamma}1_\Omega$, where Ω is a subset of B_R ($\bar{\Omega} \subsetneqq \mathrm{supp}(q) \subsetneqq B_R$) and $\tilde{\gamma}$ is a constant in $[0, 1/2]$.
(4) Let η, τ be two non-negative piecewise constant functions related to γ. Let $\tilde{\eta}, \tilde{\tau}$ be two positive constants, $\tau(x) = \tilde{\tau}$ if $\gamma(x) \neq 0$ and $\tau(x) = 0$ if $\gamma(x) = 0$. $\eta(x) = \tilde{\eta}$ if $\gamma(x) \neq 1/2$ and $\eta(x) = 0$ if $\gamma(x) = 1/2$.

Remark 3.1. All assumptions in Assumption 1 are based on the physical model. For example, if we assume that $\gamma(x) = 0$ in (1.5), then $\tau(x) = 0$ for $\sin(\pi\gamma(x)) = 0$ and if we assume that $\gamma(x) = 1/2$ in (1.5), then $\eta(x) = 0$ for $\cos(\pi\gamma(x)) = 0$.

Figure 1 presents the assumptions stated in Assumption 1 for the relation between the area with attenuating media Ω, the support of the scatterer $\mathrm{supp}(q)$ and the circle with radius R clearly.

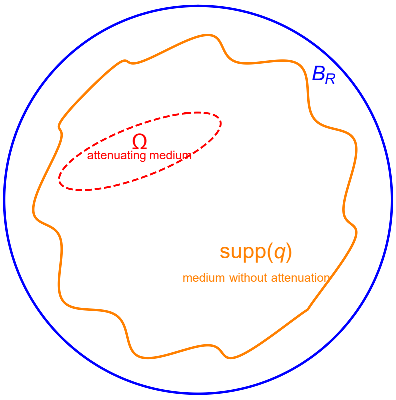

FIGURE 1. The relation between the area with attenuating media Ω, the support of the scatterer $\mathrm{supp}(q)$, and the circle with radius R.

Because one advantage of the space fractional wave equation is that it can separate amplitude loss effect and dispersion effect, we can study the loss-dominated equation and dispersion-dominated equation separately.

3.1. **Loss-dominated model.** Based on the time-domain equation (1.7), we can easily derive the loss-dominated fractional Helmholtz equation as follow:

$$\Delta u + i\omega\tau(-\Delta)^{\gamma+1/2}u + k^2(1+q(x))u = 0 \quad \text{in } \mathbb{R}^2. \tag{3.1}$$

As usual, the scatterer is illuminated by a plane incident field

$$u^{\mathrm{inc}}(x) = e^{ikx\cdot\mathbf{d}}, \tag{3.2}$$

where $\mathbf{d} = (\cos(\theta), \sin(\theta)) \in \mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the incident direction and $\theta \in (0, 2\pi)$ is the incident angle. Evidently, the incident field satisfies

$$\Delta u^{\mathrm{inc}} + k^2 u^{\mathrm{inc}} = 0 \quad \text{in } \mathbb{R}^2. \tag{3.3}$$

Before setting up the scattering problem, we need the following formula:

$$(-\Delta)^{\alpha} e^{ikx\cdot\mathbf{d}} = k^{2\alpha} e^{ikx\cdot\mathbf{d}} \quad \text{with } 0 < \alpha < 1. \tag{3.4}$$

The total field u consists of the incident field u^{inc} and the scattered field u^s:

$$u = u^{\mathrm{inc}} + u^s. \tag{3.5}$$

It follows form (3.1), (3.3), (3.5) and formula (3.4) that the scattered field satisfies

$$\Delta u^s + i\omega\tau(-\Delta)^{\gamma+1/2}u^s + k^2(1+q(x))u^s = (-k^2 q(x) - i\omega\tau k^{2\gamma+1})u^{\mathrm{inc}} \tag{3.6}$$

in $\mathbb{R}^2$. By our assumption, function γ is zero outside Ω which is contained in a ball with radius R, so the scattered field as usual should satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty} \sqrt{r}\,(\partial_r u^s - iku^s) = 0, \tag{3.7}$$

where $r = |x|$. In the domain $\mathbb{R}^2 \backslash \bar{\Omega}$, equation (3.6) is reduced to

$$\Delta u^s + k^2 u^s = -k^2 q u^{\mathrm{inc}}. \tag{3.8}$$

Relying on the classical scattering theory [2], we know that the solution of equation (3.6) in $\mathbb{R}^2\backslash\bar{B}_R$ can be written under the polar coordinates as follows:

$$u^s(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \hat{u}_n^s e^{in\theta}, \tag{3.9}$$

where $H_n^{(1)}$ is the Hankel function of the first kind with order n and

$$\hat{u}_n^s = (2\pi)^{-1}\int_0^{2\pi} u^s(R,\theta)e^{-in\theta}d\theta.$$

Let $\mathbb{B} : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ be the Dirichlet-to-Neumann (DtN) operator defined as follows: for an $u^s \in H^{1/2}(\partial B_R)$,

$$(\mathbb{B}u^s)(R,\theta) = k\sum_{n\in\mathbb{Z}} \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \hat{u}_n^s e^{in\theta}. \tag{3.10}$$

Then, the solution in (3.9) satisfies the following transparent boundary condition:

$$\partial_{\mathbf{n}} u^s = \mathbb{B}u^s \quad \text{on } \partial B_R, \tag{3.11}$$

where $\mathbf{n}$ is the unit outward normal on ∂B_R. The problem can then be converted to the bounded domain. Since we consider the bounded domain problem, the fractional Laplace operator may need to be adapted to the regional fractional Laplace operator introduced in Section 2.2. Following Assumption 1, we write the bounded elliptic problem as

$$\begin{cases} \Delta u^s + i\omega\tau A_\Omega^{\tilde{\gamma}+1/2}u^s + k^2(1+q)u^s = (-k^2q - i\omega\tau k^{2\gamma+1})u^{\text{inc}} \quad \text{in } B_R, \\ \partial_{\mathbf{n}} u^s = \mathbb{B}u^s \quad \text{on } \partial B_R. \end{cases} \tag{3.12}$$

Now, the key step is that how to construct an appropriate weak form for problem (3.12). Since Laplace operator appears in equation (3.12), we expect that the interior regularity of u will be high enough to ensure

$$\mathcal{N}^{2-2(\tilde{\gamma}+1/2)}u(z) = \mathcal{N}^{1-2\tilde{\gamma}}u(z) = \lim_{t\downarrow 0} \mathbf{n}(\mathbf{z})\cdot\nabla u(z - t\mathbf{n}(z))t^{1-2\tilde{\gamma}} = 0 \tag{3.13}$$

with $z \in \partial\Omega$. Based on these considerations, $A_\Omega^{\tilde{\gamma}+1/2}$ may be more appropriately defined as an operator with fractional Neumann boundary condition. Inspired by the method used in [24, 23, 25, 46] and the bilinear closed form defined in (2.3), we define the following bilinear form for $u, \varphi \in H^{\tilde{\gamma}+1/2}(\Omega)$:

$$\mathcal{E}_\Omega^N(u,\varphi) = \frac{C_{2,\tilde{\gamma}+1/2}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2\tilde{\gamma}}}\,dxdy. \tag{3.14}$$

Let A_L be the closed linear operator associated with the closed elliptic form $\mathcal{E}_\Omega^N$ in the sense that

$$\begin{cases} D(A_L) := \{u \in H^{\tilde{\gamma}+1/2}(\Omega),\, \exists\, v \in L^2(\Omega), \\ \qquad\qquad \mathcal{E}_\Omega^N(u,\varphi) = (v,\varphi), \quad \forall\, \varphi \in H^{\tilde{\gamma}+1/2}(\Omega)\}, \\ A_L u = v. \end{cases} \tag{3.15}$$

Remark 3.2. The operator A_L can be considered as a realization of the operator $A_\Omega^{\tilde{\gamma}+1/2}$ on $L^2(\Omega)$ with the fractional Neumann type boundary condition $\mathcal{N}^{2-2(\tilde{\gamma}+1/2)}u =$

0 on $\partial\Omega$. Specifically, if Ω has a C^2 boundary, then based on the ideas of Proposition 6.1 in [46],we find that $D(A_L) \cup C^2_{2\tilde{\gamma}+1}(\bar{\Omega}) = \{u \in C^2_{2\tilde{\gamma}+1}(\bar{\Omega}), \mathcal{N}^{1-2\tilde{\gamma}}u(z) = 0 \text{ on } \partial\Omega\}$.

Based on these considerations, equation (3.12) should have the following form:

$$\begin{cases} \Delta u^s + i\omega\tau A_L u^s + k^2(1+q)u^s = (-k^2 q - i\omega\tau k^{2\gamma+1})u^{\mathrm{inc}} & \text{in } B_R, \\ \partial_n u^s = \mathbb{B}u^s \quad \text{on } \partial B_R. \end{cases} \tag{3.16}$$

Define $a:\, H^1(B_R) \times H^1(B_R) \to \mathbb{C}$ as

$$\begin{aligned} a(u^s, \varphi) = \int_{B_R} \nabla u^s \cdot \nabla\bar{\varphi} dx - i\omega\tau \mathcal{E}^N_\Omega(u^s, \bar{\varphi}) \\ - k^2 \int_{B_R} (1+q(x)) u^s \bar{\varphi}\, dx - \int_{\partial B_R} \mathbb{B}u^s \bar{\varphi} dS, \end{aligned} \tag{3.17}$$

then define $b:\, H^1(B_R) \to \mathbb{C}$ as

$$b(\varphi) = \int_{B_R} (k^2 q(x) + i\omega\tau k^{2\tilde{\gamma}+1}) u^{\mathrm{inc}} \bar{\varphi}\, dx. \tag{3.18}$$

Following (3.15) and (3.16), we can easily obtain the variational form of equation (3.12) as follows:

$$a(u^s, \varphi) = b(\varphi) \quad \forall\, \varphi \in H^1(B_R). \tag{3.19}$$

For a given scatterer q, fractional order function γ and an incident field u^{inc}, we define the map $\mathfrak{S}(q, \gamma, u^{\mathrm{inc}})$ by $u^s = \mathfrak{S}(q, \gamma, u^{\mathrm{inc}})$, where u^s is the solution of the problem (3.12) or the variational problem (3.19). It is easily seen that the map $\mathfrak{S}(q, \gamma, u^{\mathrm{inc}})$ is linear with respect to u^{inc} but is nonlinear with respect to q, in addition, γ is assumed to be known in the fractional scattering problem. Hence, we may denote $\mathfrak{S}(q, \gamma, u^{\mathrm{inc}})$ by $\mathfrak{S}(q)u^{\mathrm{inc}}$.

Theorem 3.3. *Let $0 < \tilde{\gamma} < 1/2$. If the wavenumber k is sufficiently small, then the variational problem (3.19) admits a unique weak solution in $H^1(B_R)$ and $\mathfrak{S}(q)$ is a bounded linear map from $L^2(B_R)$ to $H^1(B_R)$. Furthermore, there is a constant C depends on B_R and $\|q\|_{L^\infty(B_R)}$, such that*

$$\|\mathfrak{S}(q)u^{inc}\|_{H^1(B_R)} \le Ck\|u^{inc}\|_{L^2(B_R)}. \tag{3.20}$$

The proof is inspired by the method used in [2, 3, 4, 5] for the integer order Helmholtz equation, so we only give a sketch here for concise.

Proof. Define

$$\begin{aligned} a_1(u^s, \varphi) &= (\nabla u^s, \nabla\varphi) - i\omega\tau\mathcal{E}^N_\Omega(u^s, \varphi) - \langle \mathbb{B}u^s, \varphi\rangle, \\ a_2(u^s, \varphi) &= -((1+q)u^s, \varphi). \end{aligned}$$

It is obviously that $a = a_1 + k^2 a_2$. Since $\mathcal{E}^N_\Omega(u^s, u^s) \ge 0$, we can obtain

$$\begin{aligned} |a_1(u^s, u^s)| &\ge C\|\nabla u^s\|^2_{L^2(B_R)} + |\omega\tau|\mathcal{E}^N_\Omega(u^s, u^s) + C\|u^s\|^2_{L^2(\partial B_R)} \\ &\ge C\|u^s\|^2_{H^1(B_R)}, \end{aligned}$$

where we used Theorem 2.6.4 in [39]. Afterward, we define an operator $\mathfrak{A}$: $L^2(B_R) \to H^1(B_R)$ by

$$a_1(\mathfrak{A}u^s, \varphi) = a_2(u^s, \varphi), \quad \forall\, \varphi \in H^1(B_R).$$

Using the Lax-Milgram lemma, it follows that $\|\mathfrak{A}u^s\|_{H^1(B_R)} \leq C\|u^s\|_{L^2(B_R)}$. Define a function $w \in L^2(B_R)$ by requiring $w \in H^1(B_R)$ and satisfying

$$a_1(w, \varphi) = b(\varphi) \quad \forall\, \varphi \in H^1(B_R). \tag{3.21}$$

It follows from the Lax-Milgram lemma again that

$$\|w\|_{H^1(B_R)} \leq C(k^2\|q\|_{L^\infty(B_R)} + |\omega|\tau k^{2\tilde{\gamma}+1})\|u^{\mathrm{inc}}\|_{L^2(B_R)}. \tag{3.22}$$

Using the operator $\mathfrak{A}$, we can see that problem (3.19) is equivalent to finding $u^s \in L^2(B_R)$ such that

$$(I + k^2\mathfrak{A})u^s = w. \tag{3.23}$$

When the wavenumber is small enough, the operator $I+k^2\mathfrak{A}$ has a uniform bounded inverse. Then, we have the estimate $\|u^s\|_{L^2(B_R)} \leq C\|w\|_{L^2(B_R)}$. Rearranging (3.23), we have $u^s = w - k^2\mathfrak{A}u^s$, so we obtain

$$\|u^s\|_{H^1(B_R)} \leq \|w\|_{H^1(B_R)} + Ck^2\|u^s\|_{L^2(B_R)} \leq Ck\|u^{\mathrm{inc}}\|_{L^2(B_R)},$$

where we used (3.22) in the second inequality. □

In order to obtain a similar result for a general wavenumber $k > 0$, we need the following uniqueness result.

Lemma 3.4. *Given the scatterer $q \in L^\infty(B_R)$, the direct scattering problem (3.16) has at most one solution.*

Proof. It suffices to show that $u^s = 0$ in B_R if $u^{\mathrm{inc}} = 0$ (no source term). From the Green's formula and fractional Gauss-Green formula (Lemma 2.3), we have

$$\begin{aligned} 0 &= \int_{B_R} u^s(\Delta\bar{u}^s - i\omega\tau A_L\bar{u}^s) - \bar{u}^s(\Delta u^s + i\omega\tau A_L u^s)\, dx \\ &= \int_{\partial B_R} u^s\frac{\partial\bar{u}^s}{\partial\mathbf{n}} - \bar{u}^s\frac{\partial u^s}{\partial\mathbf{n}}dS - i\, C_{n,\tilde{\gamma}+1/2}\omega\tau\int_\Omega\int_\Omega \frac{|u^s(x) - u^s(y)|^2}{|x-y|^{n+2\tilde{\gamma}+1}}dx\, dy \\ &= -i\left(2\mathrm{Im}\int_{\partial B_R}\bar{u}^s\mathbb{B}u^s\, dS + C_{n,\tilde{\gamma}+1/2}\omega\tau\int_\Omega\int_\Omega\frac{|u^s(x) - u^s(y)|^2}{|x-y|^{n+2\tilde{\gamma}+1}}dx\, dy\right). \end{aligned}$$

For the last equality of the above formula, the second term in the bracket is non-negative. Based on the same ideas used in the proof of Theorem 2.6.5 in [39], we obtain that $u^s = 0$ on ∂B_R. The boundary condition (3.11) yields further $\frac{\partial u^s}{\partial\mathbf{n}} = 0$ on ∂B_R. Therefore, we easily see that $u^s = 0$ in $\mathbb{R}^2\backslash B_R$. Let us recall that for $u^{\mathrm{inc}} = 0$, we have

$$\begin{cases} \Delta u^s + i\omega\tau A_L u^s + k^2(1+q)u^s = 0 & \text{in } B_R \\ \partial_n u^s = \mathbb{B}u^s & \text{on } \partial B_R. \end{cases} \tag{3.24}$$

By taking the absolute value on both sides of the above equation, we obtain

$$|\Delta u^s|^2 + \omega^2\tau^2|A_L u^s|^2 \leq |k^2(1+q)u^s| \quad \text{for } x \in B_R.$$

Hence, it is obviously that

$$|\Delta u^s(x)|^2 \leq |k^2(1+q(x))u^s(x)| \quad \text{for } x \in B_R.$$

From the results in [27], we obtain $u^s = 0$ in B_R. □

With the above lemma, we can obtain the following result for general $k > 0$ by using the Fredholm alternative theorem.

Theorem 3.5. *Given the scatterer $q \in L^\infty(B_R)$, the variational problem (3.19) admits a unique weak solution in $H^1(B_R)$ for all $k > 0$, and $\mathfrak{S}(q)$ is a bounded linear map from $L^2(B_R)$ to $H^1(B_R)$. Furthermore, the estimate*

$$\|\mathfrak{S}u^{inc}\|_{H^1(B_R)} \leq C\|u^{inc}\|_{L^2(B_R)}, \tag{3.25}$$

holds, where the constant C depends on k, B_R, and $\|q\|_{L^\infty(B_R)}$.

Theorem 3.6. *Assume that $q_1, q_2 \in L^\infty(B_R)$. Then,*

$$\|\mathfrak{S}(q_1)u^{inc} - \mathfrak{S}(q_2)u^{inc}\|_{H^1(B_R)} \leq C\|q_1 - q_2\|_{L^\infty(B_R)}\|u^{inc}\|_{L^2(B_R)}, \tag{3.26}$$

where the constant C depends on k, B_R, and $\|q_2\|_{L^\infty(B_R)}$.

Proof. Let $u_1^s = \mathfrak{S}(q_1)u^{\mathrm{inc}}$ and $u_2^s = \mathfrak{S}(q_2)u^{\mathrm{inc}}$. It follows that for $j = 1, 2$

$$\Delta u_j^s + i\omega\tau A_L u_j^s + k^2(1 + q_j)u_j^s = (-k^2 q_j - i\omega\tau k^{2\gamma+1})u^{\mathrm{inc}}.$$

By setting $\delta u^s = u_1^s - u_2^s$, we have

$$\Delta\delta u^s + i\omega\tau A_L\delta u^s + k^2(1 + q_1)\delta u^s = -k^2(q_1 - q_2)(u^{\mathrm{inc}} + u_2^s).$$

The function δu^s also satisfies the boundary condition

$$\partial_n\delta u^s = \mathbb{B}\delta u^s \quad \text{on } \partial B_R.$$

By using similar methods for proving Theorem 3.5, we obtain

$$\|\delta u^s\|_{H^1(B_R)} \leq C\|q_1 - q_2\|_{L^\infty(B_R)}\|u^{\mathrm{inc}} + u_2^s\|_{L^2(B_R)}.$$

Using Theorem 3.5 for u_2^s, we have $\|u_2^s\|_{H^1(B_R)} \leq C\|u^{\mathrm{inc}}\|_{L^2(B_R)}$, which gives

$$\|\mathfrak{S}(q_1)u^{\mathrm{inc}} - \mathfrak{S}(q_2)u^{\mathrm{inc}}\|_{H^1(B_R)} \leq C\|q_1 - q_2\|_{L^\infty(B_R)}\|u^{\mathrm{inc}}\|_{L^2(B_R)},$$

where the constant C depends on k, B_R, and $\|q_2\|_{L^\infty(B_R)}$. ☐

3.2. **Dispersion-dominated model.** Based on the time-domain equation (1.6), we can easily obtain the dispersion-dominated fractional Helmholtz equation as follows:

$$(-\Delta)^{\gamma+1}u - k^2(1 + q)u = 0 \quad \text{in } \mathbb{R}^2. \tag{3.27}$$

Given that model (3.27) is obviously a higher-order elliptic equation, we may transfer this model to a lower-order elliptic system. As mentioned in the above section, the total field u consists of the incident field u^{inc} and the scattered field u^s:

$$u = u^{\mathrm{inc}} + u^s, \tag{3.28}$$

with $u^{\mathrm{inc}}(x) = e^{ikx\cdot\mathbf{d}}$. Using formula (3.4), we will obtain

$$(-\Delta)^{\gamma+1}u^{\mathrm{inc}} - k^{2\gamma+2}u^{\mathrm{inc}} = 0. \tag{3.29}$$

We can easily see that the scattered field u^s satisfies

$$(-\Delta)^{\gamma+1}u^s - k^2(1 + q)u^s = (k^2 q + k^2 - k^{2\gamma+2})u^{\mathrm{inc}}. \tag{3.30}$$

Since $2\gamma + 2 > 2$, equation (3.30) is a high-order elliptic equation that has been relatively ignored by the general theories of fractional elliptic equations. For high-order equations, they can be transformed into some elliptic systems, which will consequently generate cross-correlated terms and may prevent the usage of unique continuation results. Therefore, equation (3.30) seems more difficult than the loss-dominated equation.

By our assumption, there has attenuation effect in domain $\Omega \subset \mathrm{supp}(q) \subset B_R$ and no attenuation effect in $B_R \backslash \Omega$. Hence, we may see that the operator $(-\Delta)^\gamma$ results in the "perturbation" of the non-attenuation equation. Given that the fractional equation (3.30) can be reduced to (3.8) in $\mathbb{R}^2 \backslash \Omega$, the operator defined in (3.10) remains valid. Similar to the loss-dominated case, we consider the bounded domain equation. In this case, the regional fractional Laplace operator $A_\Omega^{\tilde{\gamma}}$ needs to be used. Based on these considerations, we obtain the following elliptic system:

$$\begin{cases} (-\Delta)g^s - k^{3/2}(1+q)u^s = (k^{3/2}q + k^{3/2} - k^{2\gamma+3/2})u^{\mathrm{inc}} \quad \text{in } B_R, \\ A_\Omega^{\tilde{\gamma}} u^s - k^{1/2} g^s = 0 \quad \text{in } \Omega, \\ \dfrac{\partial}{\partial \mathbf{n}} g^s = \mathbb{B} g^s \quad \text{on } \partial B_R, \\ u^s = g^s \quad \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

In the above system and in what follows, the scattered field u^s outside Ω is defined as g^s. Because g^s satisfies a second-order elliptic equation, we expect that the interior regularity of g^s will be high enough to ensure that no boundary term exists in the fractional Gauss-Green formula (2.9). Following the ideas used in Section 7 in [46], we define

$$\mathcal{E}_\Omega^D(u^s, \psi) = \frac{C_{2,\tilde{\gamma}}}{2} \int_\Omega \int_\Omega \frac{(u^s(x) - u^s(y))(\psi(x) - \psi(y))}{|x-y|^{2+2\tilde{\gamma}}} dxdy + \int_{\partial\Omega} u^s \psi dS.$$

For $U^s = (g^s, u^s) \in H^1(B_R) \times H^{\tilde{\gamma}}(\Omega)$ and $\Phi = (\varphi, \psi) \in H^1(B_R) \times H^{\tilde{\gamma}}(\Omega)$, we define

$$\begin{aligned} a_D(U^s, \Phi) = &\int_{B_R} \nabla g^s \cdot \nabla \bar{\varphi} dx - \int_{\partial B_R} \mathbb{B} g^s \bar{\varphi} dS \\ &- k^{3/2} \int_{B_R} (1+q) u^s \bar{\varphi} dx + \mathcal{E}_\Omega^D(u^s, \bar{\psi}) - k^{1/2} \int_\Omega g^s \bar{\psi} dx, \end{aligned}$$

and

$$b_D(\Phi) = k^{3/2} \int_{B_R} (q + 1 - k^{2\gamma}) u^{\mathrm{inc}} \bar{\varphi} dx.$$

Then, we can define the weak formulation of system (3.31) as follow:

$$a_D(U^s, \Phi) = b_D(\Phi) \quad \forall \, \Phi \in H^1(B_R) \times H^{\tilde{\gamma}}(\Omega). \tag{3.32}$$

We use an iterative procedure to prove the well-posedness for the dispersion-dominated equation under the small wavenumber assumption. Different to the loss-dominated equation, we cannot employ the unique continuation result for the second-order elliptic operators [26] to prove uniqueness. Hence, the Fredholm alternative theorem cannot provide a unique solution for a general wavenumber. To achieve well-posedness without small wavenumber assumptions, a unique continuation result for the regional fractional Laplacian may needs to be constructed.

Theorem 3.7. *Let $0 < \tilde{\gamma} < 1/2$ and*

$$k < \left(\frac{\min\left(\frac{C_{2,\tilde{\gamma}}}{2}, 1\right)}{2(1 + \|q\|_{L^\infty(B_R)})(1 + \|u^{inc}\|_{L^2(B_R)})} \right)^2$$

with $C_{2,\tilde{\gamma}}$ is defined as in (2.1). The variational problem (3.32) admits a unique weak solution in $H^1(B_R) \times H^{\tilde{\gamma}}(\Omega)$.

Proof. **Step 1**: An iterative method is employed to show the existence of this problem. Let $u_0^s = g_0^s = 0$, and write the following system:

$$\begin{cases} -\Delta g_{n+1}^s = k^{3/2}(1+q)u_n^s + k^{3/2}(q+1-k^{2\gamma})u^{\mathrm{inc}} \quad \text{in } B_R, \\ A_\Omega^{\tilde{\gamma}} u_{n+1}^s = k^{1/2} g_n^s \quad \text{in } \Omega, \\ \dfrac{\partial}{\partial \mathbf{n}} g_{n+1}^s = \mathbb{B} g_{n+1}^s \quad \text{on } \partial B_R, \\ u_{n+1}^s = g_n^s \quad \text{on } \partial\Omega. \end{cases} \tag{3.33}$$

The weak form of the above system (3.33) can be written as

$$\begin{aligned} &\int_{B_R} \nabla g_{n+1}^s \cdot \nabla \bar{\varphi} dx - \int_{\partial B_R} \mathbb{B} g_{n+1}^s \bar{\varphi} dS \\ &\qquad = k^{3/2} \int_{B_R} (1+q) u_n^s \bar{\varphi} dx + k^{3/2} \int_{B_R} (q+1-k^{2\gamma}) u^{\mathrm{inc}} \bar{\varphi} dx, \end{aligned} \tag{3.34}$$

$$\mathcal{E}_\Omega^D(u_{n+1}^s, \bar{\psi}) = k^{1/2} \int_\Omega g_n^s \bar{\psi} dx. \tag{3.35}$$

Given that this system can be easily solved by using the Lax-Milgram lemma, we may obtain a series of solutions $u_n^s \in H^{\tilde{\gamma}}(\Omega)$ and $g_n^s \in H^1(B_R)$ with $n = 0, 1, 2, \cdots$.

We now need some uniform estimates of the solution series $\{g_n^s, u_n^s\}_{n=0,1,2,\cdots}$. Taking φ, ψ as equal to g_{n+1}^s, u_{n+1}^s in (3.34), we obtain

$$\begin{aligned} k\|g_{n+1}^s\|_{H^1(B_R)}^2 \le & k^{3/2}(1 + \|q\|_{L^\infty(B_R)})(\|u_n^s\|_{L^2(\Omega)} + \|g_n^s\|_{L^2(B_R)})\|g_{n+1}^s\|_{L^2(B_R)} \\ & + k^{3/2}(1 + \|q\|_{L^\infty(B_R)})\|u^{\mathrm{inc}}\|_{L^2(B_R)}\|g_{n+1}^s\|_{L^2(B_R)} \\ \le & \frac{1}{2}k^2(1 + \|q\|_{L^\infty(B_R)})^2(\|u_n^s\|_{L^2(\Omega)} + \|g_n^s\|_{L^2(B_R)} \\ & + \|u^{\mathrm{inc}}\|_{L^2(B_R)})^2 + \frac{k}{2}\|g_{n+1}^s\|_{L^2(B_R)}^2, \end{aligned}$$

where the properties of the DtN operator has been used [39] for the term on the left hand side. Then, we easily find that

$$\begin{aligned} \|g_{n+1}^s\|_{H^1(B_R)} \le & k^{1/2}(1 + \|q\|_{L^\infty(B_R)})(\|u_n^s\|_{H^{\tilde{\gamma}}(\Omega)} \\ & + \|g_n^s\|_{H^1(B_R)} + \|u^{\mathrm{inc}}\|_{L^2(B_R)}). \end{aligned} \tag{3.36}$$

Taking φ, ψ as equal to g_{n+1}^s, u_{n+1}^s in (3.35) and using (7.2) in [46], we obtain

$$\begin{aligned} C_1\|u_{n+1}^s\|_{H^{\tilde{\gamma}}(\Omega)}^2 \le k^{1/2} \int_\Omega g_n^s u_{n+1}^s dx \le k^{1/2}\|g_n^s\|_{L^2(\Omega)}\|u_{n+1}^s\|_{H^{\tilde{\gamma}}(\Omega)} \\ \le \frac{1}{2C_1} k\|g_n^s\|_{H^1(\Omega)}^2 + \frac{C_1}{2}\|u_{n+1}^s\|_{H^{\tilde{\gamma}}(\Omega)}^2, \end{aligned} \tag{3.37}$$

where $C_1 = \min\left(\frac{C_{2,\tilde{\gamma}}}{2}, 1\right)$. Then, we easily know that

$$\|u_{n+1}^s\|_{H^{\tilde{\gamma}}(\Omega)} \le \frac{k^{1/2}}{C_1}\|g_n^s\|_{H^1(B_R)}. \tag{3.38}$$

We now assume that $\|g_k^s\|_{H^1(B_R)} + \|u_k^s\|_{H^{\tilde{\gamma}}(\Omega)} \le 1$ with $k = 0, 1, \cdots, n-1$. Combining (3.36) and (3.38), we finally obtain that

$$\|g_n^s\|_{H^1(B_R)} + \|u_n^s\|_{H^{\tilde{\gamma}}(\Omega)} \le \frac{2k^{1/2}}{C_1}(1 + \|q\|_{L^\infty(B_R)})(1 + \|u^{\mathrm{inc}}\|_{L^2(B_R)}). \tag{3.39}$$

By our condition on k, we know that

$$\|g_n^s\|_{H^1(B_R)} + \|u_n^s\|_{H^{\tilde{\gamma}}(\Omega)} \leq 1. \tag{3.40}$$

Hence, we obtain that (3.40) holds for $n \in \mathbb{N}$.

From Section 7 in [40], we know that $H^{\tilde{\gamma}}(\Omega)$ and $H^1(B_R)$ are compactly embedded into space $L^2(\Omega)$ and $L^2(B_R)$, respectively. Therefore, for some functions u^s and g^s, we have

$$\begin{aligned} &u_n^s \rightharpoonup u^s \quad \text{in } H^{\tilde{\gamma}}(\Omega), \quad g_n^s \rightharpoonup g^s \quad \text{in } H^1(B_R), \\ &u_n^s \to u^s \quad \text{in } L^2(\Omega), \quad g_n^s \to g^s \quad \text{in } L^2(B_R), \end{aligned} \tag{3.41}$$

where "$\rightharpoonup$" indicates weak convergence. Adding (3.34) and (3.35) together and using the above convergence properties (3.41), we finally arrive at

$$a_D(U^s, \Phi) = b_D(\Phi), \tag{3.42}$$

with $U^s = (g^s, u^s)$ and $\Phi = (\varphi, \psi)$. Hence, a solution of system (3.32) has been found.

Step 2: Taking two solutions $U_1^s = (g_1^s, u_1^s)$ and $U_2^s = (g_2^s, u_2^s)$. Denote $\delta U^s = U_1^s - U_2^s = (\delta g^s, \delta u^s)$, then δU^s satisfies

$$\begin{cases} (-\Delta)\delta g^s - k^{3/2}(1+q)\delta u^s = 0 \quad \text{in } B_R, \\ A_\Omega^{\tilde{\gamma}} \delta u^s - k^{1/2}\delta g^s = 0 \quad \text{in } \Omega, \\ \dfrac{\partial}{\partial \mathbf{n}}\delta g^s = \mathbb{B}\delta g^s \quad \text{on } \partial B_R, \\ \delta u^s = \delta g^s \quad \text{on } \partial\Omega. \end{cases} \tag{3.43}$$

For the above system (3.43), performing the same procedure from (3.36) to (3.39), we obtain

$$\begin{aligned} &\|\delta g^s\|_{H^1(B_R)} + \|\delta u^s\|_{H^{\tilde{\gamma}}(\Omega)} \\ &\quad \leq \frac{2k^{1/2}}{C_1}(1 + \|q\|_{L^\infty(B_R)})(\|\delta g^s\|_{H^1(B_R)} + \|\delta u^s\|_{H^{\tilde{\gamma}}(\Omega)}). \end{aligned} \tag{3.44}$$

Based on our assumptions, we find

$$\|\delta g^s\|_{H^1(B_R)} + \|\delta u^s\|_{H^{\tilde{\gamma}}(\Omega)} < \|\delta g^s\|_{H^1(B_R)} + \|\delta u^s\|_{H^{\tilde{\gamma}}(\Omega)}. \tag{3.45}$$

Hence, the proof is completed. □

Remark 3.8. In Theorem 3.7, we only prove $u^s \in H^{\tilde{\gamma}}(\Omega)$ which seems can be improved. From the second equation in (3.31) and $g^s \in H^1(B_R)$, we may expect $u^s \in H^{1+\tilde{\gamma}}(\Omega) \cup H^1(B_R)$. The key point is the following fractional order elliptic equation:

$$\begin{cases} A_\Omega^{\tilde{\gamma}} u^s = kg^s \quad \text{in } \Omega, \\ u^s = g^s \quad \text{on } \partial\Omega. \end{cases} \tag{3.46}$$

Intuitively, we can obtain some higher regularity properties of u^s globally. However, the regional fractional Laplace operator used here differs from the spectral Dirichlet fractional Laplacian employed in [8], which is illustrated in [21]. Obviously, the regional fractional Laplace operator also differs from the fractional Laplacian studied in [42]. Therefore, the conclusions obtained in [8, 42] cannot be used directly. Because the interior regularity for equations with regional fractional Laplacian cannot be easily obtained [38], we speculate that intricate new techniques need to be developed apart from the existing techniques shown in [8, 42]. Hence, we will not investigate equation (3.46) further in this paper.

4. Inverse Methods

In this section, we provide the well-posedness theory for Bayesian inversion with model reduction error. As a straightforward extension, we show the relationship between the Bayesian method and the regularization method. Then, we investigate the small error limit problem, that is, whether the estimated function tends to be the true function if both the model reduction error and white noise vanish. At last, the general theory has been applied to a concrete inverse scattering problem.

4.1. Well-posedness.

Let X, Y be separable Hilbert space equipped with the Borel σ-algebra, and let $\mathcal{G}_a : X \to Y$ be a measurable mapping. Following the ideas presented in [29], we wish to solve the inverse problem of finding x from y where

$$y = \mathcal{G}_a(x) + \epsilon + \eta, \tag{4.1}$$

and $\eta \in Y$ denotes noise, ϵ denotes the model reduction error. We employ a Bayesian approach to solve this problem in which we let $(x, y) \in X \times Y$ be a random variable and compute $x|y$. We specify the assumptions on the random variable (x, y) as follows:

Assumption 2:

- Prior: $x \sim \mu_0$ measure on X and μ_0 is chosen to be a Gaussian with mean $\bar{x}$ and covariance operator $\mathcal{C}_x \in L_1^+(X)$.
- Noise: $\eta \sim \mathbb{Q}_0 = \mathcal{N}(0, \mathcal{C}_\eta)$ measure on Y with $\mathcal{C}_\eta \in L_1^+(Y)$, and $\eta \perp x$.
- Model Reduction Error: $\epsilon \sim \mathbb{R}_{\bar{\epsilon}} = \mathcal{N}(\bar{\epsilon}, \mathcal{C}_\epsilon)$ measure on Y with $\mathcal{C}_\epsilon \in L_1^+(Y)$, and $\eta \perp \epsilon$.

For simplicity, we take $Y = \mathbb{R}^J$ with $J \in \mathbb{N}^+$ in the following. Let $(E, \langle\cdot,\cdot\rangle, \|\cdot\|_E)$ be the Cameron-Martin space of the Gaussian measure μ_0 on X, and we make the following assumptions concerning the potential Φ appearing in Bayes' formula.

Assumption 3: The function $\Phi : X \times Y \to \mathbb{R}$ satisfies the following:

(1) For every $\epsilon > 0$, there is an $M \in \mathbb{R}$, such that for all $u \in X$,

$$\Phi(x; y) \geq M - \epsilon\|u\|_X^2.$$

(2) there exist $p > 0$ and for every $r > 0$ a $K_1 = K_1(r) > 0$ such that, for all $x \in X$ and $y \in Y$ with $|y| < r$,

$$\Phi(x; y) \leq K_1(1 + \|x\|_X^p);$$

(3) for every $r > 0$ there is $K_2 = K_2(r) > 0$ such that, for all $x_1, x_2 \in X$ and $y \in Y$ with $\max\{\|x_1\|_X, \|x_2\|_X, |y|\} < r$,

$$|\Phi(x_1; y) - \Phi(x_2; y)| \leq K_2\|x_1 - x_2\|_X;$$

(4) there is $q \geq 0$ and for every $r > 0$ a $K_3 = K_3(r) > 0$ such that, for all $y_1, y_2 \in Y$ with $\max\{|y_1|, |y_2|\} < r$, and for all $x\, inX$,

$$|\Phi(x; y_1) - \Phi(x; y_2)| \leq K_3(1 + \|x\|_X^q)|y_1 - y_2|.$$

Given that we cannot assume $\epsilon \perp x$ in (4.1), we then assume $(\epsilon, x) \in \mathcal{H} := Y \times X$ that is distributed according to a Gaussian measure $\mathcal{N}((\bar{\epsilon}, \bar{x}), \mathcal{C})$. Denote

$$\mathcal{C}_x = \mathbb{E}(x - \bar{x}) \otimes (x - \bar{x}), \quad \mathcal{C}_\epsilon = \mathbb{E}(\epsilon - \bar{\epsilon}) \otimes (\epsilon - \bar{\epsilon}),$$
$$\mathcal{C}_{x\epsilon} = \mathbb{E}(x - \bar{x}) \otimes (\epsilon - \bar{\epsilon}), \quad \mathcal{C}_{\epsilon x} = \mathbb{E}(\epsilon - \bar{\epsilon}) \otimes (x - \bar{x}).$$

According to Theorem 6.20 in [44] (which results can also be found in [22]), we find that $\epsilon|x \sim \mathcal{N}(\bar{\epsilon}_x, \mathcal{C}_{\epsilon|x})$, where

$$\bar{\epsilon}_x = \bar{\epsilon} + \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}(x - \bar{x}), \quad \mathcal{C}_{\epsilon|x} = \mathcal{C}_\epsilon - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}\mathcal{C}_{x\epsilon}.$$

Define

$$\nu = \epsilon + \eta, \tag{4.2}$$

we have

$$\nu|x = \epsilon|x + \eta, \tag{4.3}$$

and $\nu|x \sim \mathcal{N}(\bar{\nu}_x, \mathcal{C}_{\nu|x})$, where

$$\bar{\nu}_x = \bar{\epsilon} + \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}(x - \bar{x}), \quad \mathcal{C}_{\nu|x} = \mathcal{C}_\eta + \mathcal{C}_\epsilon - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}\mathcal{C}_{x\epsilon}. \tag{4.4}$$

Thus, we obtain

$$y|x \sim \tilde{\mathbb{Q}} := \mathcal{N}(\mathcal{G}(x) + \bar{\nu}_x, \mathcal{C}_{\nu|x}).$$

We assume throughout the following that $\tilde{\mathbb{Q}} \ll \mathbb{Q}_0$ for x μ_0-a.s. Thus, for some potential $\Phi : X \times Y \to \mathbb{R}$, we have

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}_0}(y) = \frac{1}{Z(y)} \exp(-\Phi(x; y)). \tag{4.5}$$

Thus, for a fixed x, $\Phi(x; \cdot) : Y \to \mathbb{R}$ is measurable.

We define ς_0 to be the product measure

$$\varsigma_0(dx, dy) = \mu_0(dx)\mathbb{Q}_0(dy). \tag{4.6}$$

We assume in what follows that $\Phi(\cdot, \cdot)$ is ς_0 measurable. Then, the random variable (x, y) is distributed according to measure $\varsigma(dx, dy) = \mu_0(dx)\tilde{\mathbb{Q}}(dy)$. Furthermore, it then follows that $\varsigma \ll \varsigma_0$ with

$$\frac{d\varsigma}{d\varsigma_0}(x, y) = \frac{1}{Z(y)} \exp(-\Phi(x; y)). \tag{4.7}$$

By similar methods used in the proof of Theorems 4.3 and 4.4 in [19], we can prove the following theorem. The details of this proof are omitted for conciseness.

Theorem 4.1. *Assume that $\Phi : X \times Y \to \mathbb{R}$ is ς_0 measurable, Assumption 2 holds and that, for y $\mathbb{Q}_0$-a.s.,*

$$Z(y) := \int_X \exp(-\Phi(x; y))\mu_0(dx) > 0. \tag{4.8}$$

Then, the conditional distribution of $x|y$ exists under ς, and is denoted by μ^y. Furthermore $\mu^y \ll \mu_0$ and, for y ς-a.s.,

$$\frac{d\mu^y}{d\mu_0}(x) = \frac{1}{Z(y)} \exp(-\Phi(x; y)). \tag{4.9}$$

Moreover, the measure μ^y is Lipschitz in the data y, with respect to the Hellinger distance: if μ^y and $\mu^{y'}$ are two measures given by (4.9) with data y and y', then there is $C = C(r) > 0$ such that, for all y, y' with $\max\{|y|, |y'|\} \leq r$,

$$d_{Hell}(\mu^y, \mu^{y'}) \leq C|y - y'|. \tag{4.10}$$

Consequently all polynomially bounded functions of $x \in X$ are continuous in y. In particular the mean and covariance operator are continuous in y.

Remark 4.2. Let ν be a common reference measure of measures μ and μ'. The Hellinger distance used in Theorem 4.1 is defined by

$$d_{\text{Hell}}(\mu, \mu') = \left(\frac{1}{2}\int\left(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}}\right)^2 d\nu\right)^{1/2}.$$

Remark 4.3. When Y is assumed to be a general Hilbert space, Theorem 4.1 still holds if we add the following assumptions:

(1) $\text{Im}(\mathcal{C}_\eta^{1/2}) = \text{Im}(\mathcal{C}_{\nu|x}^{1/2})$ and $E := \text{Im}(\mathcal{C}_\eta^{1/2})$;
(2) $\mathcal{G}(x) + \bar{\nu}_x \in E$;
(3) the operator $T := (\mathcal{C}_\eta^{-1/2}\mathcal{C}_{\nu|x}^{1/2})(\mathcal{C}_\eta^{-1/2}\mathcal{C}_{\nu|x}^{1/2})^* - I$ is Hilbert-Schmidt in $\bar{E}$.

If the operators $\mathcal{C}_\eta$ and $\mathcal{C}_{\nu|x}$ commute with each other, then these conditions can be simplified further by employing Theorem 2.9 in [17].

Remark 4.4. By applying some small modifications as stated in [19], Theorem 4.1 still holds when X is a separable Banach space.

In the last part of this section, we explain the relations between the Bayesian methods and regularization methods. For this, the MAP estimators and the Onsager-Machlup functional play an important role, which can be seen from the work [20, 18, 26]. Similar to [18], we define a function $I : X \to \mathbb{R}$ by

$$I(x) = \begin{cases} \Phi(x; y) + \frac{1}{2}\|x - \bar{x}\|_E^2 & \text{if } x - \bar{x} \in E, \text{ and} \\ +\infty & \text{else,} \end{cases} \tag{4.11}$$

where E denotes the Cameron-Martin space of the Gaussian measure μ_0 on X. The MAP estimate of a measure μ can be defined as follows.

Definition 4.5. Let

$$M^\epsilon = \sup_{x \in X} \mu(B_\epsilon(x)).$$

Any point $\hat{x} \in X$ satisfying

$$\lim_{\epsilon \to 0} \frac{\mu(B_\epsilon(\hat{x}))}{M^\epsilon} = 1$$

is a MAP estimate for the measure μ.

With these definitions, we present the following results.

Theorem 4.6. *Suppose that Assumption 2 hold. Assume also that there exists an $M \in \mathbb{R}$ such that $\Phi(x; y) \geq M$ for any $x \in X$.*

- *Let $z^\delta = argmax_{z \in X}\, \mu^y(B_\delta(z))$. There is a $\bar{z} \in E$ and a subsequence of $\{z^\delta\}_{\delta > 0}$ which converges to $\bar{z}$ strongly in X.*

- *The limit $\bar{z}$ is a MAP estimator and a minimizer of I.*

Corollary 4.7. *Under the conditions of Theorem 4.6, we have the following.*

- *Any MAP estimator, given by Definition 4.5, minimizes the Onsager-Machlup functional I.*
- *Any $z^* \in E$ which minimizes the Onsager-Machlup functional I is a MAP estimator for measure μ^y appeared in Theorem 4.1.*

Following Assumption 2, the proofs of Theorem 4.6 and Corollary 4.7 are exactly the same as those in [18]. Therefore, we only provide these two results in this paper.

4.2. **Small error limits.** This section is devoted to a small error limit problem, which could be seen as a result of posterior consistency: the idea that the posterior concentrates near the truth that give rise to the data in the small error limits. The present study is inspired by the work [6, 18]. For notational simplicity, we assume that $\bar{x} = 0$ throughout this section. We also let $\mathcal{G}$ be the forward operator without model reduction error and let $\mathcal{G}_n$ be the forward operator with model reduction error $\frac{1}{n}\epsilon_n$ with $n \in \mathbb{N}$, where $\epsilon_n \sim \mathcal{N}(\bar{\epsilon}, \mathcal{C}_\epsilon)$ is defined similarly as that in Assumption 2. In the following, we denote the truth by $x^\dagger$, let X be a separable Hilbert space, and assume that $Y = \mathbb{R}^J$. The problem can be written as

$$y_n = \mathcal{G}_n(x^\dagger) + \frac{1}{n}\epsilon_n + \frac{1}{n}\eta_n \tag{4.12}$$

where $n \in \mathbb{N}$ and $\eta_n \sim \mathbb{Q}_0 = \mathcal{N}(0, \mathcal{C}_\eta)$ is defined similarly as in Assumption 2. Similar to (4.2) and (4.3), we can define ν_n, $\nu_n|x$. Then we have $\nu_n|x \sim \mathcal{N}(\bar{\nu}_x^n, \mathcal{C}_{\nu|x}^n)$ where

$$\begin{aligned} \bar{\nu}_x^n &= \frac{1}{n}\left(\bar{\epsilon} + \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}(x - \bar{x})\right) = \frac{1}{n}\bar{\nu}_x, \\ \mathcal{C}_{\nu|x}^n &= \frac{1}{n^2}\left(\mathcal{C}_\eta + \mathcal{C}_\epsilon - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}\mathcal{C}_{x\epsilon}\right) = \frac{1}{n^2}\mathcal{C}_{\nu|x}. \end{aligned} \tag{4.13}$$

Assume μ_0 satisfy Assumption 2, we have the following formula for the posterior measure:

$$\frac{d\mu^{y_n}}{d\mu_0}(x) \propto \exp\left(-\frac{n^2}{2}|y_n - \mathcal{G}(x) - \bar{\nu}_x^n|_{\mathcal{C}_{\nu|x}}^2\right) \tag{4.14}$$

If we assume that $\mathcal{G}, \mathcal{G}_n$ are uniformly Lipschitz continuous on bounded sets, then based on Theorem 4.6 and Corollary 4.7, the MAP estimate of the above measure are the minimizers of

$$I_n(x) := \|x\|_E^2 + n^2|y_n - G_n(x) - \bar{\nu}_x^n|_{\mathcal{C}_{\nu|x}}^2, \tag{4.15}$$

where E denotes the Cameron-Martin space of the Gaussian measure μ_0 on X. With these preparations, we can show the main result of this section as follows.

Theorem 4.8. *Assume that $\mathcal{G}_n, \mathcal{G} : X \to \mathbb{R}^J$ are uniformly Lipschitz on bounded sets and $x^\dagger \in E$. For every $x \in E$, we assume*

$$\lim_{n\to+\infty} |\mathcal{G}_n(x) - \mathcal{G}(x)| = 0. \tag{4.16}$$

For every $n \in \mathbb{N}$, let $x_n \in E$ be a minimizer of I_n given by (4.15). Then, there exist a $x^* \in E$ and a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges weakly to x^* in E, almost surely. For any such x^*, we have $\mathcal{G}(x^*) = \mathcal{G}(x^\dagger)$.

Proof. For two column vectors $a, b \in \mathbb{R}^J$, denote $\langle a, b\rangle_{\mathcal{C}_{\nu|x}} := a^T \cdot \mathcal{C}_{\nu|x}^{-1} \cdot b$, where a^T represents the transpose of a. Based on (4.12) and (4.13), we obtain

$$\begin{aligned}I_n =&\|x\|_E^2 + n^2\left|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x) + \frac{1}{n}\epsilon_n - \frac{1}{n}\bar{\epsilon} + \frac{1}{n}\eta_n - \frac{1}{n}\mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x\right|_{\mathcal{C}_{\nu|x}}^2 \\ =&\|x\|_E^2 + n^2|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x)|_{\mathcal{C}_{\nu|x}}^2 + |\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x|_{\mathcal{C}_{\nu|x}}^2 \\ &+ 2n\langle \mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x), \epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x\rangle_{\mathcal{C}_{\nu|x}}\end{aligned}$$

Define

$$\begin{aligned}J_n =&\frac{1}{n^2}\|x\|_E^2 + |\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x)|_{\mathcal{C}_{\nu|x}}^2 + \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x|_{\mathcal{C}_{\nu|x}}^2 \\ &+ \frac{2}{n}\langle \mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x), \epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x\rangle_{\mathcal{C}_{\nu|x}}.\end{aligned}$$

We obtain $\mathrm{argmin}_x\, I_n = \mathrm{argmin}_x\, J_n$. Define $x_n \in E$ as $x_n = \mathrm{argmin}_{x\in E}\, J_n(x)$. The existence of x_n obviously follows from Theorem 5.4 in [44]. Based on the definition of x_n, we have

$$\begin{aligned}\frac{1}{n^2}&\|x_n\|_E^2 + |\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|_{\mathcal{C}_{\nu|x}}^2 + \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}}^2 \\ &\quad + \frac{2}{n}\langle \mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n), \epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n\rangle_{\mathcal{C}_{\nu|x}} \\ &\quad \leq \frac{1}{n^2}\|x^\dagger\|_E^2 + \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x^\dagger|_{\mathcal{C}_{\nu|x}}^2.\end{aligned}$$

Some simple calculations yields

$$\begin{aligned}\frac{1}{n^2}\|x_n\|_E^2+&|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|_{\mathcal{C}_{\nu|x}}^2 + \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}}^2 \\ \leq&\frac{1}{n^2}\|x^\dagger\|_E^2 + \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x^\dagger|_{\mathcal{C}_{\nu|x}}^2 \\ &+ \frac{2}{n}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|_{\mathcal{C}_{\nu|x}}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}}.\end{aligned} \tag{4.17}$$

Using Young's inequality, we have

$$\begin{aligned}&\frac{2}{n}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|_{\mathcal{C}_{\nu|x}}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}} \\ \leq&\frac{m-1}{m}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|_{\mathcal{C}_{\nu|x}}^2 + \frac{m}{m-1}\frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}}^2\end{aligned} \tag{4.18}$$

for a large enough real number m which will be specified later. Substituting (4.18) into (4.17), we have

$$\begin{aligned}\frac{1}{n^2}\|x_n\|_E^2+\frac{1}{m}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|_{\mathcal{C}_{\nu|x}}^2 \leq& \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x^\dagger|_{\mathcal{C}_{\nu|x}}^2 \\ &+ \frac{1}{n^2}\|x^\dagger\|_E^2 + \frac{1}{(m-1)n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}}^2.\end{aligned} \tag{4.19}$$

We then focus on the third term on the right-hand side of the above inequality. Simple calculations yield the following:

$$\begin{aligned}&\frac{1}{(m-1)n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x_n|_{\mathcal{C}_{\nu|x}}^2 \\ &\qquad\qquad \leq \frac{2}{m-1}\frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n|_{\mathcal{C}_{\nu|x}}^2 + \frac{2}{m-1}\frac{1}{n^2}C_1\|x_n\|_E^2.\end{aligned} \tag{4.20}$$

Here, we take a sufficiently large m such that $\frac{2}{m-1}C_1 \leq \frac{1}{2}$. By substituting (4.20) into (4.19), we obtain

$$\begin{aligned}\frac{1}{2n^2}\|x_n\|_E^2 + \frac{1}{m}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|^2_{\mathcal{C}_{\nu|x}} \leq \frac{2}{(m-1)n^2}|\epsilon_n - \bar{\epsilon} + \eta_n|^2_{\mathcal{C}_{\nu|x}} \\ + \frac{1}{n^2}\|x^\dagger\|_E^2 + \frac{1}{n^2}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x^\dagger|^2_{\mathcal{C}_{\nu|x}}.\end{aligned} \tag{4.21}$$

Taking expectation on both sides of the above inequality, we obtain

$$\frac{1}{2n^2}\mathbb{E}\|x_n\|_E^2 + \frac{1}{m}\mathbb{E}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|^2_{\mathcal{C}_{\nu|x}} \leq \frac{1}{n^2}\left(\|x^\dagger\|_E^2 + \frac{2}{m-1}K_1 + K_2\right),$$

where

$$K_1 := \mathbb{E}|\epsilon_n - \bar{\epsilon} + \eta_n|^2_{\mathcal{C}_{\nu|x}}, \quad K_2 := \mathbb{E}|\epsilon_n - \bar{\epsilon} + \eta_n - \mathcal{C}_{\epsilon x}\mathcal{C}_x^{-1}x^\dagger|^2_{\mathcal{C}_{\nu|x}}.$$

Obviously, K_1 and K_2 are bounded and independent of n. Hence, we have

$$\mathbb{E}|\mathcal{G}_n(x^\dagger) - \mathcal{G}_n(x_n)|^2_{\mathcal{C}_{\nu|x}} \to 0 \quad \text{as } n \to \infty \tag{4.22}$$

and

$$\mathbb{E}\|x_n\|_E^2 \leq 2\|x^\dagger\|_E^2 + \frac{4}{m-1}K_1 + 2K_2. \tag{4.23}$$

Similar to the proof of (4.4) in [18], by (4.23), we obtain that there exists $x^* \in E$ and a subsequence $\{x_{n_k(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{x\in\mathbb{N}}$ such that

$$\mathbb{E}\langle x_{n_k(k)}, v\rangle_E \to \mathbb{E}\langle x^*, v\rangle_E \quad \text{for any } v \in E. \tag{4.24}$$

By (4.22), we have $|\mathcal{G}_{n_k(k)}(x^\dagger) - G_{n_k(k)}(x_{n_k(k)})|_{\mathcal{C}_{\nu|x}} \to 0$ in probability as $k \to \infty$. Therefore, there exists a subsequence $\{x_{m(k)}\}$ of $\{x_{n_k(k)}\}$ such that

$$\mathcal{G}_{m(k)}(x^\dagger) - \mathcal{G}_{m(k)}(x_{m(k)}) \to 0 \quad \text{a.s. as } k \to \infty.$$

Following our hypothesis, we know that $\mathcal{G}_{m(k)}(x^\dagger) \to \mathcal{G}(x^\dagger)$ as $k \to \infty$. Hence, we have

$$\mathcal{G}_{m(k)}(x_{m(k)}) \to \mathcal{G}(x^\dagger) \quad \text{a.s. as } k \to \infty.$$

From (4.24), we obtain $\langle x_{m(k)} - x^*, v\rangle_E \to 0$ in probability as $k \to \infty$, and so there exists a subsequence $\{x_{\hat{m}(k)}\}$ of $\{x_{m(k)}\}$ such that $x_{\hat{m}(k)}$ converges weakly to x^* in E almost surely as $k \to \infty$. Because E is compactly embedded in X, this implies that $x_{\hat{m}(k)} \to x^*$ in X almost surely as $k \to \infty$. Since

$$|\mathcal{G}(x^*) - \mathcal{G}_{\hat{m}(k)}(x_{\hat{m}(k)})| \leq |\mathcal{G}(x^*) - \mathcal{G}_{\hat{m}(k)}(x^*)| + |\mathcal{G}_{\hat{m}(k)}(x^*) - \mathcal{G}_{\hat{m}(k)}(x_{\hat{m}(k)})|,$$

according to hypothesis (4.16) and $\mathcal{G}_n$ are uniformly Lipschitz bounded, we obtain

$$\mathcal{G}_{\hat{m}(k)}(x_{\hat{m}(k)}) \to \mathcal{G}(x^*) \quad \text{a.s. as } k \to \infty.$$

The proof is therefore completed. □

In the above theorem, we assume that the truth $x^\dagger$ belongs to the Cameron-Martin space E. We can show a weaker convergence result when $x^\dagger$ only belongs to X.

Theorem 4.9. *Suppose that $\mathcal{G}_n$, $\mathcal{G}$ and x_n satisfy the assumptions of Theorem 4.8, and that $x^\dagger \in X$. Then, there exists a subsequence of $\{\mathcal{G}_n(x_n)\}_{n\in\mathbb{N}}$ converging to $\mathcal{G}(x^\dagger)$ almost surely.*

We show the proof of Theorem 4.8 in detail. Combining the above calculations with the arguments used in the proof of Corollary 4.3 in [18], we can easily write the complete proof of Theorem 4.9.

4.3. **Applications to an inverse scattering problem.** Before going further, we provide a hypothesis on the covariance operator.

Assumption 4: The operator A, densely defined on the Hilbert space $\mathcal{H} = L^2(B_R;\mathbb{R}^n)$, satisfies the following properties:

(1) A is positive-definite, self-adjoint and invertible;
(2) the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ of A, form an orthonormal basis for $\mathcal{H}$;
(3) there are $C^{\pm} > 0$ such that the eigenvalues satisfy $\alpha_j \approx j^{2/n}$, for all $j \in \mathbb{N}$;
(4) there is $C > 0$ such that

$$\sup_{j\in\mathbb{N}}\left(\|\phi_j\|_{L^\infty} + \frac{1}{j^{1/n}}\mathrm{Lip}(\phi_j)\right) \leq C.$$

4.3.1. *Without model reduction error.* As a warm up, let us consider the case without model reduction error, which can be covered by the theory developed in [15]. Let $B_R \subset \mathbb{R}^2$ be the ball mentioned in Section 3.1. We set $X = C_u(B_R)$, and define $V := H^1(B_R)$. Let ℓ_j with $j = 1, 2, \cdots, J$ are linear functionals on V, that means $\ell_j \in V^*$ where V^* is the dual space of V. Define

$$\tilde{q}(x) := \log(1 + q(x)), \tag{4.25}$$

and as shown in Section 3.1, we denote $u^s(x) = \mathfrak{S}(e^{\tilde{q}} - 1)u^{\mathrm{inc}}$. According to Theorem 3.5, we may know that $u^s \in H^1(B_R)$. Hence, in our setting, the unknown function x should be the function $\tilde{q}$ and the observation operator can be defined as

$$\mathcal{G}(\tilde{q}) = \{\mathcal{G}_j(\tilde{q})\}_{j=1}^J := \{\ell_j(\mathfrak{S}(e^{\tilde{q}} - 1)u^{\mathrm{inc}})\}_{j=1}^J. \tag{4.26}$$

We take a prior on $\tilde{q}$ to denote the measure $\mathcal{N}(0, A^{-s})$ with $s > 1$ where A is an operator satisfy Assumption 4 with $n = 1$. From Theorem 2.18 in [19], we obtain that $\mu_0(X) = 1$.

Denote $\mathbb{Q}_0 = \mathcal{N}(0, \Gamma)$, $\mathbb{Q}_{\tilde{q}} = \mathcal{N}(\mathcal{G}(\tilde{q}), \Gamma)$. Taking B_1 as a unit ball in X. Since

$$|\mathcal{G}(\tilde{q})| \leq \|\mathfrak{S}(e^{\tilde{q}} - 1)u^{\mathrm{inc}}\|_{H^1} \leq C < \infty, \tag{4.27}$$

and $\eta \sim \mathcal{N}(0, \Gamma)$, noting that y is $\mathbb{Q}_0$-a.s. finite, we have for some $M = M(y) < \infty$

$$\sup_{\tilde{q}\in B_1} \frac{1}{2}|\Gamma^{-1/2}(y - \mathcal{G}(\tilde{q}))|^2 < M.$$

Denote $Z = \int_X \exp(-\Phi(x; y))\mu_0(dx)$, by Theorem 6.28 in [15], we know that

$$Z \geq \int_{B_1} \exp(-M)\mu_0(d\tilde{q}) = \exp(-M)\mu_0(B_1) > 0.$$

Thus, by Theorem 2.1 in [15], we obtain

$$\frac{d\mu^y}{d\mu_0}(\tilde{q}) = \frac{1}{Z(y)}\exp(-\Phi(\tilde{q}; y)),$$

where

$$\Phi(\tilde{q}; y) = \frac{1}{2}|\Gamma^{-1/2}(y - \mathcal{G}(\tilde{q}))|^2. \tag{4.28}$$

Considering (4.27) and Theorem 3.6, we easily verify that Φ in (4.28) satisfies Assumption 3. Hence, we actually prove the following theorem.

Theorem 4.10. *For the two-dimensional inverse scattering problem related with the loss-dominated fractional Helmholtz equation (problem (1.11) with $\mathcal{G}$ given by (4.26)), if we assume $\tilde{q} = \log(1+q) \sim \mu_0$ where $\mu_0 = \mathcal{N}(0, A^{-s})$ with $s > 1$. In addition, we assume $\eta \in \mathbb{R}^J$, $\eta \sim \mathbb{Q}_0$ where $\mathbb{Q}_0 = \mathcal{N}(0, \Gamma)$. Then the posterior measure μ^y exists and absolutely continuous with respect to μ_0 with Randon-Nikodym derivative given by*

$$\frac{d\mu^y}{d\mu_0}(\tilde{q}) = \frac{1}{Z(y)} \exp(-\Phi(\tilde{q}; y)),$$

where

$$\Phi(\tilde{q}; y) = \frac{1}{2}|\Gamma^{-1/2}(y - \mathcal{G}(\tilde{q}))|^2, \quad Z(y) = \int_{C_u(B_R)} \exp(-\Phi(\tilde{q}; y))\mu_0(d\tilde{q}), \tag{4.29}$$

In addition, the measure μ^y is continuous in the Hellinger metric with respect to the data y.

Apart from this well-posedness result, the approximation [16] and MAP estimators results [18] can also be obtained under the aforementioned setting.

4.3.2. *With model reduction error.* For the fractional Helmholtz equation in some unbounded domain, we usually need to calculate it by adding some artificial boundary conditions (e.g., absorbing boundary conditions or perfectly matched layer methods). As a simple illustration, we will analyze the following absorbing boundary conditions:

$$\partial_{\mathbf{n}} u^s = iku^s \quad \text{on } \partial D, \tag{4.30}$$

where $D \subset \mathbb{R}^n$ is a bounded Lipschitz domain. With this boundary condition, our problem takes the following form:

$$\begin{cases} \Delta u^s + i\omega\tau A_L u^s + k^2(1+q)u^s = (-k^2 q - i\omega\tau k^{2\gamma+1})u^{\text{inc}} & \text{in } D \\ \partial_{\mathbf{n}} u^s = iku^s, \quad \text{on } \partial D, \end{cases} \tag{4.31}$$

where $\text{supp}(q) \subset D$. As in Section 3.1, we denote $u^s = \mathfrak{S}_a(q)u^{\text{inc}}$. The operator $\mathfrak{S}_a$ and the operator $\mathfrak{S}$ in Section 3.1 will be similar if the domain D is large enough. For operator $\mathfrak{S}_a$, Theorems 3.3, 3.5, and 3.6 can be established similarly (actually, the proof will be simpler). Denote $\tilde{\epsilon} = \mathfrak{S}_a(q)u^{\text{inc}} - \mathfrak{S}(q)u^{\text{inc}}$, then $\tilde{\epsilon}$ means the system reduction error brought by the absorbing boundary condition.

Similar to Subsection 4.3.1, define $V := H^1(D)$. Let ℓ_j with $j = 1, 2, \cdots, J$ are linear functionals on V, that means $\ell_j \in V^*$. Define

$$\tilde{q}(x) := \log(1 + q(x)), \tag{4.32}$$

then the forward operator will be defined as follows:

$$\mathcal{G}_a(\tilde{q}) = \{\mathcal{G}_a^j(\tilde{q})\}_{j=1}^J := \{\ell_j(\mathfrak{S}_a(e^{\tilde{q}} - 1)u^{\text{inc}})\}_{j=1}^J. \tag{4.33}$$

The system reduction error can be defined as

$$\epsilon = \{\ell_j(\tilde{\epsilon})\}_{j=1}^J = \{\ell_j(\mathfrak{S}_a(q)u^{\text{inc}} - \mathfrak{S}(q)u^{\text{inc}})\}_{j=1}^J. \tag{4.34}$$

Based on these considerations, our model can be formulated as

$$y = \mathcal{G}_a(\tilde{q}) + \epsilon + \eta, \tag{4.35}$$

where $\epsilon, \eta \in \mathbb{R}^J$. In our setting, the covariance operators $\mathcal{C}_\eta$ and $\mathcal{C}_{\nu|\tilde{q}}$ are symmetric matrix. Hence, we can obtain the following form of the potential Φ:

$$\Phi(\tilde{q}; y) = \frac{1}{2}|\mathcal{C}_{\nu|\tilde{q}}^{-1/2}(y - \mathcal{G}_a(\tilde{q}) - \bar{\nu}_{\tilde{q}})|^2. \tag{4.36}$$

Let $s \in \mathbb{R}^+$, $\mu_0 = \mathcal{N}(0, A^{-2(s+1)})$, and $X = H^{1+s}(D)$. Based on Lemma 6.27 in [44], we conclude that $\mu_0(X) = 1$ if A satisfies Assumption 4.

Theorem 4.11. *For the two-dimensional inverse scattering problem concerned with the loss-dominated fractional Helmholtz equation with absorbing boundary condition (problem (4.31) with $\mathcal{G}_a$ given by (4.33)), if we assume $\tilde{q} = \log(1 + q) \sim \mu_0$ where $\mu_0 = \mathcal{N}(0, A^{-2(s+1)})$ with $s > 0$. In addition, we assume $\eta \in \mathbb{R}^J$, $\eta \sim \mathbb{Q}_0$ where $\mathbb{Q}_0 = \mathcal{N}(0, \mathcal{C}_\eta)$, $(\epsilon, \tilde{q}) \in \mathcal{H} := \mathbb{R}^J \times H^{1+s}(D)$ distributed according to a Gaussian measure $\mathcal{N}((\bar{\epsilon}, 0), \mathcal{C})$. Denote ν and $\nu|\tilde{q}$ have same meaning with (4.2) and (4.3). Then the posterior measure μ^y exists and absolutely continuous with respect to μ_0 with Randon-Nikodym derivative given by*

$$\frac{d\mu^y}{d\mu_0}(\tilde{q}) = \frac{1}{Z(y)} \exp(-\Phi(\tilde{q}; y)),$$

where

$$\Phi(\tilde{q}; y) = \frac{1}{2}|\mathcal{C}_{\nu|\tilde{q}}^{-1/2}(y - \mathcal{G}_a(\tilde{q}) - \bar{\nu}_{\tilde{q}})|^2, \tag{4.37}$$

and

$$Z(y) = \int_{H^{1+s}(D)} \exp(-\Phi(\tilde{q}; y))\mu_0(d\tilde{q}).$$

In addition, the measure μ^y is continuous in the Hellinger metric with respect to the data y.

Proof. To conclude the proof of this theorem, we need to check $Z > 0$ $\mathbb{Q}_0$-a.s. and Φ defined in (4.37) satisfy Assumption 3. For the former one, notice that

$$|\mathcal{G}_a(\tilde{q})| \leq \|\mathfrak{S}_a(\tilde{q})\|_{H^1(D)} \leq C < \infty,$$

where C depends on $\|\tilde{q}\|_{L^\infty(D)}$ which can be bounded by $\|\tilde{q}\|_{H^{1+s}(D)}$. Because $\eta \sim \mathcal{N}(0, \mathcal{C}_\eta)$, y is $\mathbb{Q}_0$-a.s. finite, for some $M = M(y) < \infty$, we have

$$\frac{1}{2}|\mathcal{C}_{\nu|\tilde{q}}^{-1/2}(y - \mathcal{G}_a(\tilde{q}) - \bar{\nu}_{\tilde{q}})|^2 < M.$$

Hence, by Theorem 6.28 in [15], we know that

$$Z \geq \int_{B_1} \exp(-M)\mu_0(d\tilde{q}) = \exp(-M)\mu_0(B_1) > 0.$$

To check Φ defined in (4.37) satisfy Assumption 3, we should notice that

$$\|\mathfrak{S}_a(\tilde{q}_1) - \mathfrak{S}_a(\tilde{q}_2)\|_{H^1(D)} \leq C\|\tilde{q}_1 - \tilde{q}_2\|_{L^\infty(D)} \leq C\|\tilde{q}_1 - \tilde{q}_2\|_{H^{1+s}(D)}, \tag{4.38}$$

which can be verified easily by employing similar methods used in the proof of Theorem 3.6. Considering (4.38), Assumption 3 can be verified by some simple calculations. Hence, the proof is completed. $\square$

Remark 4.12. We provide a simple example, which only incorporates model reduction error induced by the absorbing boundary condition. Using a similar method, we may incorporate some other kinds of model reduction error (e.g., induced by perfectly matched layer).

Under the above setting, we easily know that the Onsager-Machlup function has the following form:

$$I(\tilde{q}) = \begin{cases} |\mathcal{C}_{\nu|\tilde{q}}^{-1/2}(y - \mathcal{G}_a(\tilde{q}) - \bar{\nu}_{\tilde{q}})|^2 + \|\tilde{q}\|_E^2 & \text{if } \tilde{q} \in E, \text{ and} \\ +\infty & \text{else,} \end{cases} \tag{4.39}$$

with $E = A^{-(s+1)}H^{1+s}(D)$. According to Theorems 4.6 and 4.7, we can calculate the minimizers of function $I(\tilde{q})$ to obtain some appropriate estimators. Based on this observation, it seems that we can design algorithms by employing ideas used in [5, 31]. However, the present work focuses on the theoretical foundations. For designing practical algorithms, we will report it in our future work.

5. Conclusion

Based on fractional Helmholtz equations, we study two scattering problems related to the loss- and dispersion-dominated fractional Helmholtz equations. For the former, we establish well-posedness for a general wavenumber $k > 0$ and prove the Lipschitz continuity of the solution with respect to the scatterer. For the latter, because the problem seems intricate, we only prove well-posedness for a sufficiently small wavenumber. For general wavenumbers, the problem needs to be investigated further and some unique continuation results of fractional Laplace operators may need to be established.

In order to study an inverse scattering problem related to the loss-dominated fractional Helmholtz equation, we generalize the traditional infinite-dimensional Bayesian method to the infinite-dimensional Bayesian model error method which allows a part of the noise to depend on the target function (the function needs to be estimated). A result similar to posterior consistency has been obtained, and the relationship between Bayesian methods and regularization methods has been discussed. In the end, the general theory has been applied to an inverse scattering problem of the loss-dominated fractional Helmholtz equation.

There are numerous further problems, e.g., designing an algorithm for inverse problems with this new model; generalizing our theory to incorporate the variable Besov prior [28].

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