A CARLEMAN ESTIMATE OF SOME ANISOTROPIC SPACE-FRACTIONAL DIFFUSION EQUATIONS

JUNXIONG JIA AND BANGYU WU

Abstract. This paper is concerned with Carleman estimates for some anisotropic space-fractional diffusion equations, which are important tools for investigating the corresponding control and inverse problems. By employing a special weight function and the nonlocal vector calculus, we prove a Carleman estimate and apply it to build a stability result for a backward diffusion problem.

1. Introduction

Fractional diffusion equations (FDE) have been studied from the aspects of partial differential equations [6, 7] and inverse problems [2, 10]. Comparing to the classical diffusion equations, the nonlocal operators appeared in FDE produce many challenging problems. One of these problems is how to construct a Carleman type estimate. It is well known that the Carleman estimate is a key tool for both the direct problems and inverse problems on PDEs models [1, 4]. For time-fractional equations, Xu et al. [9] prove a Carleman estimate by transforming time-fractional diffusion equations to some integer-order diffusion equations. Then, this result has been extended and used to an inverse coefficient problem by Ren and Xu [8].

However, the results of [8, 9] are focused on time-fractional equations which describe the sub-diffusion phenomenon. Recently, Jin and Rundell [5] point out that the study of space-fractional inverse problems, either theoretical or numerical, is fairly scarce and difficult. To some extent we fill this gap by establishing a Carleman estimate for equations with anisotropic fractional-space operator.

The paper is organized as follows. In Section 2, we provide a brief introduction about nonlocal vector calculus. In Section 3, we prove a Carleman estimate. In Section 4, a backward diffusion problem has been studied.

2. Anisotropic space-fractional diffusion equations

In this section, firstly, let us provide a brief review of the nonlocal vector calculus [3]. Denote n as the space dimension, $\Omega \subset \mathbb{R}^n$ is a bounded open set. Given vector mappings $\nu, \alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ with α antisymmetric, the action of the nonlocal divergence operator $\mathcal{D}$ on ν is defined as

$$\mathcal{D}(\nu)(x) := \int_{\mathbb{R}^n} (\nu(x,y) + \nu(y,x)) \cdot \alpha(x,y)dy \quad \text{for } x \in \mathbb{R}^n. \tag{2.1}$$

Given a mapping $u : \mathbb{R}^n \to \mathbb{R}$, the adjoint operator $\mathcal{D}^*$ of $\mathcal{D}$ is defined as

$$\mathcal{D}^*(u)(x,y) = -(u(y) - u(x))\alpha(x,y) \quad \text{for } x, y \in \mathbb{R}^n. \tag{2.2}$$

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From (2.1) and (2.2), one easily deduces that if $a(t,x,y) = a(t,y,x)$ denotes a second-order tensor satisfying $a = a^T$, then

$$\mathcal{D}(a \cdot \mathcal{D}^* u)(x) = -2 \int_{\mathbb{R}^n} (u(y) - u(x))\alpha(x,y) \cdot (a(t,x,y) \cdot \alpha(x,y))dy \quad \text{for } x \in \mathbb{R}^n.$$

In the following, we denote $\gamma(t,x,y) := \alpha(x,y) \cdot (a(t,x,y) \cdot \alpha(x,y))$. Given an open subset $\Omega \subset \mathbb{R}^n$, the corresponding interaction domain is defined as

$$\Omega_{\mathcal{I}} := \{y \in \mathbb{R}^n \backslash \Omega \quad \text{such that } \alpha(x,y) \neq 0 \text{ for some } x \in \Omega\}. \tag{2.3}$$

Corresponding to the divergence operator $\mathcal{D}(\nu) : \mathbb{R}^n \to \mathbb{R}$ defined in (2.1), we define the action of the nonlocal interaction operator $\mathcal{N}(\nu) : \mathbb{R}^n \to \mathbb{R}$ on ν by

$$\mathcal{N}(\nu)(x) := -\int_{\Omega \cup \Omega_{\mathcal{I}}} (\nu(x,y) + \nu(y,x)) \cdot \alpha(x,y)dy \quad \text{for } x \in \Omega_{\mathcal{I}}. \tag{2.4}$$

With these notations, we have the generalized nonlocal Green's first identity

$$\int_{\Omega} v\mathcal{D}(a \cdot \mathcal{D}^* u)dx - \int_{\Omega \cap \Omega_{\mathcal{I}}} \int_{\Omega \cap \Omega_{\mathcal{I}}} \mathcal{D}^* v \cdot (a \cdot \mathcal{D}^* u)dydx = \int_{\Omega_{\mathcal{I}}} v\mathcal{N}(a \cdot \mathcal{D}^* u)dx. \tag{2.5}$$

Based on the above statements, anisotropic space-fractional diffusion equation with the homogeneous "Dirichlet" volume-constrained condition can be written as

$$\begin{cases} \partial_t u + \mathcal{D}(a \cdot \mathcal{D}^* u) = f(t,x) & \text{on } \Omega \times (0,T), \\ u(t,x) = 0 & \text{on } \Omega_{\mathcal{I}} \times (0,T), \\ u(0,x) = u_0(x) & \text{on } \Omega \cup \Omega_{\mathcal{I}}. \end{cases} \tag{2.6}$$

Similarly, equation with the homogeneous "Neumann" volume-constrained condition could be written as

$$\begin{cases} \partial_t u + \mathcal{D}(a \cdot \mathcal{D}^* u) = f(t,x) & \text{on } \Omega \times (0,T), \\ \mathcal{N}(a \cdot \mathcal{D}^* u) = 0 & \text{on } \Omega_{\mathcal{I}} \times (0,T), \\ u(0,x) = u_0(x) & \text{on } \Omega \cup \Omega_{\mathcal{I}}, \\ \displaystyle\int_{\Omega \cup \Omega_{\mathcal{I}}} u dx = 0. \end{cases} \tag{2.7}$$

Let us specify the functions $\alpha(\cdot,\cdot)$, $a(\cdot,\cdot,\cdot)$ in the definitions of $\mathcal{D}$ and $\mathcal{D}^*$. Assume β to be a constant between 0 and 1. Define $B_\epsilon(x) := \{y \in \Omega \cup \Omega_{\mathcal{I}} : |y - x| \leq \epsilon\}$, and denote $1_{B_\epsilon(x)}(\cdot)$ to be an indicator function which takes value 1 in $B_\epsilon(x)$. We choose

$$\alpha(x,y) = \frac{y - x}{|y - x|^{n/2+\beta+1}} 1_{B_\epsilon(x)}(y), \quad a(t,x,y) = (a_{ij}(t,x,y))_{1 \leq i,j \leq n} \tag{2.8}$$

with $a_{ij} \in C^1([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$, $a_{ij} = a_{ji}$, and in addition, we assume

$$0 < a_*|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t,x,y)\xi_i\xi_j \leq a^*|\xi|^2, \quad \sum_{i,j=1}^n \partial_t a_{ij}(t,x,y)\xi_i\xi_j \leq a^*|\xi|^2 \tag{2.9}$$

for all $\xi \in \mathbb{R}^n$ and $y \in B_\epsilon(x)$. Denote $\gamma(t,x,y) := \alpha(x,y) \cdot a(t,x,y) \cdot \alpha(x,y)$. Then we know that

$$\gamma(t,x,y) = \frac{(y - x) \cdot a(t,x,y) \cdot (y - x)}{|y - x|^{n+2\beta+2}} 1_{B_\epsilon(x)}(y). \tag{2.10}$$

Remark 2.1. Choosing $\epsilon = \infty$ and $a(t,x,y)$ to be the identity matrix, we find that the operator $\mathcal{D}(a\cdot\mathcal{D}^*\cdot)$ is just the operator $(-\Delta)^\beta$. Hence, equations (2.6) and (2.7) incorporate the following equation

$$\partial_t u(x,t) + (-\Delta)^\beta u(x,t) = f(x,t) \quad \text{in } \Omega\times(0,T). \tag{2.11}$$

In the sequel, we always denote $\tilde{\Omega} := \Omega\cup\Omega_{\mathcal{I}}$, $Q := \Omega\times(0,T)$ and $\tilde{Q} := \tilde{\Omega}\times(0,T)$ and define $L^2(0,T;\tilde{H}^{2\beta}(\tilde{\Omega}))$ as a space which includes functions in the following set

$$\left\{ u\in L^2(\tilde{\Omega}) \,:\, \int_0^T\int_\Omega \left|\int_{\tilde{\Omega}}(u(y,t)-u(x,t))\gamma(t,x,y)dy\right|^2 dxdt < \infty \right\}$$

with

$$\|u\|^2_{L^2(0,T;\tilde{H}^{2\beta(\cdot)}(\tilde{\Omega}))} := \int_0^T\int_\Omega \left|\int_{\tilde{\Omega}}(u(y,t)-u(x,t))\gamma(t,x,y)dy\right|^2 dxdt.$$

3. A Carleman estimate

In this section, we denote $L(u) = \partial_t u + \mathcal{D}(a\cdot\mathcal{D}^*u)$ and define $\varphi(t) := e^{\lambda t}$ where $\lambda > 0$ is fixed suitably.

Theorem 3.1. *(Carleman estimate) We set $\varphi(t) = e^{\lambda t}$. Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0,\lambda_0) > 0$ such that*

$$\begin{aligned}
&\int_Q \left\{\frac{1}{s\varphi}\left(|\partial_t u|^2 + |\mathcal{D}(a\cdot\mathcal{D}^*u)|^2\right) + s\lambda^2\varphi u^2\right\} e^{2s\varphi}dxdt \\
&\qquad + \lambda\int_0^T\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{|u(y,t)-u(x,t)|^2}{|y-x|^{n+2\beta}}e^{2s\varphi}dydxdt \\
&\qquad \leq C\int_{\tilde{Q}}|L(u)|^2e^{2s\varphi}dxdt + Ce^{C(\lambda)s}\left(\|u(\cdot,T)\|^2_{H^\beta(\tilde{\Omega})} + \|u(\cdot,0)\|^2_{H^\beta(\tilde{\Omega})}\right)
\end{aligned}$$

for all $s > s_0$ and all $u \in C([0,T];H^\beta(\tilde{\Omega}))\cap H^1(0,T;L^2(\Omega))\cap L^2(0,T;\tilde{H}^{2\beta}(\tilde{\Omega}))$ satisfying $u = 0$ in $\Omega_{\mathcal{I}}\times(0,T)$, or $\mathcal{N}(a\cdot\mathcal{D}^*u) = 0$ in $\Omega_{\mathcal{I}}\times(0,T)$.

Remark 3.2. Solutions mentioned in Theorem 3.1 do exist, which can be easily verified by using the method developed in [3].

Proof. Set $v = e^{s\varphi}u, Pv = e^{s\varphi}L(e^{-s\varphi}v) = e^{s\varphi}f$. Assume that $u|_{\Omega_{\mathcal{I}}} = 0$ or $\mathcal{N}(a\cdot\mathcal{D}^*u)|_{\Omega_{\mathcal{I}}} = 0$. Obviously, we obtain $Pv = \partial_t v - (s\lambda\varphi v - \mathcal{D}(a\cdot\mathcal{D}^*v)) = e^{s\varphi}f$. In addition, we have

$$\begin{aligned}
\|e^{s\varphi}f\|^2_{L^2(Q)} &= \int_Q|\partial_t v|^2dxdt + 2\int_Q\partial_t v\Big(-s\lambda\varphi v + \mathcal{D}(a\cdot\mathcal{D}^*v)\Big)dxdt \\
&\quad + \int_Q|s\lambda\varphi v - \mathcal{D}(a\cdot\mathcal{D}^*v)|^2dxdt \\
&\geq \int_Q|\partial_t v|^2dxdt + 2\int_Q\partial_t v\mathcal{D}(a\cdot\mathcal{D}^*v)dxdt + 2\int_Q\partial_t v(-s\lambda\varphi v)dxdt \qquad (3.1)\\
&\equiv \int_Q|\partial_t v|^2dxdt + \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Thus

$$\int_Q f^2e^{2s\varphi}dxdt \geq \mathrm{I}_1+\mathrm{I}_2, \quad \int_Q|\partial_t v|^2dxdt \leq \int_Q f^2e^{2s\varphi}dxdt + |\mathrm{I}_1+\mathrm{I}_2|. \tag{3.2}$$

In the following, $C_j > 0$ ($j \in \mathbb{N}$) denote generic constants which are independent of s and λ. Because s and λ are assumed to be large enough constants, without loss of generality, we can assume $s > 1$ and $\lambda > 1$. For the term I_1, we have

$$\begin{aligned}
|\mathrm{I}_1| =& \left| 2\int_0^T \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \mathcal{D}^* \partial_t v \cdot a \cdot \mathcal{D}^* v dy dx dt + 2\int_0^T \int_{\Omega_{\mathcal{I}}} \partial_t v \mathcal{N}(a \cdot \mathcal{D}^* v) dx dt \right| \\
=& \left| 2\int_0^T \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} (\partial_t v(y,t) - \partial_t v(x,t))\gamma(t,x,y)(v(y,t) - v(x,t)) dy dx dt \right| \\
=& \Bigg| - \int_0^T \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \partial_t \gamma(t,x,y)(v(y,t) - v(x,t))^2 dy dx dt \\
& \qquad + \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \gamma(t,x,y)(v(y,t) - v(x,t))^2 dy dx \Big|_{t=0}^{t=T} \Bigg| \\
\le& C_1 \|v\|^2_{L^2(0,T;H^\beta(\tilde{\Omega}))} + C_1 \|v(\cdot,T)\|^2_{H^\beta(\tilde{\Omega})} + C_1 \|v(\cdot,0)\|^2_{H^\beta(\tilde{\Omega})},
\end{aligned} \tag{3.3}$$

where (2.5) and (2.9) have been used. For the term I_2, we have

$$\begin{aligned}
\mathrm{I}_2 =& - s\lambda \int_Q 2(\partial_t v)\, v\, \varphi dx dt = s\lambda \int_Q v^2 \partial_t \varphi dx dt - s\lambda \Big(\int_\Omega \varphi v^2 dx \Big) \Big|_{t=0}^{t=T} \\
\ge& s\lambda^2 \int_Q \varphi v^2 dx dt - s\lambda \int_\Omega \left(e^{\lambda T} |v(x,T)|^2 + |v(x,0)|^2 \right) dx.
\end{aligned} \tag{3.4}$$

From the first inequality in (3.2), and estimate (3.3) and (3.4), we obtain

$$\begin{aligned}
\|e^{s\varphi} f\|^2_{L^2(Q)} \ge& s\lambda^2 \int_Q \varphi v^2 dx dt - C_1 \|v\|^2_{L^2(0,T;H^\beta(\tilde{\Omega}))} - C_1 \|v(\cdot,T)\|^2_{H^\beta(\tilde{\Omega})} \\
& - C_1 \|v(\cdot,0)\|^2_{H^\beta(\tilde{\Omega})} - s\lambda \int_{\tilde{\Omega}} \left(e^{\lambda T} |v(x,T)|^2 + |v(x,0)|^2 \right) dx.
\end{aligned} \tag{3.5}$$

In the following, we estimate $\|v\|^2_{L^2(0,T;H^\beta(\tilde{\Omega}))}$. Obviously, we have

$$\begin{aligned}
\int_Q (Pv) v dx dt =& \int_Q v \partial_t v dx dt - \int_Q s\lambda \varphi v^2 dx dt + \int_Q v \mathcal{D}(a \cdot \mathcal{D}^* v) dx dt \\
\equiv& J_1 + J_2 + J_3.
\end{aligned} \tag{3.6}$$

For the term J_1, we find that

$$|J_1| = \left| \int_Q v \partial_t v dx dt \right| = \left| \frac{1}{2} \int_Q \partial_t (v^2) dx dt \right| \le \frac{1}{2} \int_{\tilde{\Omega}} \left(|v(x,T)|^2 + |v(x,0)|^2 \right) dx.$$

For the term J_2, we have $|J_2| = \left| -\int_Q s\lambda \varphi v^2 dx dt \right| \le C_2 \int_Q s\lambda \varphi v^2 dx dt$. At last, for the term J_3, using (2.5), we have

$$\begin{aligned}
J_3 =& \int_0^T \int_{\tilde{\Omega}} v \cdot \mathcal{D}(a \cdot \mathcal{D}^* v) dx dt = \int_0^T \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \mathcal{D}^* v \cdot a(t,x,y) \cdot \mathcal{D}^* v dy dx dt \\
=& \int_0^T \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} (v(y,t) - v(x,t))^2 \gamma(t,x,y) dy dx dt \\
\ge& a_* \int_0^T \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{(v(y,t) - v(x,t))^2}{|y-x|^{n+2\beta}} dy dx dt.
\end{aligned}$$

From (3.6) and the above estimates on J_1, J_2 and J_3, we obtain

$$\begin{aligned}\int_Q \lambda(Pv)vdxdt \geq & a_* \int_0^T\int_{\tilde\Omega}\int_{\tilde\Omega}\lambda\frac{(v(y,t)-v(x,t))^2}{|y-x|^{n+2\beta}}dydxdt - C_2\int_Q s\lambda^2\varphi v^2 dxdt \\ & -\frac{1}{2}\lambda\int_{\tilde\Omega}\left(|v(x,T)|^2+|v(x,0)|^2\right)dx. \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}\int_Q \lambda(Pv)vdxdt \leq & \|Pv\|_{L^2(Q)}\left(\lambda\|v\|_{L^2(Q)}\right) \leq \frac{1}{2}\|Pv\|^2_{L^2(Q)} + \frac{\lambda^2}{2}\|v\|^2_{L^2(Q)} \\ \leq & \frac{1}{2}\|fe^{s\varphi}\|^2_{L^2(Q)} + \frac{\lambda^2}{2}\|v\|^2_{L^2(Q)}. \end{aligned} \tag{3.8}$$

Hence, (3.7) and (3.8) yield

$$\begin{aligned} a_*\int_0^T\int_{\tilde\Omega}\int_{\tilde\Omega}\lambda\frac{(v(y,t)-v(x,t))^2}{|y-x|^{n+2\beta}}dydxdt \leq & C_2\int_Q s\lambda^2\varphi v^2dxdt + \frac{1}{2}\|fe^{s\varphi}\|^2_{L^2(\tilde Q)} \\ & + \frac{\lambda^2}{2}\|v\|^2_{L^2(Q)} + \frac{1}{2}\lambda\int_{\tilde\Omega}\left(|v(x,T)|^2+|v(x,0)|^2\right)dx. \end{aligned}$$

Estimating the first term on the right-hand side by (3.5), we obtain

$$\begin{aligned} a_*\int_0^T\int_{\tilde\Omega}\int_{\tilde\Omega}\lambda\frac{(v(y,t)-v(x,t))^2}{|y-x|^{n+2\beta}}dydxdt \leq & \, C_3\|fe^{s\varphi}\|^2_{L^2(\tilde Q)} + C_3\|v\|^2_{L^2(0,T;H^\beta(\tilde\Omega))} \\ & + C_3\lambda^2\|v\|^2_{L^2(Q)} + C_3\lambda\Big(\|v(\cdot,T)\|^2_{L^2(\tilde\Omega)}+\|v(\cdot,0)\|^2_{L^2(\tilde\Omega)}\Big) + C_3\|v(\cdot,T)\|^2_{H^\beta(\tilde\Omega)} \\ & + C_3\|v(\cdot,0)\|^2_{H^\beta(\tilde\Omega)} + C_3 s\lambda\left(e^{\lambda T}\|v(\cdot,T)\|^2_{L^2(\tilde\Omega)}+\|v(\cdot,0)\|^2_{L^2(\tilde\Omega)}\right). \end{aligned} \tag{3.9}$$

Considering estimates (3.5) and (3.9), we obtain

$$\begin{aligned} & s\lambda^2\int_Q \varphi v^2 dxdt + a_*\int_0^T\int_{\tilde\Omega}\int_{\tilde\Omega}\lambda\frac{|v(y,t)-v(x,t)|^2}{|y-x|^{n+2\beta}}dydxdt \\ & \leq C_4\|fe^{s\varphi}\|^2_{L^2(\tilde Q)} + C_4\|v\|^2_{L^2(0,T;H^\beta(\tilde\Omega))} + C_4\lambda^2\|v\|^2_{L^2(Q)} + C_4\|v(\cdot,T)\|^2_{H^\beta(\tilde\Omega)} \\ & + C_4\|v(\cdot,0)\|^2_{H^\beta(\tilde\Omega)} + C_4 s\lambda\left(e^{\lambda T}\|v(\cdot,T)\|^2_{L^2(\tilde\Omega)}+\|v(\cdot,0)\|^2_{L^2(\tilde\Omega)}\right). \end{aligned} \tag{3.10}$$

Now, we take $s>0$, $\lambda>0$ large enough to absorb the second and third terms on the right-hand side into the left-hand side, then we obtain

$$\begin{aligned} \int_Q s\lambda^2\varphi v^2dxdt + & \int_0^T\int_{\tilde\Omega}\int_{\tilde\Omega}\lambda\frac{|v(y,t)-v(x,t)|^2}{|y-x|^{n+2\beta}}dydxdt \\ & \leq C_5\|fe^{s\varphi}\|^2_{L^2(\tilde Q)} + C_5 e^{C(\lambda)s}\left(\|v(\cdot,T)\|^2_{H^\beta(\tilde\Omega)}+\|v(\cdot,0)\|^2_{H^\beta(\tilde\Omega)}\right). \end{aligned}$$

Because $v=e^{s\varphi}u$, in addition, we have

$$\begin{aligned} \int_Q s\lambda^2\varphi u^2 e^{2s\varphi}dxdt + & \int_0^T\int_{\tilde\Omega}\int_{\tilde\Omega}\lambda\frac{|u(y,t)-u(x,t)|^2}{|y-x|^{n+2\beta}}e^{2s\varphi}dydxdt \\ & \leq C_5\|fe^{s\varphi}\|^2_{L^2(\tilde Q)} + C_5 e^{C(\lambda)s}\left(\|u(\cdot,T)\|^2_{H^\beta(\tilde\Omega)}+\|u(\cdot,0)\|^2_{H^\beta(\tilde\Omega)}\right). \end{aligned} \tag{3.11}$$

Since $\partial_t u = -s\lambda\varphi e^{-s\varphi}v + e^{-s\varphi}\partial_t v$, we obtain $\frac{1}{s\varphi}|\partial_t u|^2 e^{2s\varphi} \le 2s\lambda^2\varphi v^2 + \frac{2}{s\varphi}|\partial_t v|^2$. By the second inequality in (3.2), inequality (3.11) and estimates for $\mathrm{I}_1, \mathrm{I}_2$, we find

$$\begin{aligned}\int_Q \frac{1}{s\varphi}|\partial_t u|^2 e^{2s\varphi}dxdt \le& C\int_{\tilde{Q}} f^2 e^{2s\varphi}dxdt \\ &+ Ce^{C(\lambda)s}\left(\|u(\cdot,T)\|^2_{H^\beta(\tilde{\Omega})} + \|u(\cdot,0)\|^2_{H^\beta(\tilde{\Omega})}\right).\end{aligned} \tag{3.12}$$

From $\mathcal{D}(a\cdot\mathcal{D}^*u) = f - \partial_t u$, we can finish the proof by using (3.11) and (3.12). $\square$

4. Applications to an inverse problem

The *backward in time problem* can be briefly described as: Let $0 \le t_0 < T$. For system (2.6) or (2.7), determine $u(x,t_0)$, $x \in \Omega$ from $u(x,T)$, $x \in \Omega \cup \Omega_{\mathcal{I}}$.

For this problem, there are many studies when $t_0 > 0$ or $t_0 = 0$. As a simple application, we prove a conditional stability estimate for $\|u(\cdot,t_0)\|_{L^2(\Omega)}$ when $t_0 > 0$.

Theorem 4.1. *Let u to be a solution of system (2.6) or (2.7) satisfying $u \in C([0,T];H^\beta(\tilde{\Omega}))\cap L^2(0,T;\tilde{H}^{2\beta}(\tilde{\Omega})), \partial_t u \in L^2(0,T;L^2(\Omega))$. For $t_0 \in (0,T)$, there exist constants $\theta \in (0,1)$ and $C > 0$ depending on t_0, a_*, a^*, T, Ω and $\Omega_{\mathcal{I}}$ such that*

$$\|u(\cdot,t_0)\|_{L^2(\Omega)} \le C\|u\|^{1-\theta}_{L^2(\tilde{Q})}\|u(\cdot,T)\|^\theta_{H^\beta(\Omega\cup\Omega_{\mathcal{I}})}, \tag{4.1}$$

where θ depends on t_0 and $\theta(t_0)$ increases as $t_0 \to T$.

Proof. We choose t_1, t_2 such that $0 < t_2 < t_1 < t_0$, take $\delta_k = e^{\lambda t_k}$, $k = 0,1,2$ and choose a function $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ if $t > t_1$, and $\chi(t) = 0$ if $t > t_2$. Now, we use Theorem 3.1 by similar ideas as in the proof of Theorem 9.2 in [10] to conclude that $\|u(\cdot,t_0)\|_{L^2(\Omega)} \le C\|u\|^{1-\theta}_{L^2(\tilde{Q})}\|u(\cdot,T)\|^\theta_{H^\beta(\tilde{\Omega})}$ with $\theta = \frac{2(\delta_0-\delta_1)}{C+2(\delta_0-\delta_1)}$. $\square$

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School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China;

E-mail address: jjx323@xjtu.edu.cn

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China;

E-mail address: bangyuwu@xjtu.edu.cn