A CARLEMAN ESTIMATE OF SOME ANISOTROPIC
SPACE-FRACTIONAL DIFFUSION EQUATIONS

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Abstract. This paper is concerned with Carleman estimates for some anisotropic space-fractional diffusion equations, which are important tools for investigating the corresponding control and inverse problems. By employing a special weight function and the nonlocal vector calculus, we prove a Carleman estimate and apply it to build a stability result for a backward diffusion problem.

1. Introduction

Fractional diffusion equations (FDE) have been studied from the aspects of partial differential equations [6, 7] and inverse problems [2, 10]. Comparing to the classical diffusion equations, the nonlocal operators appeared in FDE produce many challenging problems. One of these problems is how to construct a Carleman type estimate. It is well known that the Carleman estimate is a key tool for both the direct problems and inverse problems on PDEs models [1, 4]. For time-fractional equations, Xu et al. [9] prove a Carleman estimate by transforming time-fractional diffusion equations to some integer-order diffusion equations. Then, this result has been extended and used to an inverse coefficient problem by Ren and Xu [8]. However, the results of [8, 9] are focused on time-fractional equations which describe the sub-diffusion phenomenon. Recently, Jin and Rundell [5] point out that the study of space-fractional inverse problems, either theoretical or numerical, is fairly scarce and difficult. To some extent we fill this gap by establishing a Carleman estimate for equations with anisotropic fractional-space operator.

The paper is organized as follows. In Section 2, we provide a brief introduction about nonlocal vector calculus. In Section 3, we prove a Carleman estimate. In Section 4, a backward diffusion problem has been studied.

2. Anisotropic space-fractional diffusion equations

In this section, firstly, let us provide a brief review of the nonlocal vector calculus [3]. Denote $n$ as the space dimension, $\Omega \subset \mathbb{R}^n$ is a bounded open set. Given vector mappings $\nu, \alpha : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ with $\alpha$ antisymmetric, the action of the nonlocal divergence operator $\mathcal{D}$ on $\nu$ is defined as

$$\mathcal{D}(\nu)(x) := \int_{\mathbb{R}^n} (\nu(x,y) + \nu(y,x)) \cdot \alpha(x,y) \, dy \quad \text{for} \ x \in \mathbb{R}^n. \quad (2.1)$$

Given a mapping $u : \mathbb{R}^n \rightarrow \mathbb{R}$, the adjoint operator $\mathcal{D}^*$ of $\mathcal{D}$ is defined as

$$\mathcal{D}^*(u)(x,y) = -(u(y) - u(x))\alpha(x,y) \quad \text{for} \ x,y \in \mathbb{R}^n. \quad (2.2)$$

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From (2.1) and (2.2), one easily deduces that if \( a(t, x, y) = a(t, y, x) \) denotes a second-order tensor satisfying \( a = a^T \), then

\[
D(a \cdot D^* u)(x) = -2 \int_{\mathbb{R}^n} (u(y) - u(x))\alpha(x, y) \cdot (a(t, x, y) \cdot \alpha(x, y))dy \quad \text{for} \quad x \in \mathbb{R}^n.
\]

In the following, we denote \( \gamma(t, x, y) := \alpha(x, y) \cdot (a(t, x, y) \cdot \alpha(x, y)) \). Given an open subset \( \Omega \subset \mathbb{R}^n \), the corresponding interaction domain is defined as

\[
\Omega_x := \{ y \in \mathbb{R}^n \setminus \Omega \quad \text{such that} \quad \alpha(x, y) \neq 0 \quad \text{for some} \quad x \in \Omega \}.
\]

Corresponding to the divergence operator \( D(\nu) : \mathbb{R}^n \rightarrow \mathbb{R} \) defined in (2.1), we define the action of the nonlocal interaction operator \( N(\nu) : \mathbb{R}^n \rightarrow \mathbb{R} \) on \( \nu \) by

\[
N(\nu)(x) := - \int_{\mathbb{R}^n \setminus \Omega_x} (\nu(x, y) + \nu(y, x)) \cdot \alpha(x, y)dy \quad \text{for} \quad x \in \Omega_x.
\]

With these notations, we have the generalized nonlocal Green’s first identity

\[
(2.5) \quad \int_{\Omega} vD(a \cdot D^* u)dx - \int_{\Omega \setminus \Omega_x} \int_{\Omega \setminus \Omega_x} D^* v \cdot (a \cdot D^* u)dydx = \int_{\Omega_x} vN(a \cdot D^* u)dx.
\]

Based on the above statements, anisotropic space-fractional diffusion equation with the homogeneous “Dirichlet” volume-constrained condition can be written as

\[
(2.6) \quad \begin{cases}
\partial_t u + D(a \cdot D^* u) = f(t, x) & \text{on} \quad \Omega \times (0, T), \\
u(t, x) = 0 & \text{on} \quad \Omega_x \times (0, T), \\
u(0, x) = \nu_0(x) & \text{on} \quad \Omega \cup \Omega_x.
\end{cases}
\]

Similarly, equation with the homogeneous “Neumann” volume-constrained condition could be written as

\[
(2.7) \quad \begin{cases}
\partial_t u + D(a \cdot D^* u) = f(t, x) & \text{on} \quad \Omega \times (0, T), \\
N(a \cdot D^* u) = 0 & \text{on} \quad \Omega_x \times (0, T), \\
u(0, x) = \nu_0(x) & \text{on} \quad \Omega \cup \Omega_x,
\end{cases}
\]

\[
\int_{\Omega \setminus \Omega_x} ud\tau = 0.
\]

Let us specify the functions \( \alpha(\cdot, \cdot), a(\cdot, \cdot, \cdot) \) in the definitions of \( D \) and \( D^* \). Assume \( \beta \) to be a constant between 0 and 1. Define \( B_\epsilon(x) := \{ y \in \Omega \cup \Omega_x : |y - x| \leq \epsilon \} \), and denote \( 1_{B_\epsilon(x)}(\cdot) \) to be an indicator function which takes value 1 in \( B_\epsilon(x) \). We choose

\[
(2.8) \quad \alpha(x, y) = \frac{y - x}{|y - x|^{n/2 + \beta + 1}} 1_{B_\epsilon(x)}(y), \quad a(t, x, y) = (a_{ij}(t, x, y))_{1 \leq i, j \leq n}
\]

with \( a_{ij} \in C^1([0, T] \times \mathbb{R}^n \times \mathbb{R}^n) \), \( a_{ij} = a_{ji} \), and in addition, we assume

\[
(2.9) \quad 0 < a_* |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t, x, y)\xi_i \xi_j \leq a^* |\xi|^2, \quad \sum_{i,j=1}^n \partial_t a_{ij}(t, x, y)\xi_i \xi_j \leq a^* |\xi|^2
\]

for all \( \xi \in \mathbb{R}^n \) and \( y \in B_\epsilon(x) \). Denote \( \gamma(t, x, y) := \alpha(x, y) \cdot a(t, x, y) \cdot \alpha(x, y) \). Then we know that

\[
(2.10) \quad \gamma(t, x, y) = \frac{(y - x) \cdot a(t, x, y) \cdot (y - x)}{|y - x|^{n+2\beta+2}} 1_{B_\epsilon(x)}(y).
\]
Remark 3.2. Solutions mentioned in Theorem 3.1 do exist, which can be easily verified by using the method developed in [3].

Proof. Set \( v = e^{\varphi x} u, P v = e^{\varphi x} L(e^{-s \varphi} v) = e^{\varphi x} f \). Assume that \( u|_{\partial \Omega} = 0 \) or \( N(a \cdot D^* u)|_{\partial \Omega} = 0 \). Obviously, we obtain \( P v = \partial_t v - (s \lambda \varphi v + D(a \cdot D^* v)) = e^{\varphi x} f \). In addition, we have

\[
\|e^{\varphi x} f\|_{L^2(Q)}^2 = \int_Q |\partial_t v|^2 dxdt + 2 \int_Q \partial_t v \left( -s \lambda \varphi v + D(a \cdot D^* v) \right) dxdt + \int_Q |s \lambda \varphi v - D(a \cdot D^* v)|^2 dxdt
\]

\[
\geq \int_Q |\partial_t v|^2 dxdt + 2 \int_Q \partial_t v D(a \cdot D^* v) dxdt + 2 \int_Q \partial_t v (-s \lambda \varphi v) dxdt
\]

\[
\equiv \int_Q |\partial_t v|^2 dxdt + I_1 + I_2.
\]

Thus

\[
\int_Q f^2 e^{2s \varphi} dxdt \geq I_1 + I_2, \quad \int_Q |\partial_t v|^2 dxdt \leq \int_Q f^2 e^{2s \varphi} dxdt + |I_1 + I_2|.
\]
In the following, $C_j > 0$ ($j \in \mathbb{N}$) denote generic constants which are independent of $s$ and $\lambda$. Because $s$ and $\lambda$ are assumed to be large enough constants, without loss of generality, we can assume $s > 1$ and $\lambda > 1$. For the term $I_1$, we have

$$|I_1| = \left| 2 \int_0^T \int_{\Omega} \mathcal{D}^s \partial_t v \cdot a \cdot \mathcal{D}^s v dydxdt + 2 \int_0^T \int_{\Omega^z} \partial_t v N(a \cdot \mathcal{D}^s v) dx \right|$$

$$= 2 \int_0^T \int_{\Omega} \left( \partial_t v(y, t) - \partial_t v(x, t) \right) \gamma(t, x, y)(v(y, t) - v(x, t)) dydxdt$$

$$= -\int_0^T \int_{\Omega} \partial_t \gamma(t, x, y)(v(y, t) - v(x, t))^2 dydxdt + \int_0^T \int_{\Omega} \gamma(t, x, y)(v(y, t) - v(x, t))^2 dydx dt^{t=T}$$

(3.3) \[ \leq C_1 \|v\|^2_{L^2(0, T; H^s(\tilde{\Omega}))} + C_1 \|v(T)\|^2_{H^s(\tilde{\Omega})} + C_1 \|v(0)\|^2_{H^s(\tilde{\Omega})}, \]

where (2.5) and (2.9) have been used. For the term $I_2$, we have

$$I_2 = -s\lambda \int_Q 2(\partial_t v) \varphi dxdt = s\lambda \int_Q v^2 \partial_t \varphi dxdt - s\lambda \int_\Omega \varphi dxdt \bigg|^{t=T}_{t=0}$$

(3.4) \[ \geq s\lambda^2 \int_Q \varphi v^2 dxdt - s\lambda \int_\Omega (e^{\lambda T}|v(x, T)|^2 + |v(x, 0)|^2) dx. \]

From the first inequality in (3.2), and estimate (3.3) and (3.4), we obtain

$$\|e^{\lambda f}\|_{L^2(Q)} \geq s\lambda^2 \int_Q \varphi v^2 dxdt - C_1 \|v\|^2_{L^2(0, T; H^s(\tilde{\Omega}))} - C_1 \|v(T)\|^2_{H^s(\tilde{\Omega})} - C_1 \|v(0)\|^2_{H^s(\tilde{\Omega})}$$

(3.5) \[ - s\lambda \int_\Omega \big( e^{\lambda T}|v(x, T)|^2 + |v(x, 0)|^2 \big) dx. \]

In the following, we estimate $\|v\|^2_{L^2(0, T; H^s(\tilde{\Omega}))}$. Obviously, we have

$$\int_Q (Pv)v dxdt = \int_Q \varphi v dxdt - \int_Q s\lambda \varphi v^2 dxdt + \int_Q \mathcal{D}(a \cdot \mathcal{D}^s v) dxdt$$

(3.6) \[ \equiv J_1 + J_2 + J_3. \]

For the term $J_1$, we find that

$$|J_1| = \left| \int_Q v \partial_t v dxdt \right| = \frac{1}{2} \int_Q \partial_t (v^2) dxdt \leq \frac{1}{2} \int_\Omega (|v(x, T)|^2 + |v(x, 0)|^2) dx.$$

For the term $J_2$, we have $|J_2| = \left| -\int_Q s\lambda \varphi v^2 dxdt \right| \leq C_2 \int_Q s\lambda \varphi v^2 dxdt$. At last, for the term $J_3$, using (2.5), we have

$$J_3 = \int_0^T \int_{\Omega} v \mathcal{D}(a \cdot \mathcal{D}^s v) dxdt = \int_0^T \int_{\Omega} \mathcal{D}^s v \cdot a(t, x, y) \cdot \mathcal{D}^s v dydxdt$$

$$= \int_0^T \int_{\Omega} \left( (v(y, t) - v(x, t))^2 \gamma(t, x, y) dydxdt$$

$$\geq a_\ast \int_0^T \int_{\Omega} \frac{(v(y, t) - v(x, t))^2}{|y - x|^{n+2s}} dydxdt.$$


From (3.6) and the above estimates on $J_1$, $J_2$ and $J_3$, we obtain

$$
\int_Q \lambda(Pv) v dx dt \geq a_* \int_0^T \int_\Omega \int_\Omega \frac{\lambda (v(y, t) - v(x, t))^2}{|y - x|^{n+2\delta}} dy dx dt - C_2 \int_Q s \lambda^2 v^2 dx dt
$$

(3.7)

$$
- \frac{1}{2} \lambda \int_\Omega \int_\Omega (|v(x, T)|^2 + |v(x, 0)|^2) \, dx.
$$

On the other hand, we have

$$
\int_Q \lambda(Pv) v dx dt \leq \|Pv\|_{L^2(Q)} (\lambda \|v\|_{L^2(Q)}) \leq \frac{1}{2} \|Pv\|_{L^2(\tilde{\Omega})}^2 + \frac{\lambda^2}{2} \|v\|_{L^2(Q)}^2
$$

(3.8)

$$
\leq \frac{1}{2} \|fe^{|v|}\|_{L^2(Q)}^2 + \frac{\lambda^2}{2} \|v\|_{L^2(Q)}^2.
$$

Hence, (3.7) and (3.8) yield

$$
a_* \int_0^T \int_\Omega \int_\Omega \frac{\lambda (v(y, t) - v(x, t))^2}{|y - x|^{n+2\delta}} dy dx dt \leq C_2 \int_Q s \lambda^2 v^2 dx dt + \frac{1}{2} \|fe^{|v|}\|_{L^2(Q)}^2
$$

$$
+ \frac{\lambda^2}{2} \|v\|_{L^2(Q)}^2 + \frac{1}{2} \lambda \int_\Omega \int_\Omega (|v(x, T)|^2 + |v(x, 0)|^2) \, dx.
$$

Estimating the first term on the right-hand side by (3.5), we obtain

$$
a_* \int_0^T \int_\Omega \int_\Omega \frac{\lambda (v(y, t) - v(x, t))^2}{|y - x|^{n+2\delta}} dy dx dt \leq C_3 \|fe^{|v|}\|_{L^2(Q)}^2 + C_3\|v\|_{L^2(0, T; H^\delta(\tilde{\Omega}))}^2
$$

$$
+ C_4 \lambda^2 \|v\|_{L^2(\tilde{\Omega})}^2 + C_4 \lambda \left(\|v(\cdot, T)\|_{L^2(\tilde{\Omega})}^2 + \|v(\cdot, 0)\|_{L^2(\tilde{\Omega})}^2\right) + C_3 \lambda \|v(\cdot, T)\|_{H^\delta(\tilde{\Omega})}^2
$$

(3.9)

$$
+ C_3 \|v(\cdot, 0)\|_{H^3(\tilde{\Omega})}^2 + C_3 s \lambda \left(e^{\lambda T} \|v(\cdot, T)\|_{L^2(\tilde{\Omega})}^2 + \|v(\cdot, 0)\|_{L^2(\tilde{\Omega})}^2\right).
$$

Considering estimates (3.5) and (3.9), we obtain

$$
s \lambda^2 \int_Q \varphi^2 dx dt + a_* \int_0^T \int_\Omega \int_\Omega \frac{\lambda (v(y, t) - v(x, t))^2}{|y - x|^{n+2\delta}} dy dx dt
$$

$$
\leq C_4 \|fe^{|v|}\|_{L^2(\tilde{\Omega})}^2 + C_4 \|v\|_{L^2(0, T; H^\delta(\tilde{\Omega}))}^2 + C_4 \lambda^2 \|v\|_{L^2(\tilde{\Omega})}^2 + C_4 \|v(\cdot, T)\|_{H^\delta(\tilde{\Omega})}^2
$$

(3.10)

$$
+ C_1 \|v(\cdot, 0)\|_{H^3(\tilde{\Omega})}^2 + C_1 s \lambda \left(e^{\lambda T} \|v(\cdot, T)\|_{L^2(\tilde{\Omega})}^2 + \|v(\cdot, 0)\|_{L^2(\tilde{\Omega})}^2\right).
$$

Now, we take $s > 0$, $\lambda > 0$ large enough to absorb the second and third terms on the right-hand side into the left-hand side, then we obtain

$$
\int_Q s \lambda^2 v^2 dx dt + \int_0^T \int_\Omega \int_\Omega \frac{\lambda (v(y, t) - v(x, t))^2}{|y - x|^{n+2\delta}} dy dx dt
$$

$$
\leq C_3 \|fe^{|v|}\|_{L^2(\tilde{\Omega})}^2 + C_5 e^{C(\lambda)s} \left(\|v(\cdot, T)\|_{H^\delta(\tilde{\Omega})}^2 + \|v(\cdot, 0)\|_{H^\delta(\tilde{\Omega})}^2\right).
$$

Because $v = e^{\lambda \varphi} u$, in addition, we have

$$
\int_Q s \lambda^2 \varphi u^2 e^{\lambda \varphi} dx dt + \int_0^T \int_\Omega \int_\Omega \frac{\lambda (u(y, t) - u(x, t))^2}{|y - x|^{n+2\delta}} e^{\lambda \varphi} dy dx dt
$$

(3.11)

$$
\leq C_3 \|fe^{|v|}\|_{L^2(\tilde{\Omega})}^2 + C_5 e^{C(\lambda)s} \left(\|u(\cdot, T)\|_{H^\delta(\tilde{\Omega})}^2 + \|u(\cdot, 0)\|_{H^\delta(\tilde{\Omega})}^2\right).
Since \( \partial_t u = -s\lambda \varphi e^{-s\varphi}v + e^{-s\varphi} \partial_t v, \) we obtain
\[
\frac{1}{2s^2} |\partial_t u|^2 e^{2s\varphi} \leq 2s\lambda^2 \varphi^2 + \frac{\lambda^2}{2s^2} |\partial_t v|^2.
\] By the second inequality in (3.2), inequality (3.11) and estimates for \( I_1, I_2, \) we find
\[
\int_Q \frac{1}{2s^2} |\partial_t u|^2 e^{2s\varphi} \, dx \, dt \leq C \int_Q f^2 e^{2s\varphi} \, dx \, dt + C \epsilon^{C(\lambda)_0} \left( \|u(\cdot, T)\|_{H^\beta(\tilde{\Omega})}^2 + \|u(\cdot, 0)\|_{H^\beta(\tilde{\Omega})}^2 \right).
\] From \( D(a \cdot D^* u) = f - \partial_t u, \) we can finish the proof by using (3.11) and (3.12).

4. Applications to an inverse problem

The backward in time problem can be briefly described as: Let \( 0 \leq t_0 < T. \) For system (2.6) or (2.7), determine \( u(x, t_0), \) \( \chi(\cdot, t_0) \in \Omega \) from \( u(x, T), \chi(\cdot, T), x \in \Omega \cup \Omega_\xi. \)

For this problem, there are many studies when \( t_0 > 0 \) or \( t_0 = 0. \) As a simple application, we prove a conditional stability estimate for \( \|u(\cdot, t_0)\|_{L^2(\Omega)} \) when \( t_0 > 0. \)

**Theorem 4.1.** Let \( u \) to be a solution of system (2.6) or (2.7) satisfying \( u \in C([0, T]; H^2(\tilde{\Omega}) \cap L^2(0, T; H^2(\tilde{\Omega}))), \partial_t u \in L^2(0, T; L^2(\Omega)). \) For \( t_0 \in (0, T), \) there exist constants \( \theta \in (0, 1) \) and \( C > 0 \) depending on \( t_0, a_*, a^*, T, \Omega, \Omega_\xi \) such that
\[
\|u(\cdot, t_0)\|_{L^2(\Omega)} \leq C \|u\|^{1-\theta}_{L^2(\tilde{\Omega})} \|u(\cdot, T)\|^{\theta}_{H^2(\tilde{\Omega})},
\]
where \( \theta \) depends on \( t_0 \) and \( \beta(t_0) \) increases as \( t_0 \to T. \)

**Proof.** We choose \( t_1, t_2 \) such that \( 0 < t_2 < t_1 < t_0, \) take \( \delta_k = e^{\lambda k}, k = 0, 1, 2 \) and choose a function \( \chi \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \chi \leq 1, \chi(t) = 1 \) if \( t > t_1, \) and \( \chi(t) = 0 \) if \( t < t_2. \) Now, we use Theorem 3.1 by similar ideas as in the proof of Theorem 9.2 in [10] to conclude that
\[
\|u(\cdot, t_0)\|_{L^2(\Omega)} \leq C \|u\|^{1-\theta}_{L^2(\tilde{\Omega})} \|u(\cdot, T)\|^{\theta}_{H^2(\tilde{\Omega})}
\]
with \( \theta = \frac{2(\delta_0 - \delta_1)}{\delta_0 + \delta_1}, \)

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