HARNACK'S INEQUALITY FOR A SPACE-TIME FRACTIONAL DIFFUSION EQUATION AND APPLICATIONS TO AN INVERSE SOURCE PROBLEM

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Abstract. In this paper, we focus on a space-time fractional diffusion equation with the generalized Caputo's fractional derivative operator and a general space nonlocal operator (with the fractional Laplace operator as a special case). A weak Harnack's inequality has been established by using a special test function and some properties of the space nonlocal operator. Based on the weak Harnack's inequality, a strong maximum principle has been obtained which is an important characterization of fractional parabolic equations. With these tools, we establish a uniqueness result of an inverse source problem on the determination of the temporal component of the inhomogeneous term, which seems to be the first theoretical result of the inverse problem for such a general fractional diffusion model.

1. Introduction

Fractional partial differential equations grow up to be a popular research topic for its wide applications in physics [24], geological exploration [39] and so on. For mathematical properties, Baeumer, Meerschaert etc. [3, 2, 4] investigate time fractional parabolic equations from functional and probabilistic perspective. They construct stochastic solutions and discover a lot of interesting relations between fractional differential equations and stochastic process. By extending operator semigroup theory, mild solutions and some subordination principles have been obtained in [16, 19] for some types of fractional parabolic equations. For time fractional parabolic equations, Zacher [34, 35, 36, 37] construct a series of theories concerned with weak solutions and Hölder continuities of the solutions. Recently, Allen, Caffarelli and Vasseur [1] obtain many important regularity properties for a space-time fractional parabolic equation. In this paper, we focus on a general fractional diffusion equation. Before going further, let us introduce some notations. For a real number $\gamma \in \mathbb{R}$, denote $g_\gamma(t)$ by

$$g_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \tag{1.1}$$

where $\Gamma(\cdot)$ represents the usual Gamma function. The notation $\partial_t^\alpha \cdot$ denotes the Riemann-Liouville fractional derivative defined by

$$\partial_t^\alpha f(t) := \frac{d}{dt}(g_{1-\alpha} * f(\cdot))(t), \tag{1.2}$$

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where "$*$" denotes the usual convolution operator. The space-time nonlocal diffusion equation studied in this paper has the following form

$$\begin{cases} \partial_t^\alpha(u(x,t)-u_0(x)) + Lu(x,t) = f(x,t) & \text{in } \Omega\times[0,T], \\ u(x,t)=0 & \text{in } \mathbb{R}^n\backslash\Omega,\ t\geq 0, \\ u(x,0)=u_0(x) & \text{in } \Omega, \text{ for } t=0, \end{cases} \tag{1.3}$$

where $\alpha\in(0,1)$ and L is an integro-differential operator of the form

$$Lu(t,x) = \text{p.v.}\int_{\mathbb{R}^n}[u(t,x)-u(t,y)]k(x,y)dy. \tag{1.4}$$

The time-fractional operator used here could be called the generalized Caputo's fractional derivative. For more details, we refer to [26]. The kernel $k:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$, $(x,y)\mapsto k(x,y)$ is assumed to be measurable with a certain singularity at the diagonal $x=y$.

Note that in the case $k(x,y)=c_{n,\beta}/|x-y|^{n+2\beta}$ with constant $c_{n,\beta}=\frac{\beta 2^{2\beta}\Gamma(\frac{n+2\beta}{2})}{\pi^{n/2}\Gamma(1-\beta)}$, the integral-differential operator L defined in (1.4) is equal to $(-\Delta)^\beta$ which is the pseudo-differential operator with symbol $|\xi|^{2\beta}$. Thus the operator L could be seen as a generalized fractional Laplace operator. And the following space-time fractional diffusion equation is a special case of equation (1.3)

$$\begin{cases} \partial_t^\alpha(u(x,t)-u_0(x)) + (-\Delta)^\beta u(x,t) = f(x,t) & \text{in } \Omega\times[0,T], \\ u(x,t)=0 & \text{in } \mathbb{R}^n\backslash\Omega,\ t\geq 0, \\ u(x,0)=u_0(x) & \text{in } \Omega, \text{ for } t=0, \end{cases} \tag{1.5}$$

with $\alpha,\beta\in(0,1)$.

Now, let us specify the assumptions on the kernels $k(\cdot,\cdot)$. We assume the kernels k are of the form $k(x,y)=a(x,y)k_0(x,y)$ for some measurable functions $k_0:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty]$ and $a:\mathbb{R}^n\times\mathbb{R}^n\to[1/2,1]$ which are symmetric with respect to x and y.

Fix $\beta_0\in(0,1)$ and $\Lambda\geq\max(1,\beta_0^{-1})$. A kernel k belongs to $\mathcal{R}(\beta_0,\Lambda)$, if there is $\beta\in(\beta_0,1)$ such that k_0 satisfies the following properties: for some constant $C>0$, every $x_0\in\mathbb{R}^n$, $\rho>0$, $B_\rho(x_0)\subset\Omega$ and $u\in H^\beta(B_\rho(x_0))$

$$\rho^{-2}\int_{|x_0-y|\leq\rho}|x_0-y|^2k_0(x_0,y)dy + \int_{|x_0-y|>\rho}k_0(x_0,y)dy \leq \Lambda\rho^{-2\beta}, \tag{1.6}$$

$$\begin{aligned} C^{-1}\Lambda^{-1}\int_B\int_B(v(x)-v(y))^2k_0(x,y)dxdy &\leq c_{n,\beta}\int_B\int_B\frac{(v(x)-v(y))^2}{|x-y|^{n+2\beta}}dxdy \\ &\leq C\Lambda\int_B\int_B(v(x)-v(y))^2k_0(x,y)dxdy, \quad \text{where } B=B_\rho(x_0). \end{aligned} \tag{1.7}$$

Inverse problems of fractional diffusion equations are a rather new research topic, however, there are already a lot of studies on time-fractional space-integer order diffusion equations. In 2009, an inverse problem related to a one dimensional time-fractional space-integer order diffusion equation has been studied in [6]. In this paper, a uniqueness result for the coefficient and the time fractional-order has been proved by using eigenfunction expansion of the weak solution and the Gel'fand-Levitan theory. In 2010, Li and Liu [32] study backward diffusion problem for a time-fractional space-integer order diffusion equation by generalizing the total variation regularization methods. In 2011, Zhang and Xu [38] study an inverse source problem related to a time-fractional space-integer order diffusion equation by the method of the eigenfunction expansion and numerical methods are also been presented. In 2012, Jin and Rundell [17] study an inverse problem of recovering a spatially varying potential term in an one-dimensional time-fractional diffusion equation. Meanwhile, Yamamoto and Zhang [33] obtain a local stability result of an inverse source problem for a one-dimensional fractional diffusion equation of half-order in time. Then, this stability result has been extended to an inverse coefficient problem [27] by Ren and Xu. Their results based upon two Carleman estimates and transformations between time-fractional diffusion equations and integer-order diffusion equations. In 2013, Miller and Yamamoto [25] investigate an inverse problem of determining spatial coefficient related to a time-fractional space-integer order diffusion equation. In 2015, Jin and Rundell [18] provide a lengthy review article and they also show some further results about inverse problems related to the anomalous diffusion processes. At the end of review article [18], they point out that the study of space-fractional inverse problem, either theoretical or numerical, is fairly scarce. And this is partly attributed to the relatively poor understanding of forward problems for PDEs with a space fractional derivative.

Hence, only recently, there are a few investigations on the inverse problem for the more general space-time fractional diffusion equations. For space-time fractional diffusion equation with the spectral Dirichlet fractional Laplacian operator [11], uniqueness and some numerical results have been obtained in [30, 31]. For space-time fractional diffusion equation on a periodic domain, backward diffusion problem has been studied in [14] under the Bayesian statistical framework and the same backward diffusion problem has also been studied by using variable total variation regularization methods in [15].

However, there seem rare studies on the space-time fractional diffusion equation with the restricted Dirichlet fractional Laplace operator [11]. Since we impose $u = 0$ outside of Ω other than the usual boundary condition, the fractional Laplace operator appeared in (1.5) should be understood as the restricted Dirichlet fractional Laplace operator. For more details and different properties of the spectral Dirichlet fractional Laplacian operator and the restricted Dirichlet fractional Laplace operator, we refer to [11] and references therein. In this paper, we attempt to study the forward problem (1.3) more deeply. Through the tools established for the general space-time fractional diffusion equation, we obtain a uniqueness result for an inverse source problem on the determination of the temporal component of the inhomogeneous term.

More precisely, we will assume the inhomogeneous term to be of the form $\rho(t)g(x)$ with some appropriate assumptions, which will be specified in Section 5. Let $x_0 \in \Omega$ and $T > 0$ be arbitrarily given, and u be the solution to (1.3) with $u_0 = 0$. Provided that $g(\cdot)$ is known, determine $\rho(t)\,(0 \leq t \leq T)$ by the single point observation data $u(x_0, t)\,(0 \leq t \leq T)$. Recently, similar types of problems are studied in [21, 29] for time-fractional space-integer order diffusion equations. Especially, Liu, Rundell and Yamamoto [21] prove a strong maximum principle which holds almost everywhere (roughly speaking) by using the eigenfunction expansion technique. Inspired by their work, we attempt to prove a strong maximum principle for the general fractional diffusion equation (1.3) other than a time-fractional space-integer order diffusion equation. Our methods are totally different from the methods used in [21] since our maximum principle has been obtained as a direct corollary of a weak Harnack's inequality. The contributions of this paper could be summarized as follows:

- When the kernel k in the definition of L belongs to some $\mathcal{R}(\beta_0, \Lambda)$, we prove a weak Harnack's inequality, which may be the first result of Harnack' inequality for the space-time fractional diffusion equations. Specific results will be shown in Section 3.
- A strong maximum principle has been proved, which provides a useful characterization of the solutions of the space-time fractional equations. Rigorous results will be presented in Section 4. The strong maximum principle can be used to many problems, especially for some inverse problems, e.g., [12, 22].
- Under a little stronger assumptions about the kernel k, we prove a uniqueness result of the above mentioned inverse source problem. Detailed assumptions and results will be shown in Section 5.

The organization of this paper is as follows. In Section 2, some preliminary knowledge and results will be shown. These knowledge include the definition of fractional Sobolev space, the definition of Yosida approximation. Two equivalent definitions of weak solution will also be presented. In the last part of Section 2, a unique weak solution of equation (1.3) will be constructed. Then, weak Harnak's inequality has been proved in Section 3 and the proof has been divided into four steps. In Section 4, a weak and a strong maximum principle have been proved which is the primary tools for our investigation on the inverse source problems. In Section 5, more regularity properties of the weak solution have been proved under a little stronger assumptions about the kernel k defined in the definition of L. Then a fractional Duhamel's principle has been established. At last, a uniqueness result of the inverse source problem has been achieved. In Appendix, we provide some useful lemmas.

2. Preliminaries

In this section, we provide some necessary preliminary knowledge on the function space theory, Yosida approximation and equivalent definitions of weak solutions for our purposes.

Here, let us specify the assumptions about the spatial dimension in this paper. In the following parts of this paper, the spatial dimension n equal to 2 or 3 and we will not mention this assumption again in each theorem or lemma shown below.

2.1. **A short introduction to some function spaces.** Let us provide some general notations:

- We denote $W^{s,p}$ be the Sobolev space with s-times derivative belongs to L^p space. For a Banach space X, we denote ${}_0W^{s,p}([0,T];X)$ be the Sobolev space with functions vanishing at $t=0$. When $p=2$, we denote ${}_0W^{s,2}([0,T];X)$ as ${}_0H^s([0,T];X)$.
- By $\inf u$ and $\sup u$ we denote the essential infimum and the essential supremum of a given function u respectively.
- Without additional specifications, we denote $B(x_0, r)$ be a ball in $\mathbb{R}^n$ centered at x_0 with radius r. If $x_0 = 0$, we denote $B_r := B(0,r)$ for concisely.
- For a function $f \in C^1(\mathbb{R}^n)$, sometimes, we denote $\frac{d}{dt}f(t)$ as $\dot{f}(t)$.

- In all the following parts of this paper, we denote $c_{n,\beta} = \frac{\beta 2^{2\beta}\Gamma(\frac{n+2\beta}{2})}{\pi^{n/2}\Gamma(1-\beta)}$ and denote $\mathcal{S}^{n-1}$ be the surface of a unit ball in $\mathbb{R}^n$.
- The notation "$*$" denotes the usual convolution operator defined as
$$(f * g)(t) = \int_0^t f(t-s)g(s)ds$$
with $t > 0$ for two appropriate functions.
- Notation C represents a general constant, which may different from line to line.

Now, some function spaces used in this paper will be explained. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, then the Sobolev space of fractional order $s \geq 0$ is defined by

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \,:\, \frac{|u(x)-u(y)|}{|x-y|^{s+n/2}} \in L^2(\Omega \times \Omega) \right\}, \tag{2.1}$$

endowed with the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + c_{n,s} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy. \tag{2.2}$$

We denote by $H_0^s(\Omega)$ the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_{H^s(\mathbb{R}^n)}$ and by H^{-s} the dual of H_0^s.

According to the probabilistic interpretation of the space-nonlocal integral-differential operator [9, 24], the boundary condition should be changed to the exterior boundary condition which will be specified later. In order to cope with this situation, we define $H_e^s(\Omega)$ $(s \in \mathbb{R})$ as follow

$$H_e^s(\Omega) := \{u \in H^s(\mathbb{R}^n) \,:\, u = 0 \text{ in } \mathbb{R}^n \backslash \Omega\}, \tag{2.3}$$

and $L_e^p(\Omega)$ $(1 \leq p \leq \infty)$ as

$$L_e^p(\Omega) := \{u \in L_e^p(\mathbb{R}^n) \,:\, u = 0 \text{ in } \mathbb{R}^n \backslash \Omega\}. \tag{2.4}$$

For $p \in [1, \infty)$, denote

$$\begin{aligned} V_p([0,T];\Omega) := \Big\{ & u \in L^{2p}([0,T]; L_e^2(\Omega)) \cap L^2([0,T]; H_e^\beta(\Omega)) \\ & \text{such that } g_{1-\alpha} * (u - u_0) \in C([0,T]; L_e^2(\Omega)), \text{ and } (g_{1-\alpha} * (u-u_0))|_{t=0} = 0 \Big\}, \end{aligned}$$

Recalling Theorem 3.3 in [23], if Ω is a bounded Lipschitz domain and $s \geq 0$, we know that

$$H_0^s(\Omega) = H_e^s(\Omega) \quad \text{provided } s \notin \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \right\}. \tag{2.5}$$

This equivalence relation is important for our later deduction.

2.2. **The Yosida approximation.** The Yosida approximation of the time-fractional derivative operator is an important tool for analyzing regularity properties of equations with time-fractional derivative operator. For reader's convenience, we provide a short introduction. For detailed references, we refer to [34, 35, 36, 37]. Let $0 < \alpha < 1$, $1 \leq p < \infty$, $T > 0$, and X be a real Banach space. Then the fractional derivative operator defined by

$$Bu = \frac{d}{dt}(g_{1-\alpha} * u), \quad D(B) = \{u \in L^p([0,T];X) \,:\, g_{1-\alpha} * u \in {}_0W^{1,p}([0,T];X)\}.$$

Its Yosida approximation B_m, defined by $B_m = mB(m+B)^{-1}$, $m \in \mathbb{N}$, enjoy the property that for any $u \in D(B)$, one has $B_m u \to Bu$ in $L^p([0,T];X)$ as $m \to \infty$. Further, one has the representation

$$B_m u = \frac{d}{dt}(g_{1-\alpha,m} * u), \quad u \in L^p([0,T];X),\ m \in \mathbb{N},$$

where $g_{1-\alpha,m} = ms_{\alpha,m}$, and $s_{\alpha,m}$ is the unique solution of the scalar-valued Volterra equation

$$s_{\alpha,m}(t) + m(s_{\alpha,m} * g_\alpha)(t) = 1, \quad t > 0,\ m \in \mathbb{N}.$$

Let $h_{\alpha,m} \in L^1_{\text{loc}}(\mathbb{R}^+)$ be the resolvent kernel associated with mg_α, that is

$$h_{\alpha,m}(t) + m(h_{\alpha,m} * g_\alpha)(t) = mg_\alpha(t), \quad t > 0,\ m \in \mathbb{N}.$$

In addition, we have $g_{1-\alpha,m} = ms_{\alpha,m} = g_{1-\alpha} * h_{\alpha,m}$, $m \in \mathbb{N}$. Next, we list some important properties about $g_{\alpha,m}$ and $h_{\alpha,m}$:

- The kernel $g_{1-\alpha,m}$ are nonnegative and nonincreasing for all $m \in \mathbb{N}$, and $g_{1-\alpha,m} \in W^{1,1}([0,T])$;
- For any function $f \in L^p([0,T];X)$ with $1 \le p < \infty$ and X represents a Banach space, there holds $h_{\alpha,m} * f \to f$ in $L^p([0,T];X)$ as $m \to \infty$;
- $g_{1-\alpha,m} \to g_{1-\alpha}$ in $L^1([0,T])$ as $m \to \infty$ and $B_m u \to Bu$ in $L^p([0,T];X)$ as $m \to \infty$.

In all the following parts of this paper, we denote $h_m = h_{\alpha,m}$, $m \in \mathbb{N}$ for concisely.

2.3. **Concept of weak solutions.** In order to introduce the concept of weak solutions for equation (1.3) with L defined in (1.4), we define a nonlocal bilinear form associated to L by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [u(t,x) - u(t,y)][v(t,x) - v(t,y)]k(x,y)dxdy. \tag{2.6}$$

Definition 2.1. Define the following concepts regarding the domain of the solution:

(1) $Q_T := \Omega \times (0,T) \subset \mathbb{R}^{n+1}$.
(2) Lateral boundary of Q_T: $\partial_L Q_T := \partial\Omega \times [0,T]$.
(3) Parabolic boundary of Q_T: $\partial_p Q_T := (\Omega \times \{0\}) \cup \partial_L Q_T$.

We say that a function $u \in L^\infty([0,T];L^\infty(\mathbb{R}^n))$ is a weak solution (supersolution or subsolution) of (1.3) in Q_T with $f \in L^\infty(Q_T)$ and $u_0 \in L^2_e(\Omega)$, if $u \in V_p([0,T];\Omega)$ with $p \in [1,\infty)$ (defined in Section 2.1). For any (nonnegative) test function

$$\eta \in H^{1,\beta}_e(Q_T) := W^{1,2}([0,T];L^2_e(\Omega)) \cap L^2([0,T];H^\beta_e(\Omega)) \cap L^\infty([0,T];L^\infty(\mathbb{R}^n)) \tag{2.7}$$

with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega -\eta_t \left[g_{1-\alpha} * (u - u_0)\right] dxdt + \int_0^T \mathcal{E}(u,\eta)dt = (\ge \text{ or } \le) \int_0^T \int_\Omega f\eta dxdt. \tag{2.8}$$

In order to acquire some regularity information and deduce Harnack's inequality in the following sections, we would like to provide another equivalent definition of the weak solutions.

Lemma 2.2. *Let $u \in V_p([0,T];\Omega)$ be a weak solution (supersolution or subsolution) of equation (1.3) if and only if for any (nonnegative) function $\psi \in H_e^{\beta}(\Omega) \cap L^\infty(\mathbb{R}^n)$ one has*

(2.9)
$$\int_\Omega \psi \partial_t \left[g_{1-\alpha,m} * (u-u_0)\right] dx + \mathcal{E}(h_m * u, \psi) = (\geq \ \ or \ \ \leq) \int_\Omega (h_m * f)\psi dx \ \ a.e.\ t \in (0,T),\ m \in \mathbb{N}.$$

Proof. Because the proofs of weak solutions, supersolutions and subsolutions are almost same, here, we only provide the proof of weak supersolutions. The 'if' part is readily seen as follows. Given an arbitrary nonnegative $\eta \in H_e^{1,\beta}(Q_T)$ satisfying $\eta|_{t=T} = 0$, we take in (2.9) $\psi(x) = \eta(t,x)$ for any fixed $t \in (0,T)$, integrate from $t = 0$ to $t = T$, and integrate by parts with respect to the time variable. Then by using the approximating properties of the kernels h_m (details could be find in Lemma A.10 in Appdenix), we obtain (2.8). To show the 'only-if' part, we choose the test function

$$\eta(x,t) = \int_t^T h_m(\sigma - t)\varphi(\sigma, x) d\sigma = \int_0^{T-t} h_m(\sigma)\varphi(\sigma + t, x) d\sigma, \tag{2.10}$$

with arbitrary $m \in \mathbb{N}$ and nonnegative $\varphi \in H_e^{1,\beta}(Q_T)$ satisfying $\varphi|_{t=T} = 0$; η is nonnegative since φ and h_m are both nonnegative functions. For the first term in (2.8), it can be transformed to

$$\int_0^T \int_\Omega -\varphi_t \left[g_{1-\alpha,m} * (u-u_0)\right] dxdt, \tag{2.11}$$

where we used $g_{1-\alpha,m} = g_{1-\alpha} * h_m$ and the Fubini's theorem. For term $\int_0^T \mathcal{E}(u,\eta)dt$, we have

$$\begin{aligned}
&2\int_0^T \mathcal{E}(u,\eta)dt \\
=&\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_t^T h_m(\sigma - t)(u(x,t) - u(y,t))(\varphi(x,\sigma) - \varphi(y,\sigma))k(x,y) d\sigma dxdydt \\
=&\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((h_m * u)(x,t) - (h_m * u)(y,t))(\varphi(x,t) - \varphi(y,t))k(x,y) dxdydt \\
=&2\int_0^T \mathcal{E}(h_m * u, \varphi)dt.
\end{aligned}$$

Observe that $g_{1-\alpha,m} * (u-u_0) \in {}_0W^{1,2}([0,T]; L_e^2(\Omega))$. Therefore, combining (2.11) and the above equation, then integrating by parts and using $\varphi|_{t=T} = 0$ yields

(2.12)
$$\int_0^T \int_\Omega \varphi \partial_t \left[g_{1-\alpha,m} * (u-u_0)\right] dx + \mathcal{E}(h_m * u, \varphi)dt \geq \int_0^T \int_\Omega (h_m * f)\varphi dxdt,$$

for all $m \in \mathbb{N}$ and $\varphi \in H_e^{1,\beta}(Q_T)$ with $\varphi|_{t=T} = 0$. By means of a simple approximation argument, we obtain that (2.12) holds true for any φ of the form $\varphi(x,t) = \chi_{(t_1,t_2)}\psi(x)$ where $\chi_{(t_1,t_2)}$ denotes the characteristic function of the time interval (t_1, t_2), $0 < t_1 < t_2 < T$ and $\psi \in H_e^{\beta}(\Omega)$ is nonnegative. Appealing to the Lebesgue's differentiation theorem [10], the proof is complete. ∎

2.4. **Scaling property.** Let $t_0, r > 0$ and $x_0 \in \mathbb{R}^n$. Suppose $u \in V_p([0,T];\Omega)$ is a weak solution (supersolution or subsolution) of equation (1.3) in $(0, t_0 r^{2\beta/\alpha}) \times B(x_0, r)$. Changing the coordinates according to $s = t/r^{2\beta/\alpha}$ and $y = (x - x_0)/r$ and setting $\tilde{u}(s,y) := u(sr^{2\beta/\alpha}, x_0 + yr)$, $\tilde{u}_0(y) := u_0(x_0 + yr)$, $\tilde{a}(y_1, y_2) := a(x_0 + y_1 r, x_0 + y_2 r)$, $\tilde{k}_0(y_1, y_2) := r^{n+2\beta} k_0(x_0 + y_1 r, x_0 + y_2 r)$ and $\tilde{f}(s,y) := r^{2\beta} f(sr^{2\beta/\alpha}, x_0 + yr)$.

Through simple calculations, we find that $\tilde{k}_0(\cdot,\cdot)$ still satisfies inequality (1.6) and inequality (1.7). We also have

$$\partial_t^\alpha (u(t,x) - u_0(x)) = r^{-2\beta} \partial_s^\alpha (\tilde{u}(s,y) - \tilde{u}_0(y))$$

and

$$\begin{aligned} Lu(t,x) &= \int_{\mathbb{R}^n} [\tilde{u}(s,y) - \tilde{u}(s,z)] \tilde{a}(y,z) r^{-n-2\beta} \tilde{k}_0(y,z) r^n dz \\ &= r^{-2\beta} L\tilde{u}(s,y). \end{aligned}$$

Thus the problem for $u(t,x)$ is transformed to a problem for $\tilde{u}(s,y)$ in $(0,t_0) \times B(0,1)$, namely there holds (in the weak sense)

$$\partial_s^\alpha (\tilde{u} - \tilde{u}_0) + L\tilde{u} = (\geq \text{ or } \leq) \tilde{f}, \quad s \in (0, t_0), \ y \in B(0,1).$$

2.5. **Existence of weak solution.** Weak solutions have been constructed for an abstract evolutionary integro-differential equation in Hilbert spaces in [35], which provides a general framework incorporating equation (1.3). Choosing $\beta_0 \in [n/4, 1)$ and $\beta \in (\beta_0, 1)$, notice that

$$H_e^\beta(\Omega) \hookrightarrow L_e^2(\Omega) \hookrightarrow H^{-\beta}(\Omega),$$

where we used the equivalence relation (2.5).

Because

$$\mathcal{E}(u(t,\cdot), v(t,\cdot)) \leq C \|u(t,\cdot)\|_{H_e^\beta(\Omega)} \|v(t,\cdot)\|_{H_e^\beta(\Omega)},$$

and

$$\mathcal{E}(u(t,\cdot), u(t,\cdot)) \geq C(\Omega, \Lambda) \|u(t,\cdot)\|_{H_e^\beta(\Omega)},$$

where we used the fractional Poincaré inequality (Proposition 3.6 in [9]), we know that $\mathcal{E}(\cdot,\cdot)$ satisfies condition (Ha) in [35]. Hence, according to Theorem 3.1 and Theorem 3.2 proved in [35], we could obtain the following theorem.

Theorem 2.3. *Let $T > 0$, $\alpha \in (0,1)$, $\beta_0 \in [n/4, 1)$, $\Lambda > \max\{1, \beta_0^{-1}\}$ and $k \in \mathcal{R}(\beta_0, \Lambda)$. Assume $u_0 \in L_e^2(\Omega)$, $f \in L^2([0,T]; L_e^2(\Omega))$. Then problem (1.3) admits exactly one solution in the space $V_p([0,T],\Omega)$ with $1 \leq p < 2/(1-\alpha)$ and the following estimate hold*

$$\begin{aligned} \|u - u_0\|_{{}_0H^\alpha([0,T];H^{-\beta}(\Omega))} + \|u\|_{L^2([0,T];H_e^\beta(\Omega))} + \|g_{1-\alpha} * u\|_{C([0,T];L_e^2(\Omega))} \\ + \|u\|_{L^p([0,T];L_e^2(\Omega))} \leq C(\|u_0\|_{L_e^2(\Omega)} + \|f\|_{L^2([0,T];H^{-\beta}(\Omega))}), \end{aligned} \tag{2.13}$$

where $C = C(\alpha, \beta, T, n)$ is a general constant.

3. A weak Harnack's inequality

In this section, for concisely and clarity, we only prove a weak Harnack's inequality for equation (1.3) with $f = 0$ which is enough for our purpose. To formulate our result, let μ_n denotes the Lebesgue measure in $\mathbb{R}^n$ and μ_{n+1} denotes the Lebesgue measure in $\mathbb{R} \times \mathbb{R}^n$. For $\delta \in (0,1)$, $t_0 \geq 0$, $\tau > 0$, and a ball $B(x_0, r)$, define the boxes

$$
\begin{aligned}
&Q_-(t_0, x_0, r) = (t_0, t_0 + \delta\tau r^{2\beta/\alpha}) \times B(x_0, \delta r), \\
&Q_+(t_0, x_0, r) = (t_0 + (2-\delta)\tau r^{2\beta/\alpha}, t_0 + 2\tau r^{2\beta/\alpha}) \times B(x_0, \delta r).
\end{aligned}
$$

Theorem 3.1. *Let $k \in \mathcal{R}(\beta_0, \Lambda)$ for some $\beta_0 \in (n/4, 1)$ and $\Lambda \geq \max\{1, \beta_0^{-1}\}$. Let $\alpha \in (0,1)$, $T > 0$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u_0 \in L_e^2(\Omega)$. Let further $\delta \in (0,1)$, $\eta > 1$, and $\tau > 0$ be fixed. Then for any $t_0 \geq 0$ and $r > 0$ with $t_0 + 2\tau r^{2\beta/\alpha} \leq T$, and ball $B(x_0, \eta r) \subset \Omega$ and any nonnegative weak supersolution u of (1.3) in $(0, t_0 + 2\tau r^{2\beta/\alpha}) \times B(x_0, \eta r)$ with $u_0 \geq 0$ in $B(x_0, \eta r)$ and $f = 0$, there holds*

$$
\frac{1}{\mu_{n+1}(Q_-(t_0,x_0,r))} \int_{Q_-(t_0,x_0,r)} u \, d\mu_{n+1} \leq C \operatorname*{ess\,inf}_{Q_+(t_0,x_0,r)} u,
$$

where the constant $C = C(\Lambda, \delta, \tau, \eta, \alpha, \beta, n)$.

Remark 3.2. The above theorem provides a weak Harnack's inequality in the case $f = 0$, however, when f is not a zero function similar result also holds. In order to state the main idea concisely, we only show the proof of Theorem 3.1 in the following. However, just change $\tilde{u}$ to $\tilde{u} + \|f\|_{L^\infty(Q_T)}$, and notice that $\|f/\tilde{u}\|_{L^\infty(Q_T)} \leq 1$ in the following proof, we can adjust the proof appropriately as in [7] to obtain the following estimate

$$
\frac{1}{\mu_{n+1}(Q_-(t_0,x_0,r))} \int_{Q_-(t_0,x_0,r)} u \, d\mu_{n+1} \leq C \left(\operatorname*{ess\,inf}_{Q_+(t_0,x_0,r)} u + \|f\|_{L^\infty(Q_T)} \right),
$$

under the same conditions as Theorem 3.1.

Before proving this theorem, let us provide an important inequality. For $\kappa = 1 + \frac{2\beta}{3}$, $1 < p < \min\{1/(1-\alpha), 3/(2\beta)\}$ and a function $u \in V_p([t_1, t_2] \times \Omega)$, we have

$$
\|u\|_{L^{2\kappa}([t_1,t_2]\times\Omega)} \leq C(t_1, t_2, \Omega, p, \beta, n) \|u\|_{V_p([t_1,t_2]\times\Omega)}. \tag{3.1}
$$

Proof. Let $\theta = \frac{3}{3-2\beta}$, $\theta' = \frac{3}{2\beta}$, then we have

$$
\begin{aligned}
\int_{t_1}^{t_2} \int_\Omega u^{2\kappa} dxdt = &\int_{t_1}^{t_2} \int_\Omega u^2 u^{2\frac{2\beta}{3}} dxdt \\
\leq &\int_{t_1}^{t_2} \left(\int_\Omega u^{2\theta} dx \right)^{1/\theta} \left(\int_\Omega u^2 dx \right)^{1/\theta'} dt \\
\leq &C(t_1, t_2, \Omega, p, \beta, n) \left(\int_{t_1}^{t_2} \left(\int_\Omega u^2 dx \right)^p dt \right)^{\frac{1}{2p}\frac{4\beta}{3}} \\
&\times \left[\int_{t_0}^{t_1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(s,x) - u(s,y))^2}{|x-y|^{n+2\beta}} dxdyds \right. \\
&\left. + \int_{t_0}^{t_1} \int_{\mathbb{R}^n} u^2 dxds \right],
\end{aligned}
$$

where we used Lemma A.4 to deduce the third inequality. Now, recall the definition of $V_p([t_1,t_2]\times\Omega)$, the above inequality provides us the desired result. $\square$

Remark 3.3. From the above proof, notice the relation (2.5) and $\beta_0\in(n/4,1)$, we could obtain

$$\|u\|_{L^{2\kappa}([t_1,t_2]\times\Omega)}\le C(t_1,t_2,\Omega,p,\beta,n)\|u\|_{L^{2p}([t_1,t_2]\times\Omega)\cap L^2([t_1,t_2],H_0^\beta(\Omega))}. \tag{3.2}$$

Because the proof involves a lot of complex calculations, we divide the proof into four parts for clarity.

3.1. An estimate for inf u.

For $\sigma>0$ we put $\sigma B(x,r):=B(x,\sigma r)$. Recall that μ_n denotes the Lebesgue measure in $\mathbb{R}^n$.

Theorem 3.4. *Let $\Omega\subset\mathbb{R}^n$, $\alpha\in(0,1)$, $T>0$, $k\in\mathcal{R}(\beta_0,\Lambda)$ with $\beta_0\in(n/4,1)$ and $\Lambda\ge\max\{1,\beta_0^{-1}\}$. Let further $\eta>0$ and $\delta\in(0,1)$ be fixed. Then for any $t_0\in(0,T]$ and $r>0$ with $t_0-\eta r^{2\beta/\alpha}\ge 0$, and ball $B=B(x_0,r)\subset\Omega$, and any weak supersolution $u\ge\epsilon>0$ of equation (1.3) in $(0,t_0)\times B$ with $u_0\ge 0$ in B and $f=0$, there holds*

$$\operatorname*{ess\,sup}_{U_{\sigma'}} u^{-1}\le\left(\frac{C\mu_{n+1}(U_1)^{-1}}{(\sigma-\sigma')^{\tau_0}}\right)^{1/\gamma}\|u^{-1}\|_{L^\gamma(U_\sigma)},\quad \delta\le\sigma'<\sigma\le 1,\ \gamma\in(0,1].$$

Here $U_\sigma=(t_0-\sigma\eta r^{2\beta/\alpha},t_0)\times\sigma B$, $0<\sigma\le 1$, $C=C(\Lambda,\delta,\eta,\alpha,\beta_0,n)$ and $\tau_0=\tau_0(\beta,n)$.

Proof. In general, we could change coordinates as $t\to t/r^{\frac{2\beta}{\alpha}}$ and $x\to(x-x_0)/r$, thereby transforming the equation to a problem of the same type on $(0,t_0/r^{\frac{2\beta}{\alpha}})\times B(0,1)$. Hence, without loss of generality, we could assume that $r=1$ and $x_0=0$.

Choose σ' and σ such that $\delta\le\sigma'<\sigma\le 1$ and denote $B_1=\sigma B$. For $\rho\in(0,1]$, we denote $V_\rho=U_{\rho\sigma}$. Given $0<\rho'<\rho\le 1$, let $t_1=t_0-\rho\sigma\eta$ and $t_2=t_0-\rho'\sigma\eta$. Obviously, we have $0\le t_1<t_2<t_0$. Now we introduce the shifted time $s=t-t_1$ and set $\tilde f(s):=f(s+t_1)$, $s\in(0,t_0-t_1)$, for functions f defined on (t_1,t_0). Because u is a positive weak supersolution of (1.3) in $(0,t_0)\times B$, we have

$$\int_\Omega\varphi\partial_s\left(g_{1-\alpha,m}*(\tilde u-\tilde u_0)\right)dx+\mathcal{E}(h_m*\tilde u,\varphi)\ge 0,\quad \text{a.e. } s\in(0,t_0-t_1), m\in\mathbb{N},$$

for any nonnegative function $\varphi\in H_e^\beta(B)$. Because $u_0\ge 0$ in B, we then deduce that

$$\int_B\varphi\partial_s\left(g_{1-\alpha,m}*\tilde u\right)dx+\mathcal{E}(h_m*\tilde u,\varphi)\ge 0,\quad \text{a.e. } s\in(0,t_0-t_1), m\in\mathbb{N}, \tag{3.3}$$

for any nonnegative function $\varphi\in H_e^\beta(B)$. For $s\in(0,t_0-t_1)$, we choose the test function $\varphi(s,x):=\psi^{1+q}(x)\tilde u^{-q}(s,x)$ with $q>1$ and $\psi\in C_0^1(B_1)$ so that

$$\begin{gathered}0\le\psi\le 1,\quad \psi=1\text{ in }\rho' B_1,\quad \operatorname{supp}\psi\subset\rho B_1,\\ |D\psi|\le 2/(\sigma(\rho-\rho')).\end{gathered} \tag{3.4}$$

Choose $H(y) := -(1-q)^{-1}y^{1-q}$, $y > 0$ in the fundamental identity (A.1) shown in Appendix, there holds for a.e. $(s,x) \in (0, t_0 - t_1) \times B$

$$\begin{aligned} -\tilde{u}^{-q}\partial_s(g_{1-\alpha,m} * \tilde{u}) &\geq -\frac{1}{1-q}\partial_s(g_{1-\alpha,m} * \tilde{u}^{1-q}) + \left(\frac{\tilde{u}^{1-q}}{1-q} - \tilde{u}^{1-q}\right) g_{1-\alpha,m} \\ &\geq -\frac{1}{1-q}\partial_s(g_{1-\alpha,m} * \tilde{u}^{1-q}) + \frac{q}{1-q}\tilde{u}^{1-q}g_{1-\alpha,m}. \end{aligned} \tag{3.5}$$

Considering (3.5), inequality (3.3) could be transformed into the following inequality

$$\begin{aligned} -\frac{1}{1-q}\int_{B_1} \psi^{1+q}\partial_s(g_{1-\alpha,m} * \tilde{u}^{1-q})dx - \mathcal{E}(h_m * \tilde{u}, \psi^{1+q}\tilde{u}^{-q}) \\ \leq \frac{-q}{1-q}\int_{B_1} \psi^{1+q}\tilde{u}^{1-q}g_{1-\alpha,m}dx. \end{aligned} \tag{3.6}$$

Now, we choose $\phi \in C^1([0, t_0 - t_1])$ such that

$$\begin{aligned} 0 \leq \phi \leq 1, \quad \phi = 0 \text{ in } [0,(t_2 - t_1)/2], \quad \phi = 1 \text{ in } [t_2 - t_1, t_0 - t_1], \\ 0 \leq \dot{\phi} \leq 4/(t_2 - t_1). \end{aligned} \tag{3.7}$$

Multiplying (3.6) by $q - 1 > 0$ and by ϕ, and convolving the resulting inequality with g_α yields

$$\begin{aligned} \int_{B_1} g_\alpha * \left(\phi\psi^{1+q}\partial_s(g_{1-\alpha,m} * \tilde{u}^{1-q})\right) dx + (1-q)g_\alpha * \left[\mathcal{E}(h_m * \tilde{u}, \psi^{1+q}\tilde{u}^{-q})\phi\right] \\ \leq qg_\alpha * \int_{B_1} \psi^{1+q}\tilde{u}^{1-q}g_{1-\alpha,m}\phi dx, \end{aligned} \tag{3.8}$$

for a.e. $s \in (0, t_0 - t_1)$. By Lemma A.1 presented in Appendix, we have

$$\begin{aligned} \int_{B_1} g_\alpha * (\phi\partial_s(g_{1-\alpha,m} * [\psi^{1+q}\tilde{u}^{1-q}]))dx \geq \int_{B_1} \phi g_\alpha * (\partial_s(g_{1-\alpha,m} * [\psi^{1+q}\tilde{u}^{1-q}]))dx \\ - \int_0^s g_\alpha(s-\sigma)\dot{\phi}(\sigma)\left(g_{1-\alpha,m} * \int_{B_1} \psi^{1+q}\tilde{u}^{1-q}dx\right)(\sigma)d\sigma. \end{aligned} \tag{3.9}$$

Because $g_{1-\alpha,m} * [\psi^{1+q}\tilde{u}^{1-q}] \in {}_0W^{1,1}([0, t_0 - t_1], L^1_e(B_1))$ and $g_{1-\alpha,m} = g_{1-\alpha} * h_m$ as well as $g_\alpha * g_{1-\alpha} = 1$ we have

$$g_\alpha * \partial_s(g_{1-\alpha,m} * [\psi^{1+q}\tilde{u}^{1-q}]) = h_m * (\psi^{1+q}\tilde{u}^{1-q}). \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), sending $m \to \infty$, and selecting an appropriate subsequence, if necessary, we obtain

$$\begin{aligned} &\int_{B_1} \phi\psi^{1+q}\tilde{u}^{1-q}dx + (q-1)g_\alpha * \left(\mathcal{E}(\tilde{u}, -\psi^{1+q}\tilde{u}^{-q})\phi\right) \\ \leq& qg_\alpha * \int_{B_1} \psi^{1+q}\tilde{u}^{1-q}g_{1-\alpha}\phi dx \\ &+ \int_0^s g_\alpha(s-\sigma)\dot{\phi}(\sigma)\left(g_{1-\alpha} * \int_{B_1} \psi^{1+q}\tilde{u}^{1-q}dx\right)(\sigma)d\sigma, \end{aligned} \tag{3.11}$$

for a.e. $s \in (0, t_0 - t_1)$. Now, we need a careful analysis of $\mathcal{E}(\tilde{u}, -\psi^{1+q}\tilde{u}^{-q})$. Denote $\vartheta(q) = \max\{4, (6q-5)/2\}$. Using statement (1) in Lemma A.3 given in Appendix, we could deduce that

(3.12)
$$\begin{aligned}
&\mathcal{E}(\tilde{u}, -\psi^{1+q}\tilde{u}^{-q}) \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\tilde{u}(s,x)-\tilde{u}(s,y))(\psi^{1+q}(y)\tilde{u}^{-q}(s,y)-\psi^{1+q}(x)\tilde{u}^{-q}(s,x))\frac{k(x,y)}{2}dxdy \\
\geq&\frac{1}{2(q-1)}\mathrm{I}-\frac{\vartheta(q)}{2}\mathrm{II},
\end{aligned}$$

where

$$\mathrm{I}=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi(x)\psi(y)\left(\left(\frac{\tilde{u}(s,x)}{\psi(x)}\right)^{\frac{1-q}{2}}-\left(\frac{\tilde{u}(s,y)}{\psi(y)}\right)^{\frac{1-q}{2}}\right)^2 k(x,y)dxdy,$$

and

$$\mathrm{II}=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\psi(x)-\psi(y))^2\left(\left(\frac{\tilde{u}(s,x)}{\psi(x)}\right)^{1-q}-\left(\frac{\tilde{u}(s,y)}{\psi(y)}\right)^{1-q}\right)k(x,y)dxdy.$$

Considering (3.12), denote $w=\tilde{u}^{\frac{1-q}{2}}$, (3.11) could be reduced to

(3.13)
$$\begin{aligned}
&\int_{B_1}\phi\psi^{1+q}w^2dx+\frac{1}{2}g_\alpha*(\mathrm{I}\,\phi)\leq qg_\alpha*\int_{B_1}\psi^{1+q}w^2g_{1-\alpha}\phi dx \\
&+\frac{\vartheta(q)(q-1)}{2}g_\alpha*(\mathrm{II}\,\phi)+\int_0^s g_\alpha(s-\sigma)\dot{\phi}(\sigma)\left(g_{1-\alpha}*\int_{B_1}\psi^{1+q}w^2dx\right)(\sigma)d\sigma.
\end{aligned}$$

Term II could be estimated as follow

$$\begin{aligned}
\mathrm{II}\,\phi\leq&2\int_{\rho B_1}\int_{\rho B_1}(\psi(x)-\psi(y))^2\phi w^2k(x,y)dxdy \\
&+4\int_{\rho B_1}\int_{\mathbb{R}^n\setminus(\rho B_1)}(\psi(x)-\psi(y))^2\phi w^2k(x,y)dydx \qquad (3.14)\\
\leq&C_1(n,\Lambda,\delta)(\rho-\rho')^{-2\beta}\int_{\rho B_1}\phi w^2dx,
\end{aligned}$$

where (1.6) and $\sup_{x,y\in\mathbb{R}^n}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^2}\leq\frac{4}{\sigma^2(\rho-\rho')^2}$ have been used. For term I, noticing the properties of the function ψ, we have the following estimate

$$\mathrm{I}\geq\frac{c_{n,\beta}}{2C\Lambda}\int_{\rho' B_1}\int_{\rho' B_1}\frac{(w(s,x)-w(s,y))^2}{|x-y|^{n+2\beta}}dxdy. \qquad (3.15)$$

Denote

(3.16)
$$\begin{aligned}
F(s)=&\frac{1}{2}C_1(n,\Lambda,\delta)\vartheta(q)(q-1)(\rho-\rho')^{-2\beta}\int_{\rho B_1}\phi w^2dx \\
&+qg_{1-\alpha}(s)\phi(s)\int_{\rho B_1}\psi^{1+q}w^2dx+\dot{\phi}(s)\left(g_{1-\alpha}*\int_{\rho B_1}\psi^{1+q}w^2dx\right)(s)
\end{aligned}$$

Using estimates from (3.13) to (3.16), we obtain

(3.17)
$$\int_{B_1}\phi\psi^{1+q}w^2dx+\frac{c_{n,\beta}}{4C\Lambda}g_\alpha*\int_{\rho' B_1}\int_{\rho' B_1}\frac{(w(s,x)-w(s,y))^2}{|x-y|^{n+2\beta}}\phi dxdy\leq g_\alpha*F.$$

We may drop the second term in (3.17), which is nonnegative. By Young's inequality for convolution and the properties of ϕ we then infer that for all $1 < p < \min\{1/(1-\alpha), 3/(2\beta)\}$

(3.18)
$$\left(\int_{t_2-t_1}^{t_0-t_1}\left(\int_{B_1}(\psi^{\frac{1+q}{2}}(x)w(s,x))^2\right)^p ds\right)^{1/p} \le \|g_\alpha\|_{L^p([0,t_0-t_1])}\int_0^{t_0-t_1}F(s)ds.$$

By simple calculations, we easily know that $\|g_\alpha\|_{L^p([0,t_0-t_1])} \le C_2(\alpha,p,\eta) < \infty$. We will choose any of these p and fix it.

We could also drop the first term in (3.17), convolve the resulting inequality with $g_{1-\alpha}$, then obtaining

$$\|w\|^2_{L^2([t_2-t_1,t_0-t_1];H^\beta(\rho' B_1))} \le 4C\Lambda\int_0^{t_0-t_1}F(s)ds. \tag{3.19}$$

Considering (3.18),(3.19) and Remark 3.3, we infer that

$$\|w\|^2_{L^{2\kappa}([t_2-t_1,t_0-t_1]\times\rho' B_1)} \le C(n,\Lambda,p,\alpha,\eta)\int_0^{t_0-t_1}F(s)ds. \tag{3.20}$$

For a.e. $s \in (0, t_0-t_1)$, we have

$$\begin{aligned} F(s) \le& \left(\frac{C_1\vartheta(q)(q-1)}{2(\rho-\rho')^{2\beta}} + qg_{1-\alpha}((t_2-t_1)/2)\right)\int_{\rho B_1}w^2dxds \\ &+ \frac{4}{t_2-t_1}\left(g_{1-\alpha}*\int_{\rho B_1}w^2dx\right)(s). \end{aligned}$$

In addition, we obtain

(3.21)
$$\begin{aligned} \int_0^{t_0-t_1}F(s)ds \le& \left(\frac{C_1\vartheta(q)(q-1)}{2(\rho-\rho')^{2\beta}} + \frac{2^\alpha q(\sigma\eta)^{-\alpha}}{\Gamma(1-\alpha)(\rho-\rho')^\alpha}\right)\int_0^{t_0-t_1}\int_{\rho B_1}w^2dxds \\ &+ \frac{4}{(\rho-\rho')\sigma\eta}\int_0^{t_0-t_1}g_{2-\alpha}(t_0-t_1-\tau)\int_{\rho B_1}w^2dxd\tau \\ \le& C(n,\alpha,\beta,\Lambda,\delta,\eta)\frac{q}{(\rho-\rho')^2}\int_0^{t_0-t_1}\int_{\rho B_1}w^2dxds. \end{aligned}$$

Combing (3.20) and the above estimates (3.21), we deduce that

$$\|w\|_{L^{2\kappa}([t_2-t_1,t_0-t_1]\times\rho' B_1)} \le C(n,\alpha,\beta,\Lambda,\delta,\eta,p)\frac{q}{\rho-\rho'}\|w\|_{L^2([0,t_0-t_1]\times\rho B_1)}, \tag{3.22}$$

where $\kappa = 1+\frac{2\beta}{3} > 1$. Because $w = \tilde{u}^{\frac{1-q}{2}}$ and by transforming back to the time variable t, we find that (3.22) is equivalent to

$$\left(\int_{V_{\rho'}}u^{(1-q)\kappa}dxdt\right)^{\frac{1}{2\kappa}} \le \frac{C(n,\alpha,\beta,\Lambda,\delta,\eta,p)q}{\rho-\rho'}\left(\int_{V_\rho}u^{(1-q)}dxdt\right)^{\frac{1}{2}}.$$

Taking $\gamma = q-1$, we have

$$\|u^{-1}\|_{L^{\gamma\kappa}(V_{\rho'})} \le \left(\frac{C^2(1+\gamma)^2}{(\rho-\rho')^2}\right)^{1/\gamma}\|u^{-1}\|_{L^\gamma(V_\rho)}, \quad 0<\rho'<\rho\le 1,\ \gamma>0.$$

Using Lemma A.5 with $\bar{p} = 1$, there will be a constant $M = M(\Lambda, \delta, \eta, \alpha, \beta, p, n)$ and $\tau_0 = \tau_0(\beta, n)$ such that

$$\operatorname*{ess\,sup}_{V_\theta} u^{-1} \leq \left(\frac{M_0}{(1-\theta)^{\tau_0}} \right)^{1/\gamma} \|u^{-1}\|_{L^\gamma(V_1)} \quad \text{for all } \theta \in (0,1), \ \gamma \in (0,1].$$

Then if we take $\theta = \frac{\sigma'}{\sigma}$ and notice that $\frac{1}{1-\theta} = \frac{\sigma}{\sigma - \sigma'} \leq \frac{1}{\sigma - \sigma'}$, we obtain

$$\operatorname*{ess\,sup}_{U_{\sigma'}} u^{-1} \leq \left(\frac{M_0}{(\sigma - \sigma')^{\tau_0}} \right)^{1/\gamma} \|u^{-1}\|_{L^\gamma(U_\sigma)}, \quad \gamma \in (0,1].$$

Now, the proof is complete. □

3.2. **An estimate for small positive moments of u.** The aim of this subsection is to estimate the L^1-norm of supersolutions u from above by the L^1-norm of u^γ for small values of $\gamma > 0$.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$, $\alpha \in (0,1)$, $T > 0$, $k \in \mathcal{R}(\beta_0, \Lambda)$ with $\beta_0 \in (n/4, 1)$ and $\Lambda \geq \max\{1, \beta_0^{-1}\}$. Let further $\eta > 0$ and $\delta \in (0,1)$ be fixed. Then for any $t_0 \in [0,T)$ and $r > 0$ with $t_0 + \eta r^{2\beta/\alpha} \leq T$, and ball $B = B(x_0, r) \subset \Omega$, and any nonnegative weak supersolution u of (1.3) in $(0, t_0 + \eta r^{2\beta/\alpha}) \times B$ with $u_0 \geq 0$ in B and $f = 0$, there holds*

$$\|u\|_{L^1(U'_{\sigma'})} \leq \left(\frac{C\mu_{n+1}(U'_1)}{(\sigma - \sigma')^{\tau_0}} \right)^{1/\gamma - 1} \|u\|_{L^\gamma(U'_\sigma)}, \quad \delta \leq \sigma' < \sigma \leq 1, \ 0 < \gamma \leq \kappa^{-1}.$$

Here $U'_\sigma = (t_0, t_0 + \sigma\eta r^{2\beta/\alpha}) \times \sigma B$, $C = C(\Lambda, \delta, \eta, \alpha, \beta, n)$, and $\tau_0 = \tau_0(\beta, n)$.

Proof. The proof of this theorem is similar to the proof of Theorem 3.4. Without loss of generality, we assume $r = 1$. Replacing u with $u + \epsilon$ and u_0 with $u_0 + \epsilon$ and eventually letting $\epsilon \to 0^+$, we could assume that u is bounded away from zero.

Fix σ', σ such that $\delta \leq \sigma' < \sigma \leq 1$ and let $B_1 = \sigma B$. For $\rho \in (0,1]$, we set $V'_\rho = U'_{\rho\sigma}$. Given $0 < \rho' < \rho \leq 1$, let $t_1 = t_0 + \rho'\sigma\eta$ and $t_2 = t_0 + \rho\sigma\eta$, so $0 \leq t_0 < t_1 < t_2$. We shift the time by means of $s = t - t_0$ and set $\tilde{f}(s) := f(s + t_0)$, $s \in (0, t_2 - t_0)$, for functions f defined on (t_0, t_2).

Let $\gamma \in (0, \kappa^{-1}]$ and $q = 1 - \gamma \in [1 - \kappa^{-1}, 1)$, then repeating the proof of (3.5) will lead to the following inequality

(3.23)
$$-\tilde{u}^{-1}\partial_s(g_{1-\alpha,m} * \tilde{u}) \geq \frac{-1}{1-q}\partial_s(g_{1-\alpha,m} * \tilde{u}^{1-q}), \quad \text{a.e. } (s,x) \in (0, t_2 - t_0) \times B.$$

Taking $\varphi(s,x) = \psi^2(x)\tilde{u}^{-q}(s,x)$ with $\psi \in C_0^1(B_1)$ as in the proof of Theorem 3.4, we infer that

$$-\frac{1}{1-q}\int_{B_1} \partial_s \left(g_{1-\alpha,m} * (\psi^2 \tilde{u}^{1-q})\right) dx + \mathcal{E}(h_m * \tilde{u}, -\psi^2 \tilde{u}^{-q}) \leq 0, \tag{3.24}$$

for a.e. $s \in (0, t_2 - t_0)$. Next, we choose a function $\phi \in C^1([0, t_2 - t_0])$ such that

$$\begin{gathered} 0 \leq \phi \leq 1, \quad 0 \leq -\dot{\phi} \leq \frac{4}{t_2 - t_1}, \\ \phi = 1 \text{ in } [0, t_1 - t_0], \quad \phi = 0 \text{ in } [t_1 - t_0 + (t_2 - t_1)/2, t_2 - t_0]. \end{gathered} \tag{3.25}$$

Multiplying (3.24) by $1-q>0$ and by $\phi(s)$, and applying Lemma A.2 presented in Appendix to the first term gives

$$-\int_{B_1}\partial_s(g_{1-\alpha,m}*(\phi\psi^2\tilde{u}^{1-q}))dx+(1-q)\phi\mathcal{E}(\tilde{u},-\psi^2\tilde{u}^{-q}) \\ \leq \int_0^s \dot{g}_{1-\alpha,m}(s-\tau)(\phi(s)-\phi(\tau))\left(\int_{B_1}\psi^2\tilde{u}^{1-q}dx\right)(\tau)d\tau+R_m(s), \tag{3.26}$$

where

$$R_m(s)=(1-q)\phi\left[\mathcal{E}(h_m*\tilde{u},\psi^2\tilde{u}^{-q})-\mathcal{E}(\tilde{u},\psi^2\tilde{u}^{-q})\right].$$

Now, as in the proof of Theorem 3.4, we denote $w=\tilde{u}^{\frac{1-q}{2}}$. Here, we estimate term $\mathcal{E}(\tilde{u},-\psi^2\tilde{u}^{-q})$ firstly as follow

$$\mathcal{E}(\tilde{u},-\psi^2\tilde{u}^{-q})=\frac{1}{2}\mathrm{I}+\mathrm{II}, \tag{3.27}$$

where

$$\mathrm{I}=\int_{\rho B_1}\int_{\rho B_1}(\tilde{u}(s,x)-\tilde{u}(s,y))(\psi^2(y)\tilde{u}^{-q}(s,y)-\psi^2(x)\tilde{u}^{-q}(s,x))k(x,y)dxdy,$$

and

$$\mathrm{II}=\int_{\rho B_1}\int_{\mathbb{R}^n\setminus\rho B_1}(\tilde{u}(s,x)-\tilde{u}(s,y))(-\psi^2(x)\tilde{u}^{-q}(s,x))k(x,y)dydx.$$

For II, using (1.6), the positivity of $\tilde{u}$ and the fact that $\frac{(\psi(x)-\psi(y))^2}{|x-y|^2}\leq C(\delta)(\rho-\rho')^{-2}$, we could estimate as follow

$$\mathrm{II}\geq C(\delta,\Lambda)(\rho-\rho')^{-2\beta}\int_{\rho B_1}w^2(s,x)dx. \tag{3.28}$$

Considering (3.27) and (3.28), inequality (3.26) can be changed to

$$\begin{aligned} &-\int_{B_1}\partial_s(g_{1-\alpha,m}*[\phi\psi^2w^2])dx+(1-q)\frac{1}{2}\phi\cdot\mathrm{I} \\ &\leq \int_0^s \dot{g}_{1-\alpha,m}(s-\tau)(\phi(s)-\phi(\tau))\left(\int_{B_1}\psi^2\tilde{u}^{1-q}dx\right)(\tau)d\tau \\ &\quad +C(\delta,\Lambda)(1-q)(\rho-\rho')^{-2\beta}\phi(s)\int_{\rho B_1}w^2dx+R_m(s). \end{aligned} \tag{3.29}$$

Applying Lemma A.3 (2) we could estimate I as follow

$$\begin{aligned} \mathrm{I}\geq&\zeta_1(q)\int_{\rho B_1}\int_{\rho B_1}\left[\psi(x)w(s,x)-\psi(y)w(s,y)\right]^2k(x,y)dxdy \\ &-\zeta_2(q)\int_{\rho B_1}\int_{\rho B_1}(\psi(x)-\psi(y))^2(w^2(s,x)+w^2(s,y))k(x,y)dxdy, \end{aligned} \tag{3.30}$$

where $\zeta_1(q)$, $\zeta_2(q)$ are defined as in Lemma A.3. Because

$$(1-q)\zeta_1(q)=\frac{2q}{3}\geq\frac{2}{3}\frac{\beta_0}{n+2}=:c_1=c_1(n,\beta_0), \tag{3.31}$$

and

$$\begin{aligned}(3.32)\qquad &\int_{\rho B_1}\int_{\rho B_1}\left[\psi(x)w(s,x)-\psi(y)w(s,y)\right]^2 k(x,y)dxdy\\ &\qquad\geq \int_{\rho' B_1}\int_{\rho' B_1}\left[w(s,x)-w(s,y)\right]^2 k(x,y)dxdy,\end{aligned}$$

then from (3.29), (3.30), we arrive at

$$\begin{aligned}(3.33)\qquad &-\int_{B_1}\partial_s(g_{1-\alpha,m}*[\phi\psi^2w^2])dx+\frac{1}{2}c_1\mathrm{III}\\ &\leq \int_0^s \dot{g}_{1-\alpha,m}(s-\tau)(\phi(s)-\phi(\tau))\left(\int_{B_1}\psi^2\tilde{u}^{1-q}dx\right)(\tau)d\tau\\ &+(1-q)\zeta_2(q)\phi\int_{\rho B_1}\int_{\rho B_1}(\psi(x)-\psi(y))^2(w^2(s,x)+w^2(s,y))k(x,y)dxdy\\ &+C(\delta,\Lambda)(1-q)(\rho-\rho')^{-2\beta}\phi(s)\int_{\rho B_1}w^2dx+R_m(s),\end{aligned}$$

where

$$\mathrm{III}=\phi\int_{\rho' B_1}\int_{\rho' B_1}\left[w(s,x)-w(s,y)\right]^2 k(x,y)dxdy.$$

Using (1.6) and the properties of ψ, we have

$$\begin{aligned}(3.34)\qquad &\int_{\rho B_1}\int_{\rho B_1}(\psi(x)-\psi(y))^2(w^2(s,x)+w^2(s,y))k(x,y)dxdy\\ &\qquad\leq C(\delta,\Lambda)(\rho-\rho')^{-2\beta}\int_{\rho B_1}w^2(s,x)dx.\end{aligned}$$

Because

$$(1-q)\zeta_2(q)\leq 4+9\frac{n+2}{\beta_0}=:c_2=c_2(n,\beta_0),$$

and using (3.34), we know that

(3.35)

$$\begin{aligned}&(1-q)\zeta_2(q)\phi\int_{\rho B_1}\int_{\rho B_1}(\psi(x)-\psi(y))^2(w^2(s,x)+w^2(s,y))k(x,y)dxdy\\ &\leq c_2C(\delta,\Lambda)(\rho-\rho')^{-2\beta}\phi\int_{\rho B_1}w^2(s,x)dx=c_3(\rho-\rho')^{-2\beta}\phi\int_{\rho B_1}w^2(s,x)dx,\end{aligned}$$

where $c_3:=c_3(\delta,\Lambda,n,\beta_0)$. Combining (3.33) and (3.35), we obtain

$$\begin{aligned}(3.36)\qquad &-\int_{B_1}\partial_s(g_{1-\alpha,m}*[\phi\psi^2w^2])dx+\frac{1}{2}c_1\mathrm{III}\\ &\leq \int_0^s \dot{g}_{1-\alpha,m}(s-\tau)(\phi(s)-\phi(\tau))\left(\int_{B_1}\psi^2\tilde{u}^{1-q}dx\right)(\tau)d\tau\\ &+c_4(\rho-\rho')^{-2\beta}\phi(s)\int_{\rho B_1}w^2(s,x)dx+R_m(s),\end{aligned}$$

where $c_4=c_4(\delta,\Lambda,n,\beta_0)$. Putting

$$W(s)=\int_{B_1}\phi(s)\psi^2(x)w^2(s,x)dx,$$

and denoting the right hand side of (3.36) by $F_m(s)$, it follows from (3.36) that

$$G_m(s) = \partial_s^\alpha (h_m * W)(s) + F_m(s) \geq 0, \quad \text{a.e. } s \in (0, t_2 - t_0).$$

We obviously have the following inequality

$$0 \leq h_m * W = g_\alpha * \partial_s^\alpha (h_m * W) \leq g_\alpha * G_m + g_\alpha * [-F_m(s)]^+$$

a.e. in $(0, t_2 - t_0)$. For any $1 < p < \min\{1/(1-\alpha), 3/(2\beta)\}$ and any $t_* \in [t_2 - t_0 - (t_2 - t_1)/4, t_2 - t_0]$, by Young's inequality, we obtain

$$\|h_m * W\|_{L^p([0,t_*])} \leq \|g_\alpha\|_{L^p([0,t_*])} \left(\|G_m\|_{L^1([0,t_*])} + \|[-F_m]^+\|_{L^1([0,t_*])} \right). \tag{3.37}$$

Because $t_* \leq t_2 - t_0 \leq \eta$, we have $\|g_\alpha\|_{L^p([0,t_*])} \leq C < \infty$ by some simple calculations. By positivity of G_m, we obtain

$$\|G_m\|_{L^1([0,t_*])} = (g_{1-\alpha,m} * W)(t_*) + \int_0^{t_*} F_m(s) ds. \tag{3.38}$$

Observe that $R_m \to 0$ in $L^1(0, t_2 - t_0)$ as $m \to \infty$. Hence, $\|[-F_m]^+\|_{L^1([0,t_*])} \to 0$ as $m \to \infty$. For the first term on the right hand side of (3.36), integrate for s from 0 to t_*, we have the following estimate

$$\begin{aligned}
&\int_0^{t_*} \int_0^s \dot{g}_{1-\alpha,m}(s-\tau)(\phi(s) - \phi(\tau)) \left(\int_{B_1} \psi^2 \tilde{u}^{1-q} dx \right)(\tau) d\tau ds \\
&= \int_0^{t_*} g_{1-\alpha,m}(t_* - \tau)(\phi(t_*) - \phi(\tau)) \left(\int_{B_1} \psi^2 w^2 dx \right)(\tau) d\tau \\
&\quad - \int_0^{t_*} \dot{\phi}(s) \int_0^s g_{1-\alpha,m}(s-\tau) \left(\int_{B_1} \psi^2 w^2 dx \right)(\tau) d\tau ds \\
&\leq - \int_0^{t_*} \dot{\phi}(s) \int_0^s g_{1-\alpha,m}(s-\tau) \left(\int_{B_1} \psi^2 w^2 dx \right)(\tau) d\tau ds.
\end{aligned} \tag{3.39}$$

Noticing that $g_{1-\alpha,m} * W \to g_{1-\alpha} * W$ in $L^1(0, t_2 - t_0)$ and fixing some $t_* \in [t_2 - t_0 - (t_2 - t_1)/4, t_2 - t_0]$ such that for some subsequence $(g_{1-\alpha,m} * W)(t_*) \to (g_{1-\alpha} * W)(t_*)$ as $m \to \infty$. Sending $m \to \infty$, it follows from (3.37),(3.39) that

$$\left(\int_0^{t_1 - t_0} \left(\int_{B_1} (\psi w)^2 dx \right)^p ds \right)^{1/p} \leq C \left((g_{1-\alpha} * W)(t_*) + \|F\|_{L^1([0,t_2 - t_0])} \right), \tag{3.40}$$

where

$$F(s) = -\dot{\phi}(s) \left(g_{1-\alpha} * \int_{B_1} \psi^2 w^2 dx \right)(s) + c_4 (\rho - \rho')^{-2\beta} \int_{\rho B_1} w^2(s,x) dx.$$

Dropping the first term in (3.36), integrating (3.36) over $(0, t_*)$ and taking the limit as $m \to \infty$ for the same sequence as before, we obtain

$$\int_0^{t_1 - t_0} \int_{\rho' B_1} \int_{\rho' B_1} (w(s,x) - w(s,y))^2 k(x,y) dx dy ds \leq C \int_0^{t_2 - t_0} F(s) ds. \tag{3.41}$$

Recalling Remark 3.3 and (1.7), now we can conclude from (3.40) and (3.41) that

$$\|w\|^2_{L^{2\kappa}([0,t_1 - t_0] \times \rho' B_1)} \leq C \left((g_{1-\alpha} * W)(t_*) + \|F\|_{L^1([0,t_2 - t_0])} \right). \tag{3.42}$$

Because $\phi = 0$ in $[t_1 - t_0 + (t_2 - t_1)/2, t_2 - t_0]$ and $t_* \in [t_2 - t_0 - (t_2 - t_1)/4, t_2 - t_0]$, we have

$$\begin{aligned}(g_{1-\alpha} * W)(t_*) \leq & g_{1-\alpha}((t_2 - t_1)/4) \int_0^{t_2-t_0} \int_{\rho B_1} w^2 dxds \\ = & \frac{4^\alpha}{\Gamma(1-\alpha)(\sigma\eta)^\alpha(\rho - \rho')^\alpha} \int_0^{t_2-t_0} \int_{\rho B_1} w^2 dxds.\end{aligned}$$

As in the proof of (3.21), we could obtain

$$\|F\|_{L^1([0,t_2-t_0])} \leq \frac{C(\Lambda, \delta, \eta, \beta, \alpha, n)}{(\rho - \rho')^2} \int_0^{t_2-t_0} \int_{\rho B_1} w^2 dxds.$$

Plugging the above two inequalities into (3.42), we arrive at

$$\|w\|_{L^{2\kappa}([0,t_1-t_0]\times\rho' B_1)} \leq \frac{C(\Lambda, \delta, \eta, \beta, \alpha, n)}{\rho - \rho'} \|w\|_{L^2([0,t_2-t_0]\times\rho B_1)}.$$

Remembering $\gamma = 1 - q$ and transforming the above inequality back to u to obtain

$$\|u\|_{L^{\gamma\kappa}(V'_{\rho'}, d\mu)} \leq \left(\frac{C}{(\rho - \rho')^2} \right)^{1/\gamma} \|u\|_{L^\gamma(V'_\rho, d\mu)}, \quad 0 < \rho' < \rho \leq 1, \tag{3.43}$$

where $\mu = (\eta\omega_n)^{-1}\mu_{n+1}$, ω_n the volume of the unit ball in $\mathbb{R}^n$.

Employing Lemma A.6, we know that there are constants $M_0 = M_0(\Lambda, \delta, \eta, \alpha, \beta, n)$ and $\tau_0 = \tau_0(n, \beta)$ such that

$$\|u\|_{L^{p_0}(V'_\theta, d\mu)} \leq \left(\frac{M_0}{(1-\theta)^{\tau_0}} \right)^{1/\gamma - 1} \|u\|_{L^\gamma(V'_1, d\mu)}, \quad 0 < \theta < 1.$$

If we take $\theta = \frac{\sigma'}{\sigma}$ and translate the above inequality to the Lebesgue measure, we obtain

$$\|u\|_{L^1(U'_{\sigma'})} \leq \left(\frac{M_0(\eta\omega_n)^{-1}}{(\sigma - \sigma')^{\tau_0}} \right)^{1/\gamma - 1} \|u\|_{L^\gamma(U'_\sigma)}, \quad \gamma \in (0, \kappa^{-1}]. \tag{3.44}$$

Hence, our proof is complete. □

3.3. An estimate for $\log \mathbf{u}$.

Theorem 3.6. *Let $\alpha \in (0,1)$, $T > 0$, $k \in \mathcal{R}(\beta_0, \Lambda)$ with $\beta_0 \in (n/4, 1)$ and $\Omega \subset \mathbb{R}^n$. Let further $\eta > 0$ and $\delta \in (0,1)$ be fixed. Then for any $t_0 \geq 0$ and $r > 0$ with $t_0 + \tau r^{2\beta/\alpha} \leq T$, any ball $B = B(x_0, r) \subset \Omega$, and any positive weak supersolution $u \geq \epsilon > 0$ of (1.3) in $(0, t_0 + \tau r^{2\beta/\alpha}) \times B$ with $u_0 \geq 0$ in B and $f = 0$, there is a constant $c = c(u)$ such that*

$$\mu_{n+1}(\{(t,x) \in K_- : \log u(t,x) > c + \lambda\}) \leq C r^{2\beta/\alpha} \mu_n(B) \lambda^{-1}, \quad \lambda > 0, \tag{3.45}$$

and

$$\mu_{n+1}(\{(t, x \in K_+ : \log u(t,x) < c - \lambda\}) \leq C r^{2\beta/\alpha} \mu_n(B) \lambda^{-1}, \quad \lambda > 0, \tag{3.46}$$

where $K_- := (t_0, t_0 + \eta\tau r^{2\beta/\alpha}) \times \delta B$ and $K_+ := (t_0 + \eta\tau r^{2\beta/\alpha}, t_0 + \tau r^{2\beta/\alpha}) \times \delta B$. Here the constant C depends on $\delta, \eta, \tau, n, \alpha, \beta_0, \Lambda$.

Proof. Without loss of generality, we may assume $t_0 = 0$. In fact, if $t_0 > 0$, we shift the time as $t \to t - t_0$, thereby obtaining an inequality of the same type on the time-interval $J := [0, \tau r^{2\beta/\alpha}]$. Observe that the property $g_{1-\alpha} * u \in C([0, t_0 + \tau r^{2\beta/\alpha}]; L^2(B))$ implies $g_{1-\alpha} * \tilde{u} \in C(J; L^2(B))$ for the shifted function $\tilde{u}(t,x) = u(t + t_0, x)$. Hence, we have

$$\int_B \varphi \partial_t \left(g_{1-\alpha,m} * \tilde{u}\right) dx + \mathcal{E}(h_m * \tilde{u}, \varphi) \geq 0, \quad \text{a.e. } t \in J,\ m \in \mathbb{N}, \tag{3.47}$$

for any nonnegative test function $\varphi \in H_e^1(B)$.

For $t \in J$, we choose the test function $\varphi = \psi^2 \tilde{u}^{-1}$ with $\psi \in C_0^1(B)$ such that $\operatorname{supp} \psi \subset B$, $\psi = 1$ in δB, $0 \leq \psi \leq 1$, $|D\psi| \leq 2/((1-\delta)r)$. We have

$$-\int_B \psi^2 \tilde{u}^{-1} \partial_t (g_{1-\alpha,m} * \tilde{u}) dx + \mathcal{E}(\tilde{u}, -\psi^2 \tilde{u}^{-1}) \leq R_m(t), \tag{3.48}$$

where

$$R_m(t) := \mathcal{E}(h_m * \tilde{u}, \psi^2 \tilde{u}^{-1}) - \mathcal{E}(\tilde{u}, \psi^2 \tilde{u}^{-1}).$$

Using (1.6) and properties of ψ, there holds $\mathcal{E}(\psi, \psi) \leq C_1 \mu_n(B)/r^{2\beta} < \infty$ for some constant $C_1 = C_1(n, \beta_0, \Lambda, \delta)$. Denote $w(t,x) = \log(\tilde{u}(t,x)/\psi(x))$. Now we apply Lemma A.7 and Lemma A.8 listed in Appendix to the second term of (3.48). We obtain

$$-\int_B \psi^2 \tilde{u}^{-1} \partial_t (g_{1-\alpha,m} * \tilde{u}) dx + \frac{c_2}{r^{2\beta}} \int_B (w - W)^2 \psi^2 dx \leq \frac{C_1 \mu_n(B)}{r^{2\beta}} + R_m(t), \tag{3.49}$$

where

$$W(t) := \frac{\int_B w(t,x) \psi^2(x) dx}{\int_B \psi^2(x) dx},$$

for a.e. $t \in J$. Here, the factor $r^{2\beta}$ in the second term of (3.49) comes from a simple scaling analysis. In addition, from (3.49), we infer that

$$\frac{-\int_B \psi^2 \tilde{u}^{-1} \partial_t (g_{1-\alpha,m} * \tilde{u}) dx}{\int_B \psi^2(x) dx} + \frac{c_2}{r^{2\beta} \mu_n(B)} \int_B (w - W)^2 \psi^2 dx \leq \frac{C_2}{r^{2\beta}} + S_m(t), \tag{3.50}$$

where C_2 depends on $n, \beta_0, \Lambda, \delta$ and $S_m(t) := R_m(t)/\int_B \psi^2 dx$. Now, we could use same calculations as in the proof of Theorem 3.3 in [36] to complete our proof. And for concisely, we omit the details. □

3.4. **Proof of the Harnack's inequality.** In this section, our aim is to prove Theorem 3.1. With Theorem 3.4, Theorem 3.5 and Theorem 3.6, the proof of Theorem 3.1 is conventional. However, for the completeness of this work, we provide a sketch of the proof in the following.

Without loss of generality, we assume that $u \geq \epsilon$ for some $\epsilon > 0$; otherwise replace u by $u + \epsilon$, which is a weak supersolution of (1.3) with $u_0 + \epsilon$ instead of u_0, and eventually let $\epsilon \to 0^+$.

For $0 < \sigma \leq 1$, we set $U_\sigma = (t_0 + (2 - \sigma)\tau r^{2\beta/\alpha}, t_0 + 2\tau r^{2\beta/\alpha}) \times \sigma B$ and $U'_\sigma = (t_0, t_0 + \sigma \tau r^{2\beta/\alpha}) \times \sigma B$. It is easy to find that $Q_-(t_0, x_0, r) = U'_\delta$ and $Q_+(t_0, x_0, r) = U_\delta$.

Applying Theorem 3.4, we have

$$\operatorname*{ess\,sup}_{U_{\sigma'}} u^{-1} \le \left(\frac{C\mu_{n+1}(U_1)^{-1}}{(\sigma-\sigma')^{\tau_0}}\right)^{1/\gamma} \|u^{-1}\|_{L^\gamma(U_\sigma)}, \quad \delta \le \sigma' < \sigma \le 1,\ \gamma \in (0,1].$$

Here $C = C(\Lambda, \delta, \tau, \beta_0, \alpha, n)$ and $\tau_0 = \tau_0(n, \beta)$. This implies that the first hypothesis of Lemma A.9 is satisfied by any positive constant multiple of u^{-1} with $\xi_0 = \infty$.

Consider $f_1 = u^{-1}e^{c(u)}$ where $c(u)$ is the constant from Theorem 3.6 with $K_- = U_1'$ and $K_+ = U_1$. Because $\log f_1 = c(u) - \log u$, we conclude from Theorem 3.6 that

$$\mu_{n+1}(\{(t,x) \in U_1 \,:\, \log f_1(t,x) > \lambda\}) \le M\mu_{n+1}(U_1)\lambda^{-1}, \quad \lambda > 0,$$

where $M = M(\Lambda, \delta, \tau, \eta, \alpha, \beta_0, n)$. Now, we could use Lemma A.9 with $\xi_0 = \infty$ to f_1 and the family U_σ to obtain

$$\operatorname*{ess\,sup}_{U_\delta} f_1 \le M_1$$

with $M_1 = M_1(\Lambda, \delta, \tau, \eta, \alpha, \beta_0, n)$. Changing back to the variable u, we find that

$$e^{c(u)} \le M_1 \operatorname*{ess\,inf}_{U_\delta} u. \tag{3.51}$$

On the other hand, Theorem 3.5 yields

$$\|u\|_{L^1(U'_{\sigma'})} \le \left(\frac{C\mu_{n+1}(U_1')^{-1}}{(\sigma-\sigma')^{\tau_1}}\right)^{1/\gamma-1} \|u\|_{L^\gamma(U'_\sigma)}, \quad \delta \le \sigma' < \sigma \le 1,\ 0 < \gamma \le \kappa^{-1}.$$

Here $C = C(\Lambda, \delta, \tau, \alpha, \beta_0, n)$ and $\tau_1 = \tau_1(\beta, n)$. Choosing $\xi_0 = 1$ and $\eta = \kappa^{-1}$ in Lemma A.9 and $f_2 = ue^{-c(u)}$ with $c(u)$ from above, we have $\log f_2 = \log u - c(u)$, hence, Theorem 3.6 gives

$$\mu_{n+1}(\{(t,x) \in U_1' \,;\, \log f_2 > \lambda\}) \le M\mu_{n+1}(U_1')\lambda^{-1}, \quad \lambda > 0,$$

where M is as above. Applying Lemma A.9, this time to the function f_2 and the sets U'_σ and with $\xi_0 = 1$ and $\eta = \kappa^{-1}$, we obtain

$$\|f_2\|_{L^1(U'_\delta)} \le M_2\mu_{n+1}(U_1'),$$

where $M_2 = M_2(\Lambda, \delta, \tau, \eta, \alpha, \beta_0, n)$. Changing back to the variable u, we find that

$$\mu_{n+1}(U_1')^{-1}\|u\|_{L^1(U'_\delta)} \le M_2 e^{c(u)}. \tag{3.52}$$

Finally, we combine (3.51) and (3.52) to obtain

$$\mu_{n+1}(U_1')^{-1}\|u\|_{L^1(U'_\delta)} \le M_1M_2 \operatorname*{ess\,inf}_{U_\delta} u,$$

which proves Theorem 3.1.

4. Maximum principles

In this section, we firstly state the following weak maximum principle.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$, $\alpha \in (0,1)$, $T > 0$, $k \in \mathcal{R}(\beta_0, \Lambda)$ with $\beta_0 \in (n/4, 1)$ and $\Lambda \ge \max\{1, \beta_0^{-1}\}$. Assume u be a weak supersolution of problem (1.3) with $u_0 \ge 0$ a.e. in Ω and $f \ge 0$ a.e. in $\Omega \times [0,T]$. Then $u \ge 0$ a.e. in $\mathbb{R}^n \times [0,T]$.*

Proof. Denote $u^- = \max\{-u, 0\}$ and $u^+ = \max\{u, 0\}$ and notice that

$$\int_0^T \mathcal{E}(u, u^-)dt = \int_0^T \mathcal{E}(u^+, u^-)dt - \int_0^T \mathcal{E}(u^-, u^-)dt,$$

$$\int_0^T \mathcal{E}(u^-, u^-)dt = \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u^-(x,t) - u^-(y,t))^2 k(x,y)dxdydt > 0,$$

then we have

$$\int_0^T \mathcal{E}(u, u^-)dt < \int_0^T \mathcal{E}(u^+, u^-)dt.$$

Noticing that $(u^+(x,t) - u^+(y,t))(u^-(x,t) - u^-(y,t)) \leq 0$, we obtain

$$\int_0^T \mathcal{E}(u, u^-)dt < \int_0^T \mathcal{E}(u^+, u^-)dt \leq 0. \tag{4.1}$$

With these estimates, we can follow the proof of Theorem 4.2 in [13] to obtain the required result. □

Then we show the following strong maximum principle which may has many important applications.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$, $\alpha \in (0,1)$, $T > 0$, $k \in \mathcal{R}(\beta_0, \Lambda)$ with $\beta_0 \in (n/4, 1)$ and $\Lambda \geq \max\{1, \beta_0^{-1}\}$. Let further $\eta > 0$ and $\delta \in (0,1)$ be fixed and $f = 0$ in (1.3). Take u be a weak solution of (1.3) in Q_T and assume that $-\infty < \operatorname{ess\,inf}_{Q_T} u$ and that $\operatorname{ess\,inf}_{Q_T} u \leq \operatorname{ess\,inf}_{\Omega} u_0$. Then, if for some cylinder $Q = (t_0, t_0 + \tau r^{2\beta/\alpha}) \times B(x_0, r) \subset Q_T$ with $t_0, \tau, r > 0$ and $\overline{B(x_0, r)} \subset \Omega$, we have*

$$\operatorname*{ess\,inf}_{Q} u = \operatorname*{ess\,inf}_{Q_T} u, \tag{4.2}$$

the function is constant on $(0, t_0) \times \Omega$.

Proof. Let $M = \operatorname{ess\,inf}_{Q_T} u$. Then $v := u - M$ is a nonnegative weak solution of (1.3) with u_0 replaced by $v_0 := u_0 - M \geq 0$. For any $0 \leq t_1 < t_1 + \eta r^{2\beta/\alpha} < t_0$ the weak Harnack inequality applied to v yields the following estimate

$$r^{-(n+2\beta/\alpha)} \int_{t_1}^{t_1+\eta r^{2\beta/\alpha}} \int_{B(x_0,r)} (u - M)dxdt \leq C \operatorname*{ess\,inf}_{Q}(u - M) = 0.$$

This implies that $u = M$ a.e. in $(0, t_0) \times B(x_0, r)$. As in the classical parabolic case [20], the assertion follows by a chaining argument. □

5. An inverse source problem

In this section, we focus on an inverse source problem for (1.3) under the assumption that the inhomogeneous term f takes the form of separation of variables. In addition, we add more assumptions on the kernel $k(\cdot, \cdot)$ appeared in the definition of space-nonlocal operator L. Specifically speaking, we assume

$$k(x,y) = \frac{a((x-y)/|x-y|)}{|x-y|^{n+2\beta}}. \tag{5.1}$$

Here $a \in L^1(\mathcal{S}^{n-1})$ satisfying $a(\theta) = a(-\theta)$,

$$0 < \Lambda^{-1} \leq a(\theta) \leq \Lambda, \tag{5.2}$$

and

$$0 < \Lambda^{-1} \leq \inf_{\nu \in \mathcal{S}^{n-1}} \int_{\mathcal{S}^{n-1}} |\nu \cdot \theta|^{2\beta} a(\theta) d\theta \tag{5.3}$$

for $\theta \in \mathcal{S}^{n-1}$ with Λ are some positive constants (may not be the same as in (1.6) and (1.7)). For notational convenience, denote $\mathcal{R}^p(\beta, \Lambda)$ as the space of all kernels k satisfying the above conditions.

Remark 5.1. In this section, we will always assume $k \in \mathcal{R}^p(\beta, \Lambda)$. The reason is that under the weaker assumptions $k \in \mathcal{R}(\beta_0, \Lambda)$, we could not obtain enough regularity for the solution by some conventional methods. As is well known, regularity issues under weak assumptions on kernels are important research subjects and highly nontrivial. Because this is not the main point of this paper, we will prove our results when $k \in \mathcal{R}^p(\beta, \Lambda)$. And once higher regularity properties for the solutions are available when $k \in \mathcal{R}(\beta_0, \Lambda)$, all results in this section may be adapted to this more general setting.

Problem 5.2. *Assume $n = 2$ or 3, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Let $\alpha \in (0, 1)$, $\beta_0 \in [n/4, 1)$, $\beta \in (\beta_0, 1)$, $\Lambda > 1$ and $k \in \mathcal{R}^p(\Lambda, \beta_0)$. Let $x_0 \in \Omega$ and $T > 0$ be arbitrarily given, and u be the solution to (1.3) with $u_0 = 0$ and $f(x,t) = \rho(t)g(x)$. Provided that $g(\cdot)$ is known, determine $\rho(t)\,(0 \leq t \leq T)$ by the single point observation data $u(x_0, t)\,(0 \leq t \leq T)$.*

Similar problems are investigated in [21] for a time-fractional and space-integer order diffusion equation. As in [21], spatial component g simulates e.g. a source of contaminants which may be dangerous. Usually, g is limited to a small region given by $\operatorname{supp} g \subset \Omega$. We are required to determine the time-dependent magnitude ρ by the pointwise data $u(x_0, t)\,(0 \leq t \leq T)$, where $x_0 \notin \operatorname{supp} g$ is understood as a monitoring point. For more work on similar problems, we refer to [5, 28, 29].

5.1. **Regularity of the solution.** Let us firstly recall the following lemma proved in [9].

Lemma 5.3. *Let Ω be a bounded Lipschitz domain, L is the operator defined in (1.4) with kernel $k \in \mathcal{R}^p(\beta, \Lambda)$ with $\Lambda > 1$ and $\beta \in (0, 1)$. Then, for the following nonlocal elliptic equation*

$$\begin{cases} L\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{5.4}$$

We have

1. *Equation (5.4) has a set of eigenfunctions ϕ_k forming a Hilbert basis of $L^2(\Omega)$.*
2. *If $\{\lambda_k\}_{k\in\mathbb{N}}$ is the sequence of eigenvalues associated to the eigenfunctions of L in increasing order, then*

$$\lim_{k\to\infty} \lambda_k k^{-\frac{2\beta}{n}} = C_0,$$

for some constant C_0.

Based on the above lemma, we could assume $\{\lambda_k, \phi_k(x)\}_{k=1}^{\infty}$ as the eigensystem of the operator L. Multiplying equation

$$\partial_t^\alpha (u - u_0) + Lu = f$$

by ϕ_k, denote $u_k(t) = (u(\cdot,t),\phi_k)$, $u_{0,k} = (u_0,\phi_k)$ and $f_k(t) = (f(\cdot,t),\phi_k)$, we obtain

$$\partial_t^\alpha(u_k(t) - u_{0,k}) = -\lambda_k u_k(t) + f_k(t), \quad t > 0. \tag{5.5}$$

Recalling that the operator $\partial_t^\alpha(u_k(t) - u_{0,k})$ is just the modified Caputo fractional derivative operator used in [26], and according to Lemma 1 in [26], we know that

$$u_k(t) = E_{\alpha,1}(-\lambda_k t^\alpha)u_{0,k} + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)f_k(s)ds. \tag{5.6}$$

Hence, we may have

$$\begin{aligned} u(x,t) =& \sum_{k=1}^\infty E_{\alpha,1}(-\lambda_k t^\alpha)u_{0,k}\phi_k(x) \\ &+ \sum_{k=1}^\infty \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha)f_k(s)ds\phi_k(x), \end{aligned} \tag{5.7}$$

in some sense. Actually, we could obtain the following theorem.

Theorem 5.4. *Fix $T > 0$, let $n = 2$ or 3, $\beta \in (n/4, 1)$, $\Lambda > 1$, $k \in \mathcal{R}^p(\beta,\Lambda)$, $\alpha \in (0,1)$ and Ω is a bounded Lipschitz domain. Concerning the weak solution to (1.3), we have*

(1) *Let $u_0 \in L^2(\Omega)$ and $f = 0$. Then the unique weak solution u belongs to*

$$C([0,T];L^2(\Omega)) \cap C((0,T];H_e^{2\beta}(\Omega)),$$

which can be represented as

$$u(x,t) = \sum_{k=1}^\infty E_{\alpha,1}(-\lambda_k t^\alpha)(u_0,\phi_k)\phi_k(x) \tag{5.8}$$

in $C([0,T];L^2(\Omega)) \cap C((0,T];H_e^{2\beta}(\Omega))$, where $\{(\lambda_k,\phi_k)\}_{k=1}^\infty$ is the eigensystem of L. Moreover, there exists a constant $C = C(\Omega,T,\alpha,L) > 0$ such that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\|u_0\|_{L^2(\Omega)}, \tag{5.9}$$

$$\|u(\cdot,t)\|_{H_e^{2\beta}(\Omega)} + \|\partial_t^\alpha(u(\cdot,t) - u_0(\cdot))\|_{L^2(\Omega)} \le C\|u_0\|_{L^2(\Omega)}t^{-\alpha}. \tag{5.10}$$

In addition, $u : (0,T] \to H_e^{2\beta}(\Omega)$ can be analytically extended to a sector $\{z \in \mathbb{C} : z \neq 0, |arg(z)| < \pi/2\}$.

(2) *Let $u_0 = 0$ and $f \in L^\infty([0,T];L^2(\Omega))$. Then the unique weak solution u belongs to $L^2((0,T];H_e^{2\beta}(\Omega))$ such that $\lim_{t\to 0}\|u(\cdot,t)\|_{L^2(\Omega)} = 0$.*

Proof. According to Theorem 2.3, there exists a unique weak solution under the conditions stated in both conclusions stated above. Referring to [8], we note that the Fourier symbol of the operator L is

$$A(\xi) = \int_{\mathcal{S}^{n-1}} |\xi\cdot\theta|^2 a(\theta)d\theta,$$

and it is clear that

$$0 < \Lambda^{-1}|\xi|^{2\beta} \le A(\xi) \le \Lambda|\xi|^{2\beta}. \tag{5.11}$$

Using Plancherel's theorem for Fourier transforms, we have

$$\|Lu(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |Lu(x,t)|^2 dx = \int_{\mathbb{R}^n} |A(\xi)\mathcal{F}(u)(\xi,t)|^2 d\xi.$$

Hence, using (5.11), we can conclude that

$$\Lambda^{-1/2}\|u(\cdot,t)\|_{H^{2\beta}(\mathbb{R}^n)} \leq \|Lu(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \Lambda^{1/2}\|u\|_{H^{2\beta}(\mathbb{R}^n)}. \tag{5.12}$$

Because u is a weak solution of (1.3), we know that $u = 0$ a.e. in $\mathbb{R}^n\backslash\Omega$. Hence, we obtain that

$$\Lambda^{-1/2}\|u(\cdot,t)\|_{H_e^{2\beta}(\mathbb{R}^n)} \leq \|Lu(\cdot,t)\|_{L^2(\Omega)} \leq \Lambda^{1/2}\|u\|_{H_e^{2\beta}(\mathbb{R}^n)}. \tag{5.13}$$

With these preparations, we could apply the methods used in [29] to conclude our claims. Since the proof is rather straightforward, we will omit the details for concisely. $\square$

5.2. **Fractional Duhamel's principle.** Let us recall the problem under consideration

$$\begin{cases} \partial_t^\alpha u(x,t) + Lu(x,t) = \rho(t)g(x) & \text{in } \Omega\times[0,T], \\ u(x,t) = 0 & \text{in } \mathbb{R}^n\backslash\Omega,\ t\geq 0, \\ u(x,0) = 0 & \text{in } \Omega, \text{ for } t=0, \end{cases} \tag{5.14}$$

where $g \in C^1([0,T])$, $g \in L^2_e(\Omega)$ with $g \geq 0$ and $g \not\equiv 0$.

Theorem 5.5. *Let u be the solution to (5.14), where $\rho \in C^1([0,T])$ and $g \in L^2_e(\Omega)$. Then u allows the representation*

$$u(x,t) = (\mu * v)(x,t) = \int_0^t \mu(t-s)v(x,s)ds \quad (0 < t \leq T),$$

where $v(x,t)$ solves the following homogeneous problem

$$\begin{cases} \partial_t^\alpha (v(x,t) - g) + Lv(x,t) = 0 & \text{in } \Omega\times[0,T], \\ u(x,t) = 0 & \text{in } \mathbb{R}^n\backslash\Omega,\ t\geq 0, \\ u(x,0) = g(x) & \text{in } \Omega, \text{ for } t=0, \end{cases} \tag{5.15}$$

and

$$\mu(t) := \frac{d}{dt}(g_\alpha * \rho)(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{\rho(s)}{(t-s)^{1-\alpha}}dx \quad 0 < t \leq T. \tag{5.16}$$

Proof. Because $\rho g \in L^\infty([0,T]; L^2_e(\Omega))$, equation (5.14) admits a uniqueness solution $u \in V_p([0,T],\Omega)$ with $1 \leq p < 2/(1-\alpha)$ by Theorem 2.3. In addition, we know that $u \in L^2((0,T]; H_e^{2\beta}(\Omega))$ and $\lim_{t\to 0}\|u(\cdot,t)\|_{L^2(\Omega)} = 0$. Setting

$$\tilde{u}(x,t) := \int_0^t \mu(t-s)v(x,s)ds, \tag{5.17}$$

and we may use similar deduction used in the proof of Lemma 4.1 in [21] to conclude that

$$\tilde{u} \in L^\infty((0,T]; H_e^{2\beta}(\Omega)) \subset L^2((0,T]; H_e^{2\beta}(\Omega)), \quad \lim_{t\to 0}\|\tilde{u}(\cdot,t)\|_{L^2(\Omega)} = 0.$$

From the proof of Lemma 4.1 in [21], we also know that

$$\mu(t) = \frac{1}{\Gamma(\alpha)}\left(\frac{\rho(0)}{t^{1-\alpha}} + \int_0^t \frac{\rho'(s)}{(t-s)^{1-\alpha}}ds\right), \tag{5.18}$$

and

(5.19) $$\mu \in L^1((0,T)), \quad |\mu(t)| \le Ct^{\alpha-1} \quad \text{with } 0 < t \le T.$$

By definition, we have

$$\begin{aligned}
\partial_t^\alpha(\tilde{u}(x,t) - \tilde{u}(x,0)) &= \partial_t^\alpha \tilde{u}(x,t) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t (t-s)^{-\alpha} \int_0^s \mu(\tau) v(x, s-\tau) d\tau dsp \right\} \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t \int_\tau^t (t-s)^{-\alpha} v(x, s-\tau) ds \mu(\tau) d\tau \right\} \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t \int_0^{t-\tau} (t-\tau-s)^{-\alpha} v(x,s) ds \mu(\tau) d\tau \right\} \\
&= \frac{d}{dt}(\mu * g_{1-\alpha} * v)(x,t) = \mu * \partial_t^\alpha (v - g) + \mu * g_{1-\alpha} \cdot g.
\end{aligned}$$

For the time fractional term, we have

$$\begin{aligned}
\|\partial_t^\alpha(\tilde{u}(\cdot,t) - \tilde{u}(\cdot,0))\|_{L^2(\Omega)} &= \|\partial_t^\alpha \tilde{u}(\cdot,t)\|_{L^2(\Omega)} = \|\partial_t (g_{1-\alpha} * \mu * v)(\cdot,t)\|_{L^2(\Omega)} \\
&\le \|\mu\|_{L^1(0,T)} (\|\partial_t^\alpha (v-g)\|_{L^\infty((0,T];L^2(\Omega))} + g_{1-\alpha}(t)\|g\|_{L^2(\Omega)}).
\end{aligned}$$

This implies that the above time fractional differentiation makes sense in $L^2(\Omega)$ for $0 < t \le T$. Now we illustrate $\tilde{u}$ satisfies equation (5.14). Using equation (5.15) and noticing that $\mu = \frac{d}{dt}(g_\alpha * \rho)$, we obtain

$$\begin{aligned}
\partial_t^\alpha \tilde{u}(x,t) &= -L(\mu * v) + \frac{d}{dt}(g_\alpha * \rho) * g_{1-\alpha} \cdot g \\
&= -L\tilde{u}(x,t) + \rho g.
\end{aligned}$$

Therefore, we conclude that $\partial_t^\alpha \tilde{u} + L\tilde{u} = \rho g$ and the proof is completed. □

5.3. **Uniqueness.** In this section, we provide a uniqueness theorem for Problem 5.2 as follow.

Theorem 5.6. *Under the same settings in Problem 5.2, we further assume that* $\rho \in C^1([0,T])$, $g \in L^2(\Omega)$, $g \ge 0$ *and* $g \not\equiv 0$. *Then* $u(x_0,t) = 0$ $(0 \le t \le T)$ *implies* $\rho(t) = 0$ $(0 \le t \le T)$.

With the strong maximum principle and fractional Duhamel's principle obtained in the previous section, this theorem could be proved by using similar ideas from a recent paper [21]. So we omit the proof here.

Appendix A. Some classical and technical results

A.1. **Properties of the time-fractional derivative.** In [34], the author provide an important formula that is for a sufficiently smooth function u on $(0,T)$ one has for a.e. $t \in (0,T)$,

(A.1)
$$\begin{aligned}
H'(u(t))\frac{d}{dt}(k * u)(t) &= \frac{d}{dt}(k * H(u))(t) + (-H(u(t)) + H'(u(t))u(t))k(t) \\
&\quad + \int_0^t (H(u(t-s)) - H(u(t)) - H'(u(t))[u(t-s) - u(t)])(-\dot{k}(s)) ds,
\end{aligned}$$

where $H \in C^1(\mathbb{R})$ and $k \in W^{1,1}([0,T])$. Taking $H(y) = \frac{1}{2}(y^+)^2$, for any function $u \in L^2([0,T])$, there will be a direct corollary of the above formula

$$u(t)^+ \frac{d}{dt}(k * u)(t) \geq \frac{1}{2}\frac{d}{dt}(k * (u^+)^2), \quad \text{a.e. } t \in (0,T). \tag{A.2}$$

Denote $v = -u$ and replace u in (A.2) by v, we will obtain

$$v(t)^+ \frac{d}{dt}(k * v)(t) \geq \frac{1}{2}\frac{d}{dt}(k * (v^+)^2), \quad \text{a.e. } t \in (0,T). \tag{A.3}$$

Now replacing u back into (A.3), we find that

$$u(t)^- \frac{d}{dt}(k * u)(t) \leq -\frac{1}{2}\frac{d}{dt}(k * (u^-)^2), \quad \text{a.e. } t \in (0,T). \tag{A.4}$$

The following two lemmas which could be found in [37] are important for our deduction.

Lemma A.1. *Let $T > 0$ and $\alpha \in (0,1)$. Suppose that $v \in {}_0W^{1,1}([0,T])$ and $\varphi \in C^1([0,T])$. Then*

$$(g_\alpha * (\varphi \dot{v}))(t) = \varphi(t)(g_\alpha * \dot{v})(t) + \int_0^t v(\sigma)\partial_\sigma(g_\alpha(t-\sigma)[\varphi(t) - \varphi(\sigma)])d\sigma,$$

for a.e. $t \in (0,T)$. If in addition v is nonnegative and φ is nondecreasing there holds

$$(g_\alpha * (\varphi \dot{v}))(t) \geq \varphi(t)(g_\alpha * \dot{v})(t) - \int_0^t g_\alpha(t-\sigma)\dot{\varphi}(\sigma)v(\sigma)d\sigma,$$

for a.e. $t \in (0,T)$.

Lemma A.2. *Let $T > 0$, $k \in W^{1,1}([0,T])$, $v \in L^1([0,T])$, and $\varphi \in C^1([0,T])$. Then*

$$\varphi(t)\frac{d}{dt}(k * v)(t) = \frac{d}{dt}(k * [\varphi v])(t) + \int_0^t \dot{k}(t-\sigma)(\varphi(t) - \varphi(\sigma))v(\sigma)d\sigma,$$

for a.e. $t \in (0,T)$.

A.2. **Properties of the space-fractional derivative.** The following lemmas are used in our proof and these lemmas could be found in [7, 36, 37].

Lemma A.3.

(1) *Let $q > 1$, $a, b > 0$ and $\tau_1, \tau_2 \geq 0$. Set $\vartheta(q) = \max\{4, (6q-5)/2\}$. Then*

$$\begin{aligned}(b-a)\left(\tau_1^{q+1}a^{-q} - \tau_2^{q+1}b^{-q}\right) \geq& \frac{1}{q-1}\tau_1\tau_2\left(\left(\frac{b}{\tau_2}\right)^{\frac{1-q}{2}} - \left(\frac{a}{\tau_1}\right)^{\frac{1-q}{2}}\right)^2 \\ &- \vartheta(q)(\tau_1-\tau_2)^2\left(\left(\frac{b}{\tau_2}\right)^{1-q} + \left(\frac{a}{\tau_1}\right)^{1-q}\right).\end{aligned}$$

Since $1 - q < 0$ the division by $\tau_1 = 0$ or $\tau_2 = 0$ is allowed.

(2) *Let $q \in (0,1)$, $a, b > 0$ and $\tau_1, \tau_2 \geq 0$. Set $\zeta(q) = \frac{4q}{1-q}$, $\zeta_1(q) = \frac{1}{6}\zeta(q)$ and $\zeta_2(q) = \zeta(q) + \frac{9}{q}$. Then*

$$\begin{aligned}(b-a)(\tau_1^2a^{-q} - \tau_2^2b^{-q}) \geq& \zeta_1(q)\left(\tau_2 b^{\frac{1-q}{2}} - \tau_2 a^{\frac{1-q}{2}}\right)^2 \\ &- \zeta_2(q)(\tau_2-\tau_1)^2(b^{1-q} + a^{1-q}).\end{aligned}$$

Lemma A.4. *Let $n = 2$ or 3, $\beta_0 > 0$. Then there is a constant $S > 0$ such that for any $\beta \in (\beta_0, 1)$, $R > 0$, $\sigma = \frac{3}{3-2\beta}$ and $u \in H^\beta(B_R)$ the following inequality holds:*

$$\left(\int_{B_R} |u(x)|^{2\sigma} dx\right)^{1/\sigma} \leq 2(1-\beta)S \int_{B_R}\int_{B_R} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\beta}} dxdy + SR^{-2\beta}\int_{B_R} u^2(x)dx.$$

Lemma A.5. *Let $\kappa > 1$, $\bar{p} \geq 1$, $C \geq 1$ and $\gamma > 0$. Suppose f is a μ-measurable function on U_1 such that*

$$\|f\|_{L^{\beta\kappa}(U_{\sigma'})} \leq \left(\frac{C(1+\beta)^\gamma}{(\sigma-\sigma')^\gamma}\right)^{1/\beta} \|f\|_{L^\beta(U_\sigma)}, \quad 0 < \sigma' < \sigma \leq 1, \ \beta > 0.$$

Then there exist constants $M = M(C, \gamma, \kappa, \bar{p})$ and $\gamma_0 = \gamma_0(\gamma, \kappa)$ such that

$$\operatorname*{ess\,sup}_{U_\delta} |f| \leq \left(\frac{M}{(1-\delta)^{\gamma_0}}\right)^{1/p} \|f\|_{L^p(U_1)}, \quad \text{for all } \delta \in (0,1), \ p \in (0, \bar{p}].$$

Lemma A.6. *Assume that $\mu_1(U) \leq 1$. Let $\kappa > 1$, $0 < p_0 < \kappa$, and $C \geq 1$, $\gamma > 0$. Suppose f is a Lebesgue measure function on U_1 such that*

$$\|f\|_{L^{\beta\kappa}(U_{\sigma'})} \leq \left(\frac{C}{(\sigma-\sigma)^2}\right)^{1/\beta} \|f\|_{L^\beta(U_\sigma)}, \quad 0 < \sigma' < \sigma \leq 1, \ 0 < \beta \leq \frac{p_0}{\kappa} < 1.$$

Then there exist constants $M = M(C, \gamma, \kappa)$ and $\gamma_0 = \gamma_0(\gamma, \kappa)$ such that

$$\|f\|_{L^{p_0}(U_\delta)} \leq \left(\frac{M}{(1-\delta)^{\gamma_0}}\right)^{1/p-1/p_0} \|f\|_{L^p(U_1)} \quad \text{for all } \delta \in (0,1), \ p \in \left(0, \frac{p_0}{\kappa}\right].$$

Lemma A.7. *Let $I \subset \mathbb{R}$ and $\phi : \mathbb{R}^n \to [0, \infty)$ be a continuous function satisfying* $\operatorname{supp}\phi = \bar{B}_R$ *for some $R > 0$ and $a(\phi, \phi) < \infty$. Then the following computation rule holds for $w : I \times \mathbb{R} \to [0, \infty)$:*

$$\begin{aligned}\mathcal{E}(w, -\phi^2 w^{-1}) \geq & -3\mathcal{E}(\phi,\phi) \\ & \geq \int_{B_R}\int_{B_R} \phi(x)\phi(y)\left(\log\frac{w(t,y)}{\phi(y)} - \log\frac{w(t,x)}{\phi(x)}\right)^2 k(x,y)dxdy.\end{aligned}$$

Lemma A.8. *Let $\psi : B \to [0,1]$ belongs to $C_0^1(B)$ satisfies $\psi = 1$ in δB with $\delta < 1$ and $k \in \mathcal{R}(\beta_0, \Lambda)$ for some $\beta_0 \in (0,1)$ and $\Lambda \geq 1$. Then there is a positive constant $C(n, \beta_0, \Lambda, \delta)$ such that for every $u \in L^1(B, \psi(x)dx)$*

$$\int_B [u(x) - u_\psi]^2 \psi dx \leq C \int_B\int_B [u(x)-u(y)]^2 k(x,y)(\psi(x) \wedge \psi(y))dxdy,$$

where

$$u_\psi = \frac{\int_B u(s)\psi(x)dx}{\int_B \psi(x)dx}.$$

Lemma A.9. *Let $\delta, \eta \in (0,1)$, and let γ, C be positive constants and $0 < \xi_0 \leq \infty$. Suppose f is a positive μ-measurable function on U_1 which satisfies the following two conditions:*

(1)
$$\|f\|_{L^{\xi_0}(U_{\sigma'})} \leq \left(C(\sigma-\sigma')^{-\gamma}\mu(U_1)^{-1}\right)^{1/\xi-1/\xi_0}\|f\|_{L^{\xi}(U_\sigma)},$$
for all σ, σ', β such that $0<\delta\leq\sigma'<\sigma\leq 1$ and $0<\xi\leq\min\{1,\eta\xi_0\}$.

(2)
$$\mu(\{\log f>\lambda\})\leq C\mu(U_1)\lambda^{-1}$$
for all $\lambda>0$.

Then
$$\|f\|_{L^{\xi_0}(U_\delta)}\leq M\mu(U_1)^{1/\xi_0},$$
where M depends only on δ,η,γ,C and ξ_0.

Lemma A.10. *Let u be a weak supersolution to equation (1.3). Let $\phi\in H_e^{1,\beta}(Q_T)$ be a test function. Then for every $I'\subset\subset I=[0,T]$*
$$\int_{I'}\mathcal{E}(h_m*u(t,\cdot),\phi(t,\cdot))dt\to\int_{I'}\mathcal{E}(u(t,\cdot),\phi(t,\cdot))dt,\quad \text{as } m\to\infty.$$

Proof. Let $V(t,x,y)=u(t,x)-u(t,y)$, $(h_m*V)(t,x,y)=(h_m*u)(t,x)-(h_m*u)(t,y)$ and $\Phi(t,x,y)=\phi(t,x)-\phi(t,y)$. Denote B_R is a ball with radius $R>0$, for some fixed $\epsilon>0$, denote $B:=B_{R+\epsilon}$ as a ball with radius $R+\epsilon$. Decompose the integral over $\mathbb{R}^n\times\mathbb{R}^n$ yields
$$\begin{aligned}
&\int_{I'}\mathcal{E}((h_m*u-u)(t,\cdot),\phi(t,\cdot))dt\\
=&\int_{I'}\int_B\int_B((h_m*V)(t,x,y)-V(t,x,y))\Phi(t,x,y)k(x,y)dxdydt\\
&+2\int_{I'}\int_B\phi(t,x)\int_{B^c}((h_m*V)(t,x,y)-V(t,x,y))k(x,y)dydxdt\\
=:&\mathrm{I}_m+\mathrm{II}_m.
\end{aligned}$$

For I_1, we have
$$\begin{aligned}
\mathrm{I}_m&\leq C\|(h_m*V-V)k_0^{1/2}\|_{L^2(I';L^2(B\times B))}\|\Phi k_0^{1/2}\|_{L^2(I';L^2(B\times B))}\\
&\leq C\|(h_m*V-V)k_0^{1/2}\|_{L^2(I';L^2(B\times B))}\|\Phi\|_{L^2(I';H^\beta(B))},
\end{aligned}$$
where we have used (1.7) in the second inequality. The convergence properties shown in Section 2.2 implies that the first factor of the above inequality tends to zero. Using (1.6), we could obtain
$$\begin{aligned}
\mathrm{II}_m&\leq\|\phi\|_{L^\infty(I'\times B)}\int_{I'}\|h_m*V(t,\cdot,\cdot)-V(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\int_{B_R}\int_{B^c}k_0(x,y)dydxdt\\
&\leq C\epsilon^{-2\beta}|B_R|\|\phi\|_{L^\infty(I'\times B)}\|h_m*V-V\|_{L^1(I';L^\infty(\mathbb{R}^n\times\mathbb{R}^n))}.
\end{aligned}$$
The convergence of II_m follows from the convergence properties shown in Section 2.2. Now the proof is complete. □

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