ON THE DECAY AND STABILITY OF GLOBAL SOLUTIONS TO THE 3D INHOMOGENEOUS MHD SYSTEM

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Abstract. In this paper, we investigative the large time decay and stability to any given global smooth solutions of the 3D incompressible inhomogeneous MHD systems. We proved that given a solution (a, u, B) of (1.2), the velocity field and the magnetic field decay to zero with an explicit rate, for u which coincide with incompressible inhomogeneous Navier-Stokes equations [1]. In particular, we give the decay rate of higher order derivatives of u and B which are useful to prove our main stability result. For a large solution of (1.2) denoted by (a, u, B), we proved that a small perturbation of the initial data still generates a unique global smooth solution and the smooth solution keeps close to the reference solution (a, u, B). At last, we should mention that the main results in this paper are concerned with large solutions.

1. Introduction and main results

Magnetic fields influence many fluids. Magnetohydrodynamics (MHD) is concerned with the interaction between fluid flow and magnetic field. The governing equations of nonhomogeneous MHD system can be stated as follows [16],

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \mathcal{M}) - (B \cdot \nabla) B + \nabla \Pi = 0, \\
\partial_t B - \lambda \Delta B - \operatorname{curl}(u \times B) = 0, \\
\operatorname{div} u = \operatorname{div} B = 0, \\
\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad B|_{t=0} = B_0,
\end{cases}
\tag{1.1}
$$

where ρ, $u = (u_1, u_2, u_3)$ stand for the density and velocity of the fluid, respectively, $\mathcal{M} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, Π is a scalar pressure function. B is the magnetic field. $\mu(\rho) \geq 0$ denotes the viscosity of fluid, which we assume in this paper is a positive constant. $\lambda > 0$ is also a constant, which describes the relative strength of advection and diffusion of B.

If there is no magnetic field, i.e., $B = 0$, MHD system turns to be nonhomogeneous Navier-Stokes system. Since the second equation and the third equation of (1.1) are similar, the study about MHD system has been along with that for Navier-Stokes system. Let us first recall some results about Navier stokes equations. When ρ_0 is bounded away from zero, the global existence of weak solutions was established by Kazhikov [20]. Moreover, Antontsev, Kazhikov and Monakhov [4] proved the first result on local existence and uniqueness of strong solutions. For

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the two dimensional case, they even proved that the strong solution is global. But the global existence of strong or smooth solutions in 3D is still an open problem.

Recently, Danchin proved the global existence in the Besov space framework [22]. His results show that the global in time existence of regular solutions to the inhomogeneous Navier Stokes equations in $\mathbb{R}^n$ in the optimal Besov setting, under suitable smallness conditions of the initial data. In particular, his results allow the initial densities have a jump at the interface. At the same time, Abidi, Gui, Zhang [2] proved the local well-posedness of three-dimensional incompressible inhomogeneous Navier-Stokes equations with initial data in the critical Besov spaces, without assumptions of small density variation. And they also proved the global well-posedness when the initial velocity is small in $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. For more results in this direction, see [10, 15, 14] and reference therein.

Now, let us come back to the MHD system (1.1). When we assume ρ is a constant that is to say the fluid is homogeneous, the MHD system has been extensively studied. Duraut and Lions [17] constructed a class of weak solutions with finite energy and a class of local strong solutions. Recently, Cao and Wu [7] obtained some global regularity results of the classical solutions of the MHD equations with mixed partial dissipation and magnetic diffusion. In addition, they also provided the global existence, conditional regularity and uniqueness of a weak solution of the 2D MHD equations with only magnetic diffusion. For more results in this direction, see [6, 8] and reference therein.

When the fluid is nonhomogeneous. Abidi and Paicu [3] proved that the magnetohydrodynamic system in $\mathbb{R}^N$ with variable density, variable viscosity and variable conductivity has a local weak solution in suitable Besov space if the initial density approaches a constant. They also proved that the constructed solution exist globally in time if the initial data are small enough. Huang and Wang [19] proved the global existence of strong solutions with vacuum to the 2D nonhomogeneous incompressible MHD system, as long as the initial data satisfies some compatibility conditions. In this paper, we only consider non-vacuum case.

Let $a := \frac{1}{\rho} - 1$ and take $\mu = \lambda = 1$, then the MHD system has the following form:

$$
\begin{cases}
\partial_t a + u\cdot\nabla a = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t u + u\cdot\nabla u - (1+a)(\Delta u - \nabla\Pi) - (1+a)(B\cdot\nabla)B = 0,\\
\partial_t B - \Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0,\\
\operatorname{div} u = \operatorname{div} B = 0,\\
(a,u,B)|_{t=0} = (a_0,u_0,B_0).
\end{cases}
\tag{1.2}
$$

Let $\rho := \frac{1}{1+a}$, then (ρ, u, B) solves

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\rho\partial_t u + \rho u\cdot\nabla u - \Delta u - (B\cdot\nabla)B + \nabla\Pi = 0,\\
\partial_t B - \Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0,\\
\operatorname{div} u = \operatorname{div} B = 0,\\
(\rho,u,B)|_{t=0} = (\rho_0,u_0,B_0).
\end{cases}
\tag{1.3}
$$

In what follows, we shall investigate the large time decay and stability of the above MHD system. Compared with the classical incompressible Navier-Stokes system (NS), that is, in the case when $a = 0$ and $B = 0$ in (1.2), the MHD system is more complex. Given a global large solution $u \in L^\infty_{\rm loc}([0,\infty); H^1(\mathbb{R}^3)) \cap L^2_{\rm loc}([0,\infty); H^2(\mathbb{R}^3))$, Ponce, Racke, Sideris and Titi [21] proved (NS) global stability under the additional assumption $\int_0^\infty \|\nabla u(t)\|^4_{L^2}\,dt < \infty$. Then, Gallagher, Iftimie and Planchon [18] removed the additional assumption. For the inhomogeneous Navier-Stokes equations (INS), Abidi, Gui and Zhang [1] showed its decay and stability for large solutions.

Our first result concerns the global stability of the given solution of (1.2) when the initial density ρ_0 is close to a positive constant. This is a simple generalization of Theorem 1.1 in [1].

Theorem 1.1. *Let $\bar a_0 \in B^{5/2}_{2,1}(\mathbb{R}^3)$, $\bar u_0 \in B^{3/2}_{2,1}(\mathbb{R}^3)$ and $\bar B_0 \in B^{3/2}_{2,1}(\mathbb{R}^3)$ with $\operatorname{div} \bar u_0 = \operatorname{div} \bar B_0 = 0$, and let there exist two positive constants m and M so that*

$$m \le 1 + \bar a_0 \le M. \tag{1.4}$$

We assume that $\bar a \in C([0,\infty); B^{5/2}_{2,1}(\mathbb{R}^3))$ and

$$\bar u, \bar B \in C([0,\infty); B^{3/2}_{2,1}(\mathbb{R}^3)) \cap L^1_{loc}(\mathbb{R}^+; \dot B^{7/2}_{2,1}(\mathbb{R}^3))$$

is a given solution of MHD with initial data $(\bar a_0, \bar u_0)$. Then there exist positive constants c_1, C_1 and a large enough time $T_0 := T_0(\bar a_0, \bar u_0, \bar B_0)$ so that if

$$\|\bar a_0\|_{\dot B^{3/2}_{2,1}} \exp\left\{ C_1 \int_0^{T_0} \|\nabla \bar u(\tau)\|_{\dot B^{3/2}_{2,1}}\, d\tau \right\} \le c_1, \tag{1.5}$$

a constant c_2 exists so that $(a_0, u_0, B_0) := (\bar a_0 + \tilde a_0, \bar u_0 + \tilde u_0, \bar B_0 + \tilde B_0)$ generates a unique global solution with

$$\begin{aligned} &a \in C_b([0,\infty); B^{5/2}_{2,1}(\mathbb{R}^3)),\\ &u, B \in C_b([0,\infty); B^{3/2}_{2,1}(\mathbb{R}^3)) \cap L^1([0,\infty); \dot B^{7/2}_{2,1}(\mathbb{R}^3)), \end{aligned} \tag{1.6}$$

provided that $(\tilde a_0, \tilde u_0, \tilde B_0)$ satisfies

$$\|\tilde a_0\|_{B^{3/2}_{2,1}} + \|\tilde u_0\|_{B^{1/2}_{2,1}} + \|\tilde B_0\|_{B^{1/2}_{2,1}} \le c_2. \tag{1.7}$$

In order to obtain the stability of large solutions of system (1.2), here, we need to investigate the decay properties of the velocity field u and the magnetic field B. Comparing with the INS case, our case is more complex and we need to use the coupling between the equations of u and B. In order to get the desired estimates, we need to provide the estimate of $\|\nabla a(t)\|_{L^\infty}$ and in addition to get the higher order decay properties of u and B.

Theorem 1.2. *For $p \in (1, \frac{6}{5})$, let $a_0 \in B^{5/2}_{2,1}$ and $u_0, B_0 \in L^p(\mathbb{R}^3) \cap B^2_{2,1}(\mathbb{R}^3)$ satisfying (1.4) and $\operatorname{div} u_0 = 0$ and $\operatorname{div} B_0 = 0$. We assume that*

$$a \in C([0,\infty); B^{5/2}_{2,1}), \quad u \in C([0,\infty); B^2_{2,1}(\mathbb{R}^3)) \cap L^1_{\rm loc}(\mathbb{R}^+; \dot B^4_{2,1}(\mathbb{R}^3))$$

and $B \in C([0,\infty); B^2_{2,1}(\mathbb{R}^3)) \cap L^1_{\rm loc}(\mathbb{R}^+; \dot B^4_{2,1}(\mathbb{R}^3))$ is a given global solution of (1.2) with initial data (a_0, u_0, B_0). Then there exists a positive time t_0 such that there *hold*

$$
\begin{aligned}
&\|u(t)\|_{L^2}+\|B(t)\|_{L^2}\le C\,(1+t)^{-\beta(p)},\\
&\|\nabla u(t)\|_{L^2}+\|\nabla B(t)\|_{L^2}\le C\,(1+t)^{-\frac12-\beta(p)},\quad \text{for } t\ge t_0,\\
&\int_{t_0}^{\infty}(1+t)^{(1+2\,\beta(p))^-}\Big(\|(\partial_t u,\partial_t B)\|_{L^2}^2+\|(\Delta u,\Delta B)\|_{L^2}^2+\|\nabla\Pi\|_{L^2}^2\Big)\,dt\le C,\\
&\int_{t_0}^{\infty}\Big(\|u(t)\|_{L^\infty}+\|B(t)\|_{L^\infty}+\|\nabla u(t)\|_{L^\infty}+\|\nabla B(t)\|_{L^\infty}\Big)\,dt\le C
\end{aligned}
\tag{1.8}
$$

and

$$
\begin{aligned}
\sup_{t\ge t_0}\Big(\|\nabla^2u(t)\|_{L^2}^2+\|\nabla^2B(t)\|_{L^2}^2\Big)&+\int_{t_0}^{\infty}\|\partial_t\nabla u(t)\|_{L^2}^2+\|\partial_t\nabla B(t)\|_{L^2}^2\,dt\\
&+\int_{t_0}^{\infty}\|\nabla^3u(t)\|_{L^2}^2+\|\nabla^3B(t)\|_{L^2}^2\,dt\le C,
\end{aligned}
\tag{1.9}
$$

where $\beta(p)=\frac34(\frac2p-1)$. $(1+2\beta(p))^-$ *denotes any positive number smaller than* $1+2\beta(p)$, *and the constant* C *depends on the initial data.*

With the above theorem in hand, we can use the estimates of transport diffusion equations and various interpolation theorems in Besov space to obtain the estimate of $\|(u,B)\|_{L^1(\mathbb{R}^+;\dot B^{5/2}_{2,1})}<C$. Then after complex calculations, we can obtain the global estimates of the reference solution $(\bar a,\bar u,\bar B)$. At last, by some complexed calculations, we obtain the decay properties of the perturbed solution $(a-\bar a,u-\bar u,B-\bar B)$. Using the decay properties of the reference solution and the perturbed solution, we finally prove the following theorem.

Theorem 1.3. *For* $p\in(1,\frac65)$, *let* $\bar a_0\in B^{7/2}_{2,1}(\mathbb{R}^3)$, $\bar u_0\in L^p(\mathbb{R}^3)\cap B^2_{2,1}(\mathbb{R}^3)$, $\bar B_0\in L^p(\mathbb{R}^3)\cap B^2_{2,1}(\mathbb{R}^3)$ *satisfy* $\operatorname{div}\bar u_0=\operatorname{div}\bar B_0=0$ *and (1.4). We assume that* $\bar a\in C([0,\infty);B^{7/2}_{2,1}(\mathbb{R}^3))$, $\bar u\in C([0,\infty);B^2_{2,1}(\mathbb{R}^3))\cap L^1_{\rm loc}(\mathbb{R}^+;\dot B^4_{2,1})$, $\bar B\in C([0,\infty);B^2_{2,1}(\mathbb{R}^3))$ $\cap\, L^1_{\rm loc}(\mathbb{R}^+;\dot B^4_{2,1})$ *is a given global solution of (1.2) with initial data* $(\bar a_0,\bar u_0,\bar B_0)$. *Then there exists a constant* c *so that if*

$$(\tilde a_0,\tilde u_0,\tilde B_0)\in B^{7/2}_{2,1}(\mathbb{R}^3)\times\Big(L^p(\mathbb{R}^3)\cap B^2_{2,1}(\mathbb{R}^3)\Big)\times\Big(L^p(\mathbb{R}^3)\cap B^2_{2,1}(\mathbb{R}^3)\Big)$$

with

$$A_0:=\|(\tilde u_0,\tilde B_0)\|_{H^1}+\|(\tilde u_0,\tilde B_0)\|_{L^p}+\|\tilde a_0\|_{B^{3/2}_{2,1}}\le c,$$

$(a_0,u_0,B_0):=(\bar a_0+\tilde a_0,\bar u_0+\tilde u_0,\bar B_0+\tilde B_0)$ *generates a unique global smooth solution* (a,u,B) *to (1.2) that satisfies*

$$
\begin{aligned}
&a\in C_b([0,\infty);B^{7/2}_{2,1}(\mathbb{R}^3)),\\
&u\in C_b([0,\infty);L^p(\mathbb{R}^3\cap B^2_{2,1}(\mathbb{R}^3)))\cap L^1(\mathbb{R}^+;\dot B^4_{2,1}(\mathbb{R}^4)),\\
&B\in C_b([0,\infty);L^p(\mathbb{R}^3\cap B^2_{2,1}(\mathbb{R}^3)))\cap L^1(\mathbb{R}^+;\dot B^4_{2,1}(\mathbb{R}^4)).
\end{aligned}
$$

Moreover, there holds

$$\|a-\bar a\|_{\tilde L^\infty(\mathbb{R}^+;B^{s+1}_{2,1})}\le CA_0^{\frac54-\frac12 s} \tag{1.10}$$

for any $s\in[\frac{1}{2},\frac{5}{2}]$ *and*

$$\begin{aligned}\|(u-\bar u,B-\bar B)\|_{\tilde L^\infty(\mathbb{R}^+;B^s_{2,1})}&+\|(u-\bar u,B-\bar B)\|_{L^\infty(\mathbb{R}^+;L^p)}\\&+\|(u-\bar u,B-\bar B)\|_{L^1(\mathbb{R}^+;\dot B^{s+2}_{2,1})}\le CA_0^{\frac43-\frac23 s}\end{aligned}\tag{1.11}$$

for any $s\in[\frac12,2]$.

Remark 1.4. The above theorems may not be obtained by regarding the term $B\cdot\nabla B$ as a source term in the velocity equation. The reason is that if we regard this term as a source term, we will encounter terms like $\bar B\cdot\nabla\tilde B$ and $\bar B\cdot\nabla\tilde u$ in (3.10). For the appearance of these terms, the Bootstrap argument will not work. So we consider the linear system (2.8) are necessary and the higher order decay estimates is also necessary to obtain the results for the MHD system.

The paper is organized as follows. In Section 2, we will give some notations, a brief introduction to the Besov space and some useful lemmas. In Section 3, as a warm up, we give the proof of Theorem 1.1. Then, in Section 4, we prove Theorem 1.2 in a series of propositions. Using Theorem 1.2, we obtain the global estimates of the reference solutions in Section 5. At last, we prove the decay properties of the perturbed solutions and Theorem 1.3 in Section 6.

2. Preliminaries

Throughout this paper we will use the following notations.

- For any tempered distribution u both $\widehat u$ and $\mathcal{F}u$ denote the Fourier transform of u.
- The norm in the mixed space-time Lebesgue space $L^p([0,T];L^r(\mathbb{R}^d))$ is denoted by $\|\cdot\|_{L^p_TL^r}$ (with the obvious generalization to $\|\cdot\|_{L^p_TX}$ for any normed space X).
- For X a Banach space and I an interval of $\mathbb{R}$, we denote by $C(I;X)$ the set of continuous functions on I with values in X, and by $C_b(I;X)$ the subset of bounded functions of $C(I;X)$.
- For any pair of operators P and Q on some Banach space X, the commutator $[P,Q]$ is given by $PQ-QP$.
- C stands for a "harmless" constant, and we sometimes use the notation $A\lesssim B$ as an equivalent of $A\le CB$. The notation $A\approx B$ means that $A\lesssim B$ and $B\lesssim A$.
- $\{c_{j,r}\}_{j\in\mathbb{Z}}$ a generic element of the sphere of $\ell^r(\mathbb{Z})$, and $(c_k)_{k\in\mathbb{Z}}$ (respectively, $(d_j)_{j\in\mathbb{Z}}$) a generic element of the sphere of $\ell^2(\mathbb{Z})$ (respectively, $\ell^1(\mathbb{Z})$).
- Denote γ^- be any number smaller than γ.

Then, we give a short introduction to the Besov type space. Details about Besov type space can be found in [12] or [5]. There exist two radial positive functions $\chi\in\mathcal{D}(\mathbb{R}^d)$ and $\varphi\in\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ such that

- $\chi(\xi)+\sum_{q\ge0}\varphi(2^{-q}\xi)=1$; $\forall q\ge1$, $\mathrm{supp}\chi\cap\mathrm{supp}\varphi(2^{-q}\cdot)=\phi$,
- $\mathrm{supp}\varphi(2^{-j}\cdot)\cap\mathrm{supp}\varphi(2^{-k}\cdot)=\phi$, if $|j-k|\ge2$,

For every $v\in S'(\mathbb{R}^d)$ we set

$$\Delta_{-1}v=\chi(D)v,\quad\forall q\in\mathbb{N},\quad\Delta_jv=\varphi(2^{-q}D)v\quad\text{and}\quad S_j=\sum_{-1\le m\le j-1}\Delta_m.$$

The homogeneous operators are defined by

$$\dot{\Delta}_q v = \varphi(2^{-q}D)v, \quad \dot{S}_j = \sum_{m\leq j-1} \dot{\Delta}_j v, \quad \forall q \in \mathbb{Z}.$$

One can easily verifies that with our choice of φ,

$$\Delta_j \Delta_k f = 0 \quad \text{if} \quad |j-k| \geq 2 \tag{2.1}$$

$$\Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if} \quad |j-k| \geq 5. \tag{2.2}$$

As in Bony's decomposition, we split the product uv into three parts

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_j S_{j-1} u \Delta_j v,$$

$$R(u,v) = \sum_j \Delta_j u \widetilde{\Delta}_j v$$

where $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$.

Let us now define inhomogeneous Besov spaces. For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := (2^{js}\|\Delta_j u\|_{L^p})_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in S'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := (2^{js}\|\dot{\Delta}_j u\|_{L^p})_{\ell^r} < +\infty.$$

Notice that the usual Sobolev spaces H^s coincide with $B^s_{2,2}$ for every $s \in \mathbb{R}$ and that the homogeneous spaces $\dot{H}^s$ coincide with $\dot{B}^s_{2,2}$.

We shall need some mixed space-time spaces. Let $T > 0$ and $\rho \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of distribution u such that

$$\|u\|_{L^\rho_T \dot{B}^s_{p,r}} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_{\ell^r}\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T B^s_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T \dot{B}^s_{p,r}} := (2^{js}\|\dot{\Delta}_j u\|_{L^\rho_T L^p})_{\ell^r} < +\infty,$$

which appeared firstly in [9]. Through a direct application of the Minkowski inequality, the following links between these spaces is true [5]. Let $\varepsilon > 0$, then

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\varepsilon}_{p,r}, \quad \text{if } r \geq \rho,$$

$$L^\rho_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \quad \text{if } \rho \geq r.$$

Lemma 2.1. [1] *Let v be a divergence-free vector field with $\nabla v \in L^1([0,T]; \dot{B}^{3/2}_{2,1})$. For $s \in (-\frac{5}{2}, \frac{5}{2}]$, given $f_0 \in \dot{B}^s_{2,1}$, $F \in L^1([0,T]; \dot{B}^s_{2,1})$, the transport equation*

$$\begin{cases} \partial_t f + v \cdot \nabla f = F \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ f|_{t=0} = f_0, \end{cases} \tag{2.3}$$

has a unique solution $f \in C([0,T]; \dot{B}^s_{2,1})$. Moreover, there holds for all $t \in [0,T]$

$$\begin{aligned} \|f\|_{\tilde{L}^\infty_T(\dot{B}^s_{2,1})} \le \|f_0\|_{\dot{B}^s_{2,1}} + C\int_0^t \|f(\tau)\|_{\dot{B}^s_{2,1}} \|\nabla v(\tau)\|_{\dot{B}^{3/2}_{2,1}}\, d\tau \\ + C\|F\|_{L^1_t(\dot{B}^s_{2,1})}. \end{aligned} \tag{2.4}$$

If $s \in (0, \frac{5}{2}]$, there also holds

$$\begin{aligned} \|f\|_{\tilde{L}^\infty_t(B^s_{2,1})} \le \|f_0\|_{B^s_{2,1}} + C\int_0^t \|f(t')\|_{B^s_{2,1}} \|\nabla v(t')\|_{\dot{B}^{3/2}_{2,1}}\, dt' \\ + C\|F\|_{L^1_t(B^s_{2,1})}. \end{aligned} \tag{2.5}$$

Lemma 2.2. [1] *Let $s \in (-\frac{3}{2}, 2)$, $\vec{F} = (F_1, F_2, F_3) \in L^1_T(\dot{B}^s_{2,1})$, $a \in \tilde{L}^\infty_T(\dot{H}^2)$ with $\underline{a} := \inf_{(t,x)\in[0,T]\times\mathbb{R}^3}(1 + a(t,x)) > 0$, and $\Pi \in \tilde{L}^1_T(\dot{H}^{s+1/2})$, which solves*

$$\operatorname{div}\big((1+a)\nabla\Pi\big) = \operatorname{div}\vec{F}. \tag{2.6}$$

Then there holds

$$\underline{a}\|\nabla\Pi\|_{L^1_T(\dot{B}^s_{2,1})} \lesssim \|\vec{F}\|_{L^1_T(\dot{B}^s_{2,1})} + \|a\|_{\tilde{L}^\infty_T(\dot{H}^2)}\|\nabla\Pi\|_{\tilde{L}^1_T(\dot{H}^{s-1/2})}. \tag{2.7}$$

The following lemma could be obtained through a similar method used in the proof of Proposition 3.6 in [1].

Lemma 2.3. *For $s \in (-\frac{3}{2}, 1)$, $r = 1$ or 2. Let $u_0 \in \dot{B}^s_{2,r}$, $B_0 \in \dot{B}^s_{2,r}$ and $v \in L^1_T(\dot{B}^{5/2}_{2,1})$, $w \in L^1_T(\dot{B}^{5/2}_{2,1})$ be two divergence-free vector field. Letting $f \in \tilde{L}^1_T(\dot{B}^s_{2,r})$ and $a \in L^\infty_T(\dot{H}^2) \cap L^\infty_T(\dot{H}^{s+3/2})$ with $1 + a \ge \underline{c} > 0$, we assume that $u \in L^\infty_T(\dot{B}^s_{2,r}) \cap L^1_T(\dot{B}^{s+\frac{3}{2}}_{2,r})$, $B \in L^\infty_T(\dot{B}^s_{2,r}) \cap L^1_T(\dot{B}^{s+\frac{3}{2}}_{2,r})$ and $\Pi \in L^1_T(\dot{H}^1)$ solve*

$$\begin{cases} \partial_t u - w \cdot \nabla B + v \cdot \nabla u - \Delta u + \nabla\Pi = f + a(\Delta u - \nabla\Pi), \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t B + v \cdot \nabla B - w \cdot \nabla u - \Delta B = g, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (u, B)|_{t=0} = (u_0, B_0). \end{cases} \tag{2.8}$$

Then there holds

$$\begin{aligned} &\|(u,B)\|_{\tilde{L}^\infty_T(\dot{B}^s_{2,r})} + \|(u,B)\|_{\tilde{L}^1_T(\dot{B}^{s+2}_{2,r})} \\ &\lesssim \exp\left(C\int_0^T \|v(t)\|_{\dot{B}^{5/2}_{2,1}} + \|w(t)\|_{\dot{B}^{5/2}_{2,1}}\, dt\right)\Big\{\|(u_0,B_0)\|_{\dot{B}^s_{2,r}} \\ &\qquad + \|(f,g)\|_{\tilde{L}^1_T(\dot{B}^s_{2,r})} + \|a\|_{L^\infty_T(\dot{H}^{s+3/2})}\|\nabla\Pi\|_{L^1_T(L^2)} \\ &\qquad\qquad + \|a\|_{L^\infty_T(\dot{H}^2)}\|u\|_{L^1_T(\dot{B}^{s+3/2}_{2,r})}\Big\} \end{aligned} \tag{2.9}$$

Remark 2.4. It is easy to observe from the following

$$\|\Pi\,\nabla a\|_{\tilde L^1_T(\dot B^s_{2,r})} \lesssim \|\nabla a\|_{\tilde L^\infty_T \dot H^1}\|\Pi\|_{\tilde L^1_T(\dot H^{s+1/2})} \quad \text{for all } s\in(-\frac{3}{2},1),$$

that

$$\begin{aligned}
&\|(u,B)\|_{\tilde L^\infty_T(\dot B^s_{2,r})} + \|(u,B)\|_{\tilde L^1_T(\dot B^{s+2}_{2,r})}\\
&\lesssim \exp\left(C\int_0^T \|(v(t),B(t))\|_{\dot B^{5/2}_{2,1}}\,dt\right)\Big\{\|(u_0,B_0)\|_{\dot B^s_{2,r}}\\
&\quad + \|(f,g)\|_{\tilde L^1_T(\dot B^s_{2,r})} + \|a\|_{\tilde L^\infty_T(\dot H^2)}\big(\|\Pi\|_{\tilde L^1_T(\dot H^{s+1/2})} + \|u\|_{L^1_T(\dot B^{s+3/2}_{2,r})}\big)\Big\}.
\end{aligned} \tag{2.10}$$

Remark 2.5. Note that $\operatorname{div} u=0$, taking div to the first equation of (2.8), we obtain

$$\operatorname{div}\big((1+a)\nabla\Pi\big) = \operatorname{div}\big(f + a\Delta u + w\cdot\nabla B - v\cdot\nabla u\big).$$

Then, it follows from Lemma 2.2 that for $s\in(-\frac{3}{2},\frac{3}{2})$, we have

$$\begin{aligned}
\|\nabla\Pi\|_{L^1_T(\dot B^s_{2,1})} \lesssim{}& \|v\|_{L^\infty_T(\dot H^s)}\|u\|_{L^1_T(\dot B^{5/2}_{2,1})} + \|w\|_{L^\infty_T(\dot H^s)}\|B\|_{L^1_T(\dot B^{5/2}_{2,1})}\\
&+ \|a\|_{\tilde L^\infty_T(\dot B^{3/2}_{2,1})}\|u\|_{\tilde L^1_T \dot H^{s+2}} + \|a\|_{\tilde L^\infty_T(\dot H^2)}\|\nabla\Pi\|_{\tilde L^1_T(\dot H^{s-1/2})}\\
&+ \|f\|_{\tilde L^1_T(\dot B^s_{2,1})}.
\end{aligned} \tag{2.11}$$

Remark 2.6. If the parameter $s=1$, then we have the following estimation

$$\begin{aligned}
&\|u\|_{\tilde L^\infty_T(\dot B^1_{2,1})} + \|u\|_{\tilde L^1_T(\dot B^3_{2,1})} + \|B\|_{\tilde L^\infty_T(\dot B^1_{2,1})} + \|B\|_{\tilde L^1_T(\dot B^3_{2,1})}\\
&\lesssim \exp\left(C\int_0^T \|v\|_{\dot B^{5/2}_{2,1}} + \|w\|_{\dot B^{5/2}_{2,1}}\,dt\right)\Big(\|u_0\|_{\dot B^1_{2,1}} + \|B_0\|_{\dot B^1_{2,1}}\\
&\quad + \|f\|_{\tilde L^1_T(\dot B^1_{2,1})} + \|g\|_{\tilde L^1_T(\dot B^1_{2,1})} + \|a\|_{L^\infty_T(\dot B^2_{2,1})}\|\Pi\|_{L^1_T(\dot B^{3/2}_{2,1})}\\
&\quad + \|a\|_{L^\infty_T(\dot B^2_{2,1})}\|u\|_{L^1_T(\dot B^{5/2}_{2,1})}\Big).
\end{aligned} \tag{2.12}$$

Considering the methods used in [13] and Remark 4 in [13], the above estimate could be derived easily.

3. Stability of Global Solutions with Densities Close to 1

The aim of this section is to investigate the global stability of the given solution of (1.2) with the initial density of which is close to 1, namely Theorem 1.1.

Proof. To deal with the global well-posedness of (1.2) with initial data (a_0,u_0,B_0) given by the theorem, we need some global-in-time control of the reference solution $(\bar a,\bar u,\bar B)$. In what follows, we shall always denote $\bar\rho := \frac{1}{1+\bar a}$. By a standard energy estimate to (1.3), we have

$$\frac{1}{2}\|(\sqrt{\bar\rho}\bar u(t),\bar B(t))\|^2_{L^2} + \int_0^t \|(\nabla\bar u(\tau),\nabla\bar B(\tau))\|^2_{L^2}\,d\tau = \frac{1}{2}\|(\sqrt{\bar\rho_0}\bar u_0,\bar B_0)\|^2_{L^2}. \tag{3.1}$$

From (3.1), we deduce that

$$\int_0^t \|\bar u(\tau),\bar B(\tau)\|^4_{\dot B^{1/2}_{2,1}}\,d\tau \lesssim \int_0^t \|\bar u(\tau)\|^2_{L^2}\|\nabla\bar u(\tau)\|^2_{L^2}+\|\bar B(\tau)\|^2_{L^2}\|\nabla\bar B(\tau)\|^2_{L^2}\,d\tau$$
$$\lesssim \|\bar u_0\|^4_{L^2}+\|\bar B_0\|^4_{L^2},\quad \text{for } t>0.$$

Hence, for any $\epsilon>0$, there exists $T_0=T_0(\epsilon)>0$ such that

$$\|\bar u(T_0)\|_{\dot B^{1/2}_{2,1}}<\epsilon,\quad \|\bar B(T_0)\|_{\dot B^{1/2}_{2,1}}<\epsilon. \tag{3.2}$$

On the other hand, by using similar ideas in [1], we have

$$\|\bar a\|_{\tilde L^\infty_{T_0}(\dot B^{3/2}_{2,1})}\le \|\bar a_0\|_{\dot B^{3/2}_{2,1}}\exp\left\{C\int_0^{T_0}\|\nabla\bar u(\tau)\|_{\dot B^{3/2}_{2,1}}\,d\tau\right\},$$

and

$$\|\bar a\|_{\tilde L^\infty([T_0,t];\dot B^{3/2}_{2,1})}\le \|\bar a(T_0)\|_{\dot B^{3/2}_{2,1}}+C\|\bar a\|_{L^\infty([T_0,t];\dot B^{3/2}_{2,1})}\|\nabla\bar u\|_{L^1([T_0,t];\dot B^{3/2}_{2,1})}, \tag{3.3}$$

where Lemma 2.1 and Gronwall's inequality have been used. Note that for $\bar a$ small, we can rewrite the momentum equation and magnetic field equation in (1.2) as

$$\partial_t\bar u+(\bar u\cdot\nabla)\bar u-\Delta\bar u+\nabla\bar\Pi=\bar a(\Delta\bar u-\nabla\bar\Pi)+(\bar B\cdot\nabla)\bar B+\bar a(\bar B\cdot\nabla)\bar B,$$
$$\partial_t\bar B+(\bar u\cdot\nabla)\bar B-\Delta\bar B=(\bar B\cdot\nabla)\bar u.$$

For any $t\ge T_0$, we denote

$$\bar Z(t):=\|\bar a\|_{\tilde L^\infty([T_0,t];\dot B^{3/2}_{2,1})}+\|(\bar u,\bar B)\|_{\tilde L^\infty([T_0,t];\dot B^{1/2}_{2,1})}+\|\nabla\Pi\|_{L^1([T_0,t];\dot B^{1/2}_{2,1})}$$
$$+\|(\bar u,\bar B)\|_{L^1([T_0,t];\dot B^{5/2}_{2,1})}.$$

From the product law, we get

$$\|(\bar B\cdot\nabla)\bar B\|_{L^1([T_0,t];\dot B^{1/2}_{2,1})}\lesssim\|\bar B\|_{L^\infty([T_0,t];\dot B^{1/2}_{2,1})}\|\nabla\bar B\|_{L^1([T_0,t];\dot B^{3/2}_{2,1})}.$$

The terms $(\bar u\cdot\nabla)\bar u$, $\bar a(\bar B\cdot\nabla)\bar B$, $\bar a(\Delta\bar u-\nabla\bar\Pi)$, $(\bar u\cdot\nabla)\bar B$ and $(\bar B\cdot\nabla)\bar u$ can be estimated similarly. Then, following the procedure of the proof of Theorem 1.1 in [1], we have

$$\bar Z(t)\le\|\bar a(T_0)\|_{\dot B^{3/2}_{2,1}}+\|(\bar u(T_0),\bar B(T_0))\|_{\dot B^{1/2}_{2,1}}+C\,\bar Z(t)^2+C\,\bar Z(t)^3.$$

Let

$$\bar T:=\sup_{t>T_0}\left\{t\,:\,\bar Z(t)\le 2\left(\|\bar a(T_0)\|_{\dot B^{3/2}_{2,1}}+\|(\bar u(T_0),\bar B(T_0))\|_{\dot B^{1/2}_{2,1}}\right)\right\} \tag{3.4}$$

Without loss of generality, we can assume that

$$\|\bar a(T_0)\|_{\dot B^{3/2}_{2,1}}+\|\bar u(T_0)\|_{\dot B^{1/2}_{2,1}}+\|\bar B(T_0)\|_{\dot B^{1/2}_{2,1}}\le 1. \tag{3.5}$$

Then, if $\bar T<\infty$, for $T_0<t<\bar T$, we have

$$\bar Z(t)\le\left(\|\bar a(T_0)\|_{\dot B^{3/2}_{2,1}}+\|(\bar u(T_0),\bar B(T_0))\|_{\dot B^{1/2}_{2,1}}\right)\left(1+6C\bar Z(t)\right)$$

Taking ϵ in (3.2) small enough so that $\epsilon\le\frac{1}{108C}$, then if c_1 in Theorem 1.1 is so small that $c_1\le\frac{1}{108C}$, we have

$$\bar Z(t)\le\frac32\left(\|\bar a(T_0)\|_{\dot B^{3/2}_{2,1}}+\|(\bar u(T_0),\bar B(T_0))\|_{\dot B^{1/2}_{2,1}}\right)\quad\text{for } T_0\le t\le\bar T. \tag{3.6}$$

This contradicts with the definition of (3.4), and therefore $\bar{T}=\infty$. Moreover, there hold

$$\begin{aligned}\|\bar{a}\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{3/2}_{2,1})} &\le 2\,\|\bar{a}_0\|_{\dot{B}^{3/2}_{2,1}}\exp\left\{C\int_0^{T_0}\|\nabla\bar{u}(\tau)\|_{\dot{B}^{3/2}_{2,1}}\,d\tau\right\}+4\epsilon,\\ \|(\bar{u},\bar{B})\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}&+\|(\bar{u},\bar{B})\|_{L^1(\mathbb{R}^+;\dot{B}^{5/2}_{2,1})}+\|\nabla\bar{\Pi}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}\le C.\end{aligned}\tag{3.7}$$

With (3.7), we can solve the global well-posedness with initial data $(a_0,u_0,B_0)=(\bar{a}_0+\tilde{a}_0,\bar{u}_0+\tilde{u}_0,\bar{B}_0+\tilde{B}_0)$ for $(\tilde{a}_0,\tilde{u}_0,\tilde{B}_0)$ sufficiently small. Let $\tilde{u}:=u-\bar{u}$, $\tilde{B}:=B-\bar{B}$, then $(a,\tilde{u},\tilde{B})$ solves

$$\begin{cases}\partial_t a+(\bar{u}+\tilde{u})\nabla a=0,\\ \partial_t\tilde{u}-\Delta\tilde{u}+\nabla\tilde{\Pi}=-(\bar{u}+\tilde{u})\nabla\tilde{u}-(\tilde{u}\cdot\nabla)\bar{u}+a\,(\Delta\tilde{u}-\nabla\tilde{\Pi})\\ \qquad\qquad\qquad\quad +(a-\bar{a})(\Delta\bar{u}-\nabla\bar{\Pi})+(a-\bar{a})(\bar{B}\cdot\nabla\bar{B})\\ \qquad\qquad\qquad\quad +(1+a)(\bar{B}+\tilde{B})\nabla\tilde{B}+(1+a)(\tilde{B}\cdot\nabla)\bar{B},\\ \partial_t\tilde{B}-\Delta\tilde{B}=-(\tilde{u}+\bar{u})\nabla\tilde{B}-(\tilde{u}\cdot\nabla)\bar{B}+(\tilde{B}+\bar{B})\nabla\tilde{u}+(\tilde{B}\cdot\nabla)\bar{u},\\ \mathrm{div}\tilde{u}=\mathrm{div}\tilde{B}=0,\\ (a,\tilde{u},\tilde{B})|_{t=0}=(\bar{a}_0+\tilde{a}_0,\tilde{u}_0,\tilde{B}_0).\end{cases}\tag{3.8}$$

Reformulate equation (3.8) as follows:

$$\begin{cases}\partial_t\tilde{u}-(\bar{B}\cdot\nabla)\tilde{B}+(\bar{u}\cdot\nabla)\tilde{u}-\Delta\tilde{u}+\nabla\tilde{\Pi}=\tilde{f}+a\,(\Delta\tilde{u}-\nabla\tilde{\Pi}),\\ \partial_t\tilde{B}+(\bar{u}\cdot\nabla)\tilde{B}-(\bar{B}\cdot\nabla)\tilde{u}-\Delta\tilde{B}=\tilde{g},\end{cases}\tag{3.9}$$

with

$$\begin{aligned}\tilde{f}&=-(\tilde{u}\cdot\nabla)\tilde{u}-(\tilde{u}\cdot\nabla)\bar{u}+(a-\bar{a})(\Delta\bar{u}-\nabla\bar{\Pi})+(a-\bar{a})(\bar{B}\cdot\nabla\bar{B})\\ &\quad+(\tilde{B}\cdot\nabla)\tilde{B}+(\tilde{B}\cdot\nabla)\bar{B}+a\,\tilde{B}\cdot\nabla\tilde{B}+a\,\bar{B}\cdot\nabla\tilde{B}+a\,\tilde{B}\cdot\nabla\bar{B}\\ \tilde{g}&=-(\tilde{u}\cdot\nabla)\tilde{B}-(\tilde{u}\cdot\nabla)\bar{B}+(\bar{B}\cdot\nabla)\tilde{u}+(\tilde{B}\cdot\nabla)\bar{u}.\end{aligned}$$

Let

$$\tilde{Z}(t):=\|a\|_{\tilde{L}^\infty_t(\dot{B}^{3/2}_{2,1})}+\|(\tilde{u},\tilde{B})\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})}+\|(\tilde{u},\tilde{B})\|_{L^1_t(\dot{B}^{5/2}_{2,1})}+\|\nabla\tilde{\Pi}\|_{L^1_t(\dot{B}^{1/2}_{2,1})}.$$

Similar to the methods used to obtain (4.11) in [1], we can obtain

$$\tilde{Z}(t)\le\,C\,\left(\|(\tilde{u}_0,\tilde{B}_0)\|_{\dot{B}^{1/2}_{2,1}}+\|\tilde{a}_0\|_{\dot{B}^{3/2}_{2,1}}+\|\bar{a}\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})}+\tilde{Z}(t)^2+\tilde{Z}(t)^3\right).\tag{3.10}$$

Using similar arguments employed in proving (3.6), we can show that if $\|(\tilde{u}_0,\tilde{B}_0)\|_{\dot{B}^{1/2}_{2,1}}+\|\tilde{a}_0\|_{\dot{B}^{3/2}_{2,1}}+2c_1+4\epsilon\le\frac{1}{12\,C^2}$, there holds

$$\tilde{Z}(t)\le\,2\,C\,\left(\|(\tilde{u}_0,\tilde{B}_0)\|_{\dot{B}^{1/2}_{2,1}}+\|\tilde{a}_0\|_{\dot{B}^{3/2}_{2,1}}+\|\bar{a}\|_{\tilde{L}^\infty_t(\dot{B}^{3/2}_{2,1})}\right)\quad\text{for all } t>0.\tag{3.11}$$

With (3.11), we can prove that the propagation of regularity for smoother initial data. From Lemma 2.1, inequality (3.7) and inequality (3.11), we know that $a\in\tilde{L}^\infty(\mathbb{R}^+;B^{5/2}_{2,1}(\mathbb{R}^3))$. Applying standard energy estimate to the second and the third equation of (1.2), we obtain

$$\begin{aligned}
&\|(u,B)\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,1})}+\|(u,B)\|_{L^1_t(\dot{B}^{s+2}_{2,1})}+\|\nabla\Pi\|_{L^1_t(\dot{B}^s_{2,1})}\\
\leq&\|(u_0,B_0)\|_{\dot{B}^s_{2,1}}+\|a\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})}\left(\|u\|_{L^1_t(\dot{B}^{s+2}_{2,1})}+\|\nabla\Pi\|_{L^1_t(\dot{B}^s_{2,1})}\right)\\
&+C\left(1+\|a\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})}\right)\int_0^t\|(\nabla u(\tau),\nabla B(\tau))\|_{L^\infty}\|(u(\tau),B(\tau))\|_{\dot{B}^s_{2,1}}\,d\tau\\
&+\int_0^t\|(\nabla u,\nabla B)\|_{\dot{B}^{3/2}_{2,1}}\|B\|_{\dot{B}^s_{2,1}}\,d\tau,
\end{aligned}$$

for $s\in[0,\frac{3}{2}]$ and $t>0$. Then following the proof on page 850 in [1], we can obtain our desired results. □

4. Decay in Time Estimates for the Reference Solutions

In this section, we will show the decay estimates, namely Theorem 1.2. The main ingredient of the proof will be Abidi, Gui and Zhang's approach in [1]. The difference is that we need to provide the decay estimates for higher order derivatives of momentum and magnetic fields, which is required for the global in time estimates proved in the next section.

In what follows, we shall always denote $\rho(t,x):=\frac{1}{1+a(t,x)}$, so that we can use both (1.2) and (1.3) just according to our convenience. In order to make our presentation clearly, we divided the proof of Theorem 1.2 into the following propositions:

Proposition 4.1. Under the same assumptions of Theorem 1.2, there exists $t_0>0$ and two positive constants e_1 and e_2 such that there holds

$$\begin{aligned}
\frac{d}{dt}\|(\nabla u(t),\nabla B(t))\|^2_{L^2}&+e_1\|(\partial_t u(t),\partial_t B(t))\|^2_{L^2}\\
&+e_2\|(\nabla^2 u(t),\nabla^2 B(t))\|^2_{L^2}\leq 0\quad\text{for all } t\geq t_0,
\end{aligned}\tag{4.1}$$

or consequently

$$\begin{aligned}
&\sup_{t\geq t_0}\|(\nabla u(t),\nabla B(t))\|^2_{L^2}+\int_{t_0}^\infty e_1\|(\partial_t u(t),\partial_t B(t))\|^2_{L^2}\,d\tau\\
&\qquad+\int_{t_0}^\infty e_2\|(\nabla^2 u(t),\nabla^2 B(t))\|^2_{L^2}\,d\tau\leq\|(\nabla u(t_0),\nabla B(t_0))\|^2_{L^2}.
\end{aligned}\tag{4.2}$$

Proof. Similar to the proof of Proposition 5.1 in [1], for some positive constant c, we can obtain

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla u(t),\nabla B(t))\|^2_{L^2}+c\,\|(\nabla^2 u(t),\nabla^2 B(t))\|^2_{L^2}\\
\leq&\,C\,\|(u(t),B(t))\|^{1/2}_{L^2}\|(\nabla u(t),\nabla B(t))\|^{1/2}_{L^2}\|(\nabla^2 u(t),\nabla^2 B(t))\|^2_{L^2}\\
&+C\,\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2},
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
\frac{d}{dt}\|(\nabla u(t),\nabla B(t))\|^2_{L^2}&+\|(\sqrt{\rho}\partial_t u(t),\partial_t B(t))\|^2_{L^2}\\
&\leq C\,\|(u,B)\|_{L^2}\|(\nabla u,\nabla B)\|_{L^2}\|(\nabla^2 u,\nabla^2 B)\|^2_{L^2}.
\end{aligned}\tag{4.4}$$

The above inequality (4.4) along with (4.3) ensures a positive constant e_1 such that

$$\frac{d}{dt}\|(\nabla u, \nabla B)\|_{L^2}^2 + e_1\|(\partial_t u, \partial_t B)\|_{L^2}^2 + \mathcal{A}\|(\nabla^2 u, \nabla^2 B)\|_{L^2}^2 \le 0. \tag{4.5}$$

where

$$\mathcal{A} := \frac{c}{2\,C} - \frac{1}{2}\|(u,B)\|_{L^2}^{1/2}\|(\nabla u, \nabla B)\|_{L^2}^{1/2} - C\,\|(u,B)\|_{L^2}\|(\nabla u, \nabla B)\|_{L^2}.$$

By (3.1), for any $\eta > 0$, there exists $t_0 = t_0(\eta) > 0$ such that

$$\|\nabla u(t_0)\|_{L^2} + \|\nabla B(t_0)\|_{L^2} \le \eta.$$

Now choosing $\eta > 0$ small enough such that

$$\eta^{1/2}\|(u_0, B_0)\|_{L^2}^{1/2}\left(1 + \|(u_0, B_0)\|_{L^2}^{1/2}\eta^{1/2}\right) \le \frac{c}{16\,C^2}, \tag{4.6}$$

we define

$$\tau^* := \sup\left\{t \ge t_0\ :\ \|(\nabla u(t), \nabla B(t))\|_{L^2} \le 2\eta\right\}. \tag{4.7}$$

Then, we find that $\tau^* = \infty$ by using the same procedure appeared in the proof of Proposition 5.1 in [1]. At this stage, the proof is completed. $\square$

Proposition 4.2. Under the assumptions of Theorem 1.2, there holds $u, B \in C([0,\infty); L^p(\mathbb{R}^3))$ where $p \in (1, \frac{6}{5})$.

Proof. Multiplying the equation of u^i in (1.3) by $|u^i|^{p-1}\mathrm{sgn}(u^i)$ for $i = 1,2,3$ and integrating the resulting equation over $\mathbb{R}^3$, we obtain that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}\rho|u^i|^p\,dx + \frac{4\,(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla|u^i|^{p/2}|^2\,dx \lesssim \int_{\mathbb{R}^3} B\cdot\nabla B^i\,|u^i|^{p-1}\mathrm{sgn}(u^i)\,dx + \|\nabla\Pi\|_{L^p}\|u^i\|_{L^p}^{p-1}.$$

Then, using Hölder's inequalities and considering $p \in (1, \frac{6}{5})$, we obtain

$$\frac{d}{dt}\|\rho^{1/p}u\|_{L^p}^p + \frac{4\,(p-1)}{p}\int_{\mathbb{R}^3}|\nabla|u|^{p/2}|^2\,dx \lesssim \left(\|B\|_{L^2\cap L^3}\|\nabla B\|_{L^2} + \|\nabla\Pi\|_{L^p}\right)\|u\|_{L^p}^{p-1}. \tag{4.8}$$

By similar calculations, we have

$$\frac{d}{dt}\|B\|_{L^p}^p + \frac{4\,(p-1)}{p}\int_{\mathbb{R}^3}|\nabla|B|^{p/2}|^2\,dx \lesssim \|u\|_{L^2\cap L^3}\|\nabla B\|_{L^2}\|B\|_{L^p}^{p-1} + \|B\|_{L^2\cap L^3}\|\nabla u\|_{L^2}\|B\|_{L^p}^{p-1}. \tag{4.9}$$

Summing up (4.8) and (4.9), we easily obtain that

$$\begin{aligned}\|(u,B)\|_{L^\infty_t(L^p)} \lesssim{}& \|(u_0,B_0)\|_{L^p} + \|\nabla\Pi\|_{L^1_t(L^p)} + \int_0^t \|B\|_{L^2\cap L^3}\|\nabla u\|_{L^2}\,d\tau\\ &+ \int_0^t \left(\|B\|_{L^2\cap L^3} + \|u\|_{L^2\cap L^3}\right)\|\nabla B\|_{L^2}\,d\tau\end{aligned} \tag{4.10}$$

On the other hand, applying the operator div to the first equation in (1.2), we obtain

$$\Delta\Pi = \mathrm{div}\left(-u\cdot\nabla u + (1+a)B\cdot\nabla B + a\left(\Delta u - \nabla\Pi\right)\right),$$

which together with the classical elliptic estimates implies

$$\begin{aligned}\|\nabla\Pi\|_{L^p} &\lesssim \|u\cdot\nabla u\|_{L^p} + \|a\,(\Delta u-\nabla\Pi)\|_{L^p} + \|(1+a)B\cdot\nabla B\|_{L^p}\\ &\lesssim \|u\|_{L^2\cap L^3}\|\nabla u\|_{L^2} + \|a\|_{L^2\cap L^3}\|\Delta u-\nabla\Pi\|_{L^2}\\ &\quad + (1+\|a\|_{L^\infty})\,\|B\|_{L^2\cap L^3}\|\nabla B\|_{L^2}.\end{aligned}\tag{4.11}$$

Considering (4.4), we have

$$\begin{aligned}\|(\partial_t u,\partial_t B)\|^2_{L^2_t(L^2)} &\lesssim \|(\nabla u_0,\nabla B_0)\|^2_{L^2}\\ &+ \|(u,B)\|_{L^\infty_t(L^2)}\|(\nabla u,\nabla B)\|_{L^\infty_t(L^2)}\|(\nabla^2 u,\nabla^2 B)\|^2_{L^2_t(L^2)} \le C,\end{aligned}\tag{4.12}$$

which together with (4.11) gives rise to

$$\begin{aligned}\|\Delta u-\nabla\Pi\|^2_{L^2_t(L^2)} &\lesssim \|\partial_t u\|^2_{L^2_t(L^2)} + \|u\|_{L^\infty_t(L^2)}\|\nabla u\|_{L^\infty_t(L^2)}\|\Delta u\|^2_{L^2_t(L^2)}\\ &\quad + \|B\|_{L^\infty_t(L^2)}\|\nabla B\|_{L^\infty_t(L^2)}\|\Delta B\|^2_{L^2_t(L^2)} \le C.\end{aligned}\tag{4.13}$$

Therefore, thanks to $H^1(\mathbb{R}^3)\hookrightarrow L^2\cap L^3(\mathbb{R}^3)$, together with (4.11), we get

$$\|\nabla\Pi\|_{L^1_t(L^p)} \le C(t).$$

Hence, we have

$$\|u\|_{L^\infty_t(L^p)} + \|B\|_{L^\infty_t(L^p)} \le C(t),$$

which together with (4.12) and the classical Aubin-Lions lemma implies the desired results. □

Proposition 4.3. Under the assumptions of Theorem 1.2, there holds

$$\begin{aligned}\|u(t)\|_{L^2} + \|B(t)\|_{L^2} &\le C\,(1+t)^{-\beta(p)},\\ \|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2} &\le C\,(1+t)^{1/2-\beta(p)}, \quad \text{for } t\ge t_0,\\ \int_{t_0}^{\infty}(1+t)^{(1+2\,\beta(p))^-}\Big(\|(\partial_t u,\partial_t B)\|^2_{L^2} &+ \|(\Delta u,\Delta B)\|^2_{L^2} + \|\nabla\Pi\|^2_{L^2}\Big)\,dt \le C,\end{aligned}$$

where C depends on $m, M, \|a_0\|_{L^2}, \|a_0\|_{L^\infty}, \|u_0\|_{L^p}$ and $\|u_0\|_{H^1}$.

Proof. Step 1 : Rough decay estimate of $\|u(t)\|_{L^2}$ and $\|B(t)\|_{L^2}$. The methods used here follow from the Step 1 of the proof of Proposition 5.2 in [1]. We also need to split the phase space $\mathbb{R}^3$ into two time-dependent regions. For the velocity u, we split the domain as $S_1(t) := \Big\{\xi : |\xi| \le \sqrt{\bar\rho/2}\,g(t)\Big\}$ and $S_1(t)^c$, where $\bar\rho$ and $g(t)$ are defined as in [1]. For the magnetic field B, we split the domain as $S_2(t) := \Big\{\xi : |\xi| \le \sqrt{1/2}\,g(t)\Big\}$ and $S_2(t)^c$. Then, we could obtain

$$\begin{aligned}&\frac{d}{dt}\big(\|\sqrt{\rho}u(t)\|^2_{L^2} + \|B(t)\|^2_{L^2}\big) + g^2(t)\big(\|\sqrt{\rho}u(t)\|^2_{L^2} + \|B(t)\|^2_{L^2}\big)\\ &\qquad \le \bar\rho g^2(t)\int_{S_1(t)}|\hat u(\xi,t)|^2\,d\xi + g^2(t)\int_{S_2(t)}|\hat B(\xi,t)|^2\,d\xi.\end{aligned}\tag{4.14}$$

Thanks to Proposition 4.1 and (3.1), we have

$$\begin{aligned}\Big(\int_{t_0}^{t}\|\mathcal{F}(a\,B\cdot\nabla B)\|_{L^\infty_\xi}\Big)^2 &\lesssim \Big(\int_{t_0}^{t}\|a\,B\cdot\nabla B\|_{L^1}\,dt'\Big)^2\\ &\lesssim \|a\|^2_{L^\infty_t(L^\infty)}\Big(\int_{t_0}^{t}\|B\|_{L^2}\|\nabla B\|_{L^2}\,dt'\Big)^2 \lesssim t-t_0,\end{aligned}$$

while it is easy to see that

$$\begin{aligned}\left(\int_{t_0}^{t}\|\widehat{B\otimes B}\|_{L^\infty_\xi}+\|\widehat{B\otimes u}\|_{L^\infty_\xi}\,dt'\right)^2 &\lesssim\left(\int_{t_0}^{t}\|B\|_{L^2}^2\,dt'+\int_{t_0}^{t}\|B\|_{L^2}\|u\|_{L^2}\,dt'\right)^2\\ &\lesssim (t-t_0)^2.\end{aligned}$$

With the above estimates, following the Step 1 of the proof of Proposition 5.2 in [1], we find that

$$\int_{S_1(t)}|\hat u(\xi,t)|^2\,d\xi\lesssim\langle t\rangle^{-2\beta(p)}+\langle t\rangle^{-\frac12}\lesssim\langle t\rangle^{-\frac12}\quad\text{for } t\geq t_0, \tag{4.15}$$

and

$$\int_{S_2(t)}|\hat B(\xi,t)|^2\,d\xi\lesssim\langle t\rangle^{-2\beta(p)}+\langle t\rangle^{-\frac12}\lesssim\langle t\rangle^{-\frac12}\quad\text{for } t\geq t_0. \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.14) results in

$$e^{\int_{t_0}^t g^2(t)\,dt'}\|(\sqrt{\rho}u,B)\|_{L^2}^2\lesssim\|(\sqrt{\rho}u(t_0),B(t_0))\|_{L^2}^2+\int_{t_0}^{t}e^{\int_{t_0}^{t'}g(\tau)^2\,d\tau}\langle t'\rangle^{-\frac32}\,dt'$$

Taking $g^2(t):=\frac{\alpha}{1+t}$ (with $\alpha>\frac12$) in the above inequality, we infer that

$$\left(\|\sqrt{\rho}u(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2\right)\langle t\rangle^{\alpha}\lesssim 1+\langle t\rangle^{\alpha-\frac12},$$

which gives

$$\|u(t)\|_{L^2}+\|B(t)\|_{L^2}\lesssim\langle t\rangle^{-\frac14}. \tag{4.17}$$

Step 2 : Rough decay estimate of $\|\nabla u(t)\|_{L^2}$ and $\|\nabla B(t)\|_{L^2}$. We split the phase space $\mathbb{R}^3$ into two time-dependent regions so that

$$\begin{aligned}\|\nabla^2 u(t)\|_{L^2}^2&=\int_{S(t)}|\xi|^4|\hat u(\xi,t)|^2\,d\xi+\int_{S(t)^c}|\xi|^2|\widehat{\nabla u}(\xi,t)|^2\,d\xi,\\ \|\nabla^2 B(t)\|_{L^2}^2&=\int_{S(t)}|\xi|^4|\hat B(\xi,t)|^2\,d\xi+\int_{S(t)^c}|\xi|^2|\widehat{\nabla B}(\xi,t)|^2\,d\xi,\end{aligned}$$

where $S(t):=\left\{\xi:|\xi|\leq\sqrt{\frac{1}{e_2}}\,g(t)\right\}$ and $g(t)\lesssim\langle t\rangle^{-\frac12}$. From Proposition 4.1, we know that

$$\begin{aligned}&\frac{d}{dt}\|(\nabla u,\nabla B)\|_{L^2}^2+e_1\|(\partial_t u,\partial_t B)\|_{L^2}^2+g^2(t)\|(\nabla u,\nabla B)\|_{L^2}^2\\ &\qquad\leq g^4(t)\int_{S(t)}|\hat u(\xi,t)|^2+|\hat B(\xi,t)|^2\,d\xi\lesssim g^4(t)\langle t\rangle^{-\frac12}\lesssim\langle t\rangle^{-\frac52},\end{aligned} \tag{4.18}$$

Then, following the same procedure of the proof of Proposition 5.2 in [1], we obtain

$$\|\nabla u(t)\|_{L^2}+\|\nabla B(t)\|_{L^2}\lesssim\langle t\rangle^{-\frac34}\quad\text{for } t\geq t_0, \tag{4.19}$$

and

$$\begin{aligned}&\left(\int_{t_0}^{t}\|(\partial_t u,\partial_t B)\|_{L^2}\right)^2\\ &\qquad\lesssim\int_{t_0}^{t}\langle t\rangle^{\left(\frac32\right)^-}\|(\partial_t u,\partial_t B)\|_{L^2}^2\,dt'\int_{t_0}^{t}\langle t\rangle^{-\left(\frac32\right)^-}\,dt'\lesssim 1.\end{aligned} \tag{4.20}$$

Step 3 : Improved decay estimates of $\|u(t)\|_{L^2}$, $\|\nabla u(t)\|_{L^2}$ and $\|B(t)\|_{L^2}, \|\nabla B(t)\|_{L^2}$. Thanks to (4.17), we obtain

$$\left(\int_{t_0}^{t} \|\mathcal{F}(u\otimes B)(t')\|_{L^\infty_\xi}\, dt'\right)^2 \lesssim \langle t\rangle. \tag{4.21}$$

Using (4.17) and (4.19), we obtain

$$\begin{aligned}
&\left(\int_{t_0}^{t} \|\mathcal{F}(a\, B\cdot\nabla B)(t')\|_{L^\infty_\xi}\, dt'\right)^2\\
&\lesssim \left(\int_{t_0}^{t} \|a\, B\cdot\nabla B\|_{L^1}\, dt'\right)^2 \lesssim \left(\int_{t_0}^{t} \|B\|_{L^2}\|\nabla B\|_{L^2}\, dt'\right)^2\\
&\lesssim \left(\int_{t_0}^{t} \langle t\rangle^{-\frac14}\langle t\rangle^{-\frac34}\right)^2 \lesssim (\ln\langle t\rangle)^2
\end{aligned} \tag{4.22}$$

With these new estimates and the estimates in the Step 3 of the proof of Proposition 5.2 in [1], for a small enough constant $\epsilon>0$, we can obtain

$$\int_{S(t)} |\hat u(\xi,t)|^2\, d\xi \lesssim \langle t\rangle^{-2\beta(p)} + \langle t\rangle^{-\frac32+\epsilon} \lesssim \langle t\rangle^{-2\beta(p)}, \tag{4.23}$$

$$\int_{S(t)} |\hat B(\xi,t)|^2\, d\xi \lesssim \langle t\rangle^{-2\beta(p)} + \langle t\rangle^{-\frac32} \lesssim \langle t\rangle^{-2\beta(p)}, \tag{4.24}$$

and

$$\|u(t)\|_{L^2} + \|B(t)\|_{L^2} \lesssim \langle t\rangle^{-\beta(p)}.$$

From (4.23), (4.24) and (4.18), we infer

$$\begin{aligned}
&e^{\int_{t_0}^{t} g(t')^2\, dt'}\|(\nabla u,\nabla B)\|_{L^2}^2 + e_1\int_{t_0}^{t} e^{\int_{t_0}^{t'} g(\tau)^2\, d\tau}\|(\partial_t u,\partial_t B)\|_{L^2}^2\, dt'\\
&\qquad\qquad \lesssim \|(\nabla u(t_0),\nabla B(t_0))\|_{L^2}^2 + \int_{t_0}^{t} e^{\int_{t_0}^{t'} g(\tau)\, d\tau}\langle t\rangle^{-2-2\beta(p)}\, dt'
\end{aligned}$$

Taking $g^2(t) = \frac{\alpha}{1+t}$ (with $\alpha>1$) in the above inequality, we obtain

$$\begin{aligned}
&\|(\nabla u,\nabla B)\|_{L^2}^2\langle t\rangle^{\alpha} + e_1\int_{t_0}^{t}\langle t'\rangle^{\alpha}\|(\partial_t u,\partial_t B)\|_{L^2}^2\, dt'\\
&\qquad\qquad \lesssim 1 + \int_{t_0}^{t}\langle t'\rangle^{\alpha-2-2\beta(p)}\, dt'.
\end{aligned} \tag{4.25}$$

In particular, taking $\alpha > 1+2\beta(p)$ in (4.25), we get

$$\|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2} \lesssim \langle t\rangle^{-\frac12-\beta(p)}. \tag{4.26}$$

Taking $\alpha\in(\frac32, 1+2\beta(p))$ in (4.25) results in

$$\int_{t_0}^{t}\langle t'\rangle^{(1+2\beta(p))^-}\left(\|\partial_t u(t')\|_{L^2}^2 + \|\partial_t B(t')\|_{L^2}^2\right) dt' \le C.$$

Considering the following facts

$$\begin{aligned}
\|u\cdot\nabla u\|_{L^2}^2 &\le C\,\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \le C\,\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^3,\\
\|B\cdot\nabla B\|_{L^2}^2 &\le C\,\|B\|_{L^\infty}^2\|\nabla B\|_{L^2}^2 \le C\,\|\Delta B\|_{L^2}\|\nabla B\|_{L^2}^3,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{t_0}^{\infty} \langle t'\rangle^{(1+2\beta(p))^-} \left(\|(\Delta u(t'), \Delta B(t'), \nabla\Pi(t'))\|_{L^2}^2\right) dt' \\
&\lesssim \int_{t_0}^{\infty} \langle t\rangle^{(1+2\beta(p))^-} \|(\partial_t u, \partial_t B)\|_{L^2}^2 \, dt' \\
&\qquad + \int_{t_0}^{\infty} \langle t\rangle^{-\frac{1}{2}(1+2\beta(p))^-} \|(\Delta u, \Delta B)\|_{L^2} \, dt' \leq C,
\end{aligned} \tag{4.27}$$

where we used (4.26) and (4.2). □

Remark 4.4. Thanks to (4.27), it is easy to observe that

$$\int_{t_0}^{\infty} \langle t'\rangle^{\alpha} \left(\|\Delta u(t')\|_{L^2}^2 + \|\Delta B(t')\|_{L^2}^2 + \|\nabla\Pi(t')\|_{L^2}^2\right) dt' \lesssim \langle t\rangle^{\alpha-(1+2\beta(p))}, \tag{4.28}$$

for $\alpha \in (1+2\beta(p), 2+6\beta(p))$. Moreover, without loss of generality, we may assume that

$$\|\Delta u(t_0)\|_{L^2} + \|\Delta B(t_0)\|_{L^2} \leq 1. \tag{4.29}$$

Proposition 4.5. Under the assumptions of Theorem 1.2, there holds

$$\int_{t_0}^{\infty} \left(\|u(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} + \|\nabla B(t)\|_{L^\infty}\right) dt \leq C \tag{4.30}$$

and

$$\int_{t_0}^{\infty} \left(\|\Delta u(t)\|_{L^2}^{\theta} + \|\Delta B(t)\|_{L^2}^{\theta} + \|\nabla\Pi(t)\|_{L^2}^{\theta}\right) dt \leq C, \tag{4.31}$$

for $\frac{2}{3} \leq \theta \leq 2$.

Proof. Step 1 : Estimate of $\|u_t(t)\|_{L^2}$, $\|u(t)\|_{\dot{H}^2}$ and $\|B_t(t)\|_{L^2}$, $\|B(t)\|_{\dot{H}^2}$. Notice that

$$\left|\int_{\mathbb{R}^2} B_t \cdot \nabla B \cdot u_t \, dx\right| \leq \frac{1}{8}\|\nabla u_t\|_{L^2}^2 + \frac{1}{8}\|\nabla B_t\|_{L^2}^2 + C\,\|B_t\|_{L^2}^2 \|\nabla B\|_{L^2}^4,$$

$$\left|\int_{\mathbb{R}^3} B_t \cdot (B_t \cdot \nabla u) \, dx\right| \leq \frac{1}{8}\|\nabla B_t\|_{L^2}^2 + C\,\|B_t\|_{L^2}^2 \|\nabla u\|_{L^2}^4,$$

and

$$\int_{t_0}^{t} \|\nabla B\|_{L^2}^4 \, dt' \leq \sup_{t'\in[t_0,t]} \|\nabla B(t')\|_{L^2}^2 \|\nabla B\|_{L_t^2(L^2)}^2 \leq C.$$

Using similar ideas of the proof of Proposition 5.4 in [1], we find that

$$\sup_{t\geq t_0} \left(\|\sqrt{\rho} u_t(t)\|_{L^2}^2 + \|B_t(t)\|_{L^2}^2\right) + \int_{t_0}^{\infty} \left(\|\nabla u_t\|_{L^2}^2 + \|\nabla B_t\|_{L^2}^2\right) dt' \leq C. \tag{4.32}$$

Notice from the momentum equation of (1.3) that

$$\begin{aligned}
\|\nabla^2 u(t)\|_{L^2} + \|\nabla\Pi(t)\|_{L^2} &\lesssim \|\sqrt{\rho} u_t(t)\|_{L^2} + \|(u\cdot\nabla)u(t)\|_{L^2} + \|(B\cdot\nabla)B(t)\|_{L^2} \\
&\leq C\bigg(\|\sqrt{\rho} u_t(t)\|_{L^2} + \|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}^4 \\
&\qquad + \|B(t)\|_{L^2}\|\nabla B(t)\|_{L^2}^4\bigg) + \frac{1}{6}\|(\nabla^2 u(t), \nabla^2 B(t))\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}\|\nabla^2 B(t)\|_{L^2} \leq & C\left(\|B_t(t)\|_{L^2} + \|(u\cdot\nabla)B(t)\|_{L^2} + \|(B\cdot\nabla)u(t)\|_{L^2}\right)\\ \leq & C\left(\|B_t(t)\|_{L^2} + \|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}^4 + \|B(t)\|_{L^2}\|\nabla B(t)\|_{L^2}^4\right)\\ & + \frac{1}{6}\|\nabla^2 B(t)\|_{L^2} + \frac{1}{6}\|\nabla^2 B(t)\|_{L^2},\end{aligned}$$

which along with Proposition 4.1 and (4.32) implies that

$$\sup_{t\geq t_0}\left(\|\nabla^2 u(t)\|_{L^2} + \|\nabla^2 B(t)\|_{L^2} + \|\nabla\Pi(t)\|_{L^2}\right) \leq C. \tag{4.33}$$

Step 2 : Estimate of $\int_{t_0}^{\infty}\|(u(t'), B(t'))\|_{L^\infty}\,dt'$, and $\int_{t_0}^{\infty}\|(\nabla u(t'), \nabla B(t'))\|_{L^\infty}dt'$. It is noticed that

$$\begin{aligned}&\|\nabla^2 B\|_{L^2([t_0,t];L^6)}^2\\ \leq & C\Big\{\|\nabla B_t\|_{L^2([t_0,t];L^2)}^2 + \sup_{t'\in[t_0,t]}\|\nabla u(t')\|_{L^2}\sup_{t'\in[t_0,t]}\|\nabla^2 u(t')\|_{L^2}\|\nabla^2 B\|_{L^2([t_0,t];L^2)}^2\\ & + \sup_{t'\in[t_0,t]}\|u(t')\|_{L^2}^2\sup_{t'\in[t_0,t]}\|\nabla u(t')\|_{L^2}^6\|\nabla^2 B\|_{L^2([t_0,t];L^2)}^2\\ & + \sup_{t'\in[t_0,t]}\|\nabla B(t')\|_{L^2}\sup_{t'\in[t_0,t]}\|\nabla^2 B(t')\|_{L^2}\|\nabla^2 u\|_{L^2([t_0,t];L^2)}^2\\ & + \sup_{t'\in[t_0,t]}\|B(t')\|_{L^2}^2\sup_{t'\in[t_0,t]}\|\nabla B(t')\|_{L^2}^6\|\nabla^2 u\|_{L^2([t_0,t];L^2)}^2\Big\}\\ & + \frac{1}{4}\|\nabla^2 B\|_{L^2([t_0,t];L^6)}^2 + \frac{1}{4}\|\nabla^2 u\|_{L^2([t_0,t];L^6)}^2\end{aligned}$$

and

$$\begin{aligned}\int_{t_0}^{t}\|\nabla B(t')\|_{L^\infty}\,dt' \lesssim & \int_{t_0}^{t}\|\nabla B(t')\|_{L^6}^{1/2}\|\nabla^2 B(t')\|_{L^6}^{1/2}\,dt'\\ \lesssim & \int_{t_0}^{t}\|\Delta B(t')\|_{L^2}^{1/2}\|\nabla^2 B(t')\|_{L^6}^{1/2}\,dt'\\ \lesssim & \int_{t_0}^{t}\|\nabla^2 B(t')\|_{L^6}^2\,dt' + \int_{t_0}^{t}\|\Delta B(t')\|_{L^2}^{2/3}\,dt'.\end{aligned}$$

Then, following the proof of Proposition 5.4 in [1], we obtain the desired results. □

Proposition 4.6. Under the assumptions of Theorem 1.2, there holds

$$\|\nabla a(t)\|_{L^q} \leq C\,\|\nabla a_0\|_{L^q}, \tag{4.34}$$

and

$$\|a(t)\|_{\dot{H}^2} \leq C\,\|a_0\|_{\dot{H}^2}. \tag{4.35}$$

Proof. According to the classical transport equation theory, the results in Proposition 4.6 can be obtained easily. So we omit the details for the sake of simplicity. □

Proposition 4.7. Under the assumptions of Theorem 1.2, there holds

$$\sup_{t\geq t_0}\left(\|\nabla^2 u(t)\|_{L^2}^2+\|\nabla^2 B(t)\|_{L^2}^2\right)+c_1\int_{t_0}^{\infty}\|\partial_t\nabla u(t)\|_{L^2}^2+\|\partial_t\nabla B(t)\|_{L^2}^2\,dt$$
$$+c_2\int_{t_0}^{\infty}\|\nabla^3 u(t)\|_{L^2}^2+\|\nabla^3 B(t)\|_{L^2}^2\,dt\leq C.$$

where C depend on m, M, $\|a_0\|_{L^2}$, $\|a_0\|_{L^\infty}$, $\|(u_0,B_0)\|_{L^p}$, $\|(u_0,B_0)\|_{H^2}$.

Proof. Taking spatial derivative to the momentum equation in (1.3), we have

$$\rho\partial_t\partial_j u^i+\partial_j\rho\cdot\partial_t u^i+\partial_j(\rho u)\cdot\nabla u^i$$
$$+\rho u\cdot\nabla\partial_j u^i-\Delta\partial_j u^i+\partial_j\partial_i\Pi=\partial_j(B\cdot\nabla B^i),$$

with $i=1,2,3$. Standard energy estimates yields

$$\begin{aligned}&\|\sqrt{\rho}\partial_t\nabla u(t)\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla^2 u(t)\|_{L^2}^2\\&=-\int_{\mathbb{R}^3}\nabla\rho\,\partial_t u\,\partial_t\nabla u+\nabla(\rho u)\nabla u\,\partial_t\nabla u\,dx\\&\quad-\int_{\mathbb{R}^3}(\rho u\cdot\nabla)\nabla u\,\partial_t\nabla u-\nabla(B\cdot\nabla B)\,\partial_t\nabla u\,dx.\end{aligned}\tag{4.36}$$

Similar argument gives

$$\begin{aligned}&\|\partial_t\nabla B(t)\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla^2 B(t)\|_{L^2}^2\\&=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla B)\,\partial_t\nabla B-\nabla(B\cdot\nabla u)\,\partial_t\nabla B\,dx.\end{aligned}\tag{4.37}$$

Summing (4.36), (4.37), Proposition 4.6 and (3.2) in [1], we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2 u(t)\|_{L^2}^2+\|\nabla^2 B(t)\|_{L^2}^2\right)+\|\sqrt{\rho}\partial_t\nabla u(t)\|_{L^2}^2+\|\partial_t\nabla B(t)\|_{L^2}^2\\&\lesssim\|\partial_t u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}\|\nabla^3 u\|_{L^2}+\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}\\&+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\|\nabla^2 B\|_{L^2}\|\nabla^3 B\|_{L^2}\\&+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^3 B\|_{L^2}^2+\|\nabla B\|_{L^2}^2\|\nabla^2 u\|_{L^2}\|\nabla^3 u\|_{L^2}\\&+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^3 B\|_{L^2}^2+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^3 u\|_{L^2}^2.\end{aligned}\tag{4.38}$$

On the other hand, taking partial derivative to the momentum equation in (1.3) and multiplying $\frac{1}{\rho}\Delta\nabla u$ on both sides, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^2 u(t)\|_{L^2}^2+\|\frac{1}{\sqrt{\rho}}\Delta\nabla u\|_{L^2}^2&=\int_{\mathbb{R}^3}\nabla\rho\,\partial_t u\,\frac{1}{\rho}\,\Delta\nabla u\,dx\\&+\int_{\mathbb{R}^3}\nabla(\rho u)\,\nabla u\,\frac{1}{\rho}\,\Delta\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(\nabla\Pi)\,\frac{1}{\rho}\,\Delta\nabla u\,dx\\&-\int_{\mathbb{R}^3}\nabla(B\cdot\nabla B)\,\frac{1}{\rho}\,\Delta\nabla u\,dx+\int_{\mathbb{R}^3}u\cdot\nabla(\nabla u)\,\Delta\nabla u\,dx.\end{aligned}\tag{4.39}$$

Through similar estimate, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^2 B\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2\\&=\int_{\mathbb{R}^3}\nabla(u\cdot\nabla B)\,\Delta\nabla B\,dx-\int_{\mathbb{R}^3}\nabla(B\cdot\nabla u)\,\Delta\nabla B\,dx.\end{aligned}\tag{4.40}$$

Combining (4.39) and (4.40), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\right)+\|\frac{1}{\sqrt{\rho}}\nabla^3 u\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\nabla\rho\,\partial_t u\,\frac{1}{\rho}\,\Delta\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(\rho u)\,\Delta u\,\frac{1}{\rho}\,\Delta\nabla u\,dx\\
&\quad+\int_{\mathbb{R}^3}u\cdot\nabla(\nabla u)\,\Delta\nabla u\,dx+\int_{\mathbb{R}^3}(1+a)\nabla(\nabla\Pi)\,\Delta\nabla u\,dx\\
&\quad-\int_{\mathbb{R}^3}\nabla(B\cdot\nabla B)\,\frac{1}{\rho}\,\Delta\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(u\cdot\nabla B)\,\Delta\nabla B\,dx\\
&\quad-\int_{\mathbb{R}^3}\nabla(B\cdot\nabla u)\,\Delta\nabla B\,dx.
\end{aligned}
\tag{4.41}
$$

Next, we provide the estimation of each term on the right hand side of (4.41),

$$
\int_{\mathbb{R}^3}\nabla\rho\,\partial_t u\,\frac{1}{\rho}\,\Delta\nabla u\,dx\lesssim\|1+a\|_{L^\infty}\|\nabla a\|_{L^\infty}\|\partial_t u\|_{L^2}\|\nabla^3 u\|_{L^2},
$$

$$
\begin{aligned}
\int_{\mathbb{R}^3}\nabla(\rho u)\,\nabla u\,\frac{1}{\rho}\,\Delta\nabla u\,dx\lesssim&\|1+a\|_{L^\infty}\|\nabla a\|_{L^\infty}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}\|\nabla^3 u\|_{L^2}\\
&+\left(\|\nabla^2 u\|_{L^2}+\|\nabla^3 u\|_{L^2}\right)\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^2},
\end{aligned}
$$

$$
\int_{\mathbb{R}^3}(u\cdot\nabla)\nabla u\,\Delta\nabla u\,dx\lesssim\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^3 u\|_{L^2}^2,
$$

$$
\begin{aligned}
\int_{\mathbb{R}^3}\nabla(B\cdot\nabla B)\,\frac{1}{\rho}\,\Delta\nabla u\,dx\lesssim&\|1+a\|_{L^\infty}\|B\|_{L^2}^{1/2}\|\nabla B\|_{L^2}^{1/2}\|\nabla^3 B\|_{L^2}\|\nabla^3 u\|_{L^2}\\
&+\|1+a\|_{L^\infty}\left(\|\nabla^2 B\|_{L^2}+\|\nabla^3 B\|_{L^2}\right)\|\nabla B\|_{L^2}\|\nabla^3 u\|_{L^2},
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^3}\nabla(u\cdot\nabla B)\,\Delta\nabla B\,dx\lesssim&\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^3 B\|_{L^2}^2\\
&+\left(\|\nabla^2 u\|_{L^2}+\|\nabla^3 u\|_{L^2}\right)\|\nabla B\|_{L^2}\|\nabla^3 B\|_{L^2},
\end{aligned}
$$

$$
\int_{\mathbb{R}^3}(1+a)\,\nabla(\nabla\Pi)\,\Delta\nabla u\,dx\lesssim\|1+a\|_{L^\infty}\|\nabla^2\Pi\|_{L^2}\|\nabla^3 u\|_{L^2}.
$$

Moreover, using $\operatorname{div} u=0$, we obtain

$$
\begin{aligned}
&\|\Delta\nabla u(t)\|_{L^2}+\|\nabla(\nabla\Pi(t))\|_{L^2}\le\sqrt{2}\|\Delta\nabla u-\nabla(\nabla\Pi)\|_{L^2}\\
&\quad\le\sqrt{2}\|\rho\partial_t\nabla u+\nabla\rho\,\partial_t u+\nabla(\rho u)\,\nabla u+\rho u\cdot\nabla(\nabla u)-\nabla(B\cdot\nabla B)\|_{L^2}\\
&\quad\le\|\sqrt{\rho}\partial_t\nabla u\|_{L^2}+\|\partial_t u\|_{L^2}+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}\\
&\qquad+\|B\|_{L^2}^{1/2}\|\nabla B\|_{L^2}^{1/2}\|\nabla^3 B\|_{L^2}+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^3 u\|_{L^2}\\
&\qquad+\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}+\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^{\frac{1}{2}}\|\nabla^3 B\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Summing up all the above estimations in (4.41), then after a long and tedious calculations, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2 u(t)\|_{L^2}^2+\|\nabla^2 B(t)\|_{L^2}^2\right)+\|\frac{1}{\sqrt{\rho}}\nabla^3 u(t)\|_{L^2}^2+\|\nabla^3 B(t)\|_{L^2}^2\\
&\lesssim \|\partial_t u\|_{L^2}^2+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^2\\
&\quad+\|\nabla^2 B\|_{L^2}^2\|\nabla B\|_{L^2}^2+\|\nabla B\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2+\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2\\
&\quad+\|\nabla B\|_{L^2}^4\|\nabla^2 B\|_{L^2}^2+\|\sqrt{\rho}\partial_t\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\nabla^3 u\|_{L^2}^2\\
&\quad+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^3 u\|_{L^2}^2+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^3 B\|_{L^2}^2\\
&\quad+\|\nabla B\|_{L^2}^2\|\nabla^3 B\|_{L^2}^2+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^3 B\|_{L^2}^2\\
&\quad+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^2}^2.
\end{aligned}
\tag{4.42}
$$

Performing (4.38) + $\frac{1}{2}$(4.42), we have

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla^2 u,\nabla^2 B)\|_{L^2}^2+\|(\sqrt{\rho}\partial_t\nabla u,\partial_t\nabla B)\|_{L^2}^2+\|(\frac{1}{\sqrt{\rho}}\nabla^3 u,\nabla^3 B)\|_{L^2}^2\\
&\lesssim \|\partial_t u\|_{L^2}^2+\left(\|\nabla u\|_{L^2}^4+\|\nabla B\|_{L^2}^4\right)\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\right)\\
&\quad+\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^2\\
&\quad+\|\nabla^2 B\|_{L^2}^2\|\nabla B\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\|\nabla B\|_{L^2}^2+\big(\|u\|_{L^2}\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}\\
&\quad+\|B\|_{L^2}\|\nabla B\|_{L^2}+\|\nabla B\|_{L^2}^2+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\big)\big(\|\nabla^3 u\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2\big).
\end{aligned}
\tag{4.43}
$$

Note that we can take $\eta>0$ in the proof of Proposition 4.1 smaller so that

$$
\|u_0\|_{L^2}\eta+\|B_0\|_{L^2}\eta+\|u_0\|_{L^2}^{1/2}\eta^{1/2}+\eta^2\le\frac{c}{16C^2}
$$

where C is the constant on the right hand side of (4.43), c satisfies

$$
c\left(\|\nabla^3 u\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2\right)\le\|\frac{1}{\sqrt{\rho}}\nabla^3 u\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2.
$$

From the proof of Proposition 4.1, we know that $\tau^*=\infty$ as in Proposition 4.1. Hence, there exists two constants c_1 and c_2 such that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\right)+c_1\left(\|\partial_t\nabla u\|_{L^2}^2+\|\partial_t\nabla B\|_{L^2}^2\right)\\
&\qquad\qquad\qquad+c_2\left(\|\nabla^3 u\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2\right)\\
&\lesssim \|\partial_t u\|_{L^2}^2+\left(\|\nabla u\|_{L^2}^4+\|\nabla B\|_{L^2}^4\right)\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\right)\\
&\quad+\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^2\\
&\quad+\|\nabla^2 B\|_{L^2}^2\|\nabla B\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\|\nabla B\|_{L^2}^2,
\end{aligned}
$$

for $t \geq t_0$. Integrating the above inequality from t_0 to ∞, we obtain

$$\begin{aligned}
&\sup_{t\geq t_0}\left(\|\nabla^2 u(t)\|_{L^2}^2+\|\nabla^2 B(t)\|_{L^2}^2\right)+c_1\int_{t_0}^{\infty}\|\partial_t\nabla u\|_{L^2}^2+\|\partial_t\nabla B(t)\|_{L^2}^2\,dt\\
&\qquad\qquad\qquad\qquad\qquad\qquad+c_2\int_{t_0}^{\infty}\|\nabla^3 u\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2\,dt\\
&\lesssim\|\nabla^2 u(t_0)\|_{L^2}^2+\|\nabla^2 B(t_0)\|_{L^2}^2+\int_{t_0}^{\infty}\|\partial_t u\|_{L^2}^2\,dt\\
&\quad+\sup_{t\in[t_0,\infty]}\left(\|\nabla u\|_{L^2}^4+\|\nabla B\|_{L^2}^4\right)\int_{t_0}^{\infty}\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\,dt\\
&\quad+\sup_{t\in[t_0,\infty]}\|\nabla u(t)\|_{L^2}^3\int_{t_0}^{\infty}\|\nabla^2 u\|_{L^2}\,dt+\sup_{t\in[t_0,\infty]}\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}\int_{t_0}^{\infty}\|\nabla^2 u\|_{L^2}^2\,dt\\
&\quad+\sup_{t\in[t_0,\infty]}\left(\|\nabla u\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2\right)\int_{t_0}^{\infty}\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\,dt
\end{aligned}$$

At last, by Proposition 4.3, the proof is completed. □

5. Global in Time Estimates for Reference Solutions

In this section, we prove the global in time estimates for the reference solution of (1.3). The proof will be based mainly on Theorem 1.2.

Proposition 5.1. Under the assumptions of Theorem 1.3, there holds

$$\|(\bar{u},\bar{B})\|_{\tilde{L}^{\infty}(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}+\|(\bar{u},\bar{B})\|_{L^1(\mathbb{R}^+;\dot{B}^{5/2}_{2,1})}\leq C, \tag{5.1}$$

for a constant C depending on the initial data.

Proof. For the velocity equation, it can be estimated as in the proof of Proposition 6.1 in [1]. For the magnetic field equation, we use the standard estimates of heat equation [5]. Here, we omit the details for simplicity and only provide estimates of $\|\bar{a}(\bar{B}\cdot\nabla)\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}$, which are not appeared in the proof of Proposition 6.1 in [1]. Considering Proposition 4.1 to Proposition 4.7, we know that

$$\|\bar{a}\|_{L^{\infty}(\mathbb{R}^+;\dot{B}^{3/2}_{2,1})}\lesssim\|\bar{a}\|_{L^{\infty}(\mathbb{R}^+;L^2)}+\|\bar{a}\|_{L^{\infty}(\mathbb{R}^+;\dot{H}^2)}<\infty. \tag{5.2}$$

Using product rules in Besov space, we find that

$$\|\bar{B}\cdot\nabla\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}\lesssim\|\bar{B}\|_{L^{\infty}(\mathbb{R}^+;\dot{B}^{3/2}_{2,1})}\|\nabla\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}. \tag{5.3}$$

Since

$$\begin{aligned}
\|\nabla\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}\lesssim\|\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{3/2}_{2,1})}&\lesssim\int_0^{\infty}|\bar{B}(t)|^{1/3}_{\dot{B}^{1/2}_{2,2}}|\bar{B}(t)|^{2/3}_{\dot{B}^{2}_{2,2}}\,dt\\
&\lesssim\|\bar{B}\|^{1/3}_{L^{\infty}(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}\int_0^{\infty}|\bar{B}(t)|^{2/3}_{\dot{B}^{2}_{2,2}}\,dt<+\infty,
\end{aligned}$$

we easily know that

$$\|\bar{B}\cdot\nabla\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}<\infty. \tag{5.4}$$

Combining (5.2), (5.4) and the following estimate

$$\|\bar{a}(\bar{B}\cdot\nabla)\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})}\lesssim\|\bar{a}\|_{L^{\infty}(\mathbb{R}^+;\dot{B}^{3/2}_{2,1})}\|\bar{B}\cdot\nabla\bar{B}\|_{L^1(\mathbb{R}^+;\dot{B}^{1/2}_{2,1})},$$

we obtain the boundedness of $\|\bar a(\bar B\cdot\nabla)\bar B\|_{L^1(\mathbb{R}^+;\dot B^{1/2}_{2,1})}$. For other new terms $\|\bar B\cdot\nabla\bar B\|_{L^1(\mathbb{R}^+;\dot B^{1/2}_{2,1})}$, $\|\bar u\cdot\nabla\bar B\|_{L^1(\mathbb{R}^+;\dot B^{1/2}_{2,1})}$ and $\|\bar B\cdot\nabla\bar u\|_{L^1(\mathbb{R}^+;\dot B^{1/2}_{2,1})}$, boundedness can be obtained by using similar estimates. □

Proposition 5.2. Under the assumptions of Theorem 1.3, there hold

$$\|\bar a\|_{\tilde L^\infty(\mathbb{R}^+;B^{5/2}_{2,1})}\le C \tag{5.5}$$

and

$$\|\bar u\|_{L^\infty(\mathbb{R}^+;L^p)}+\|\bar u\|_{\tilde L^\infty(\mathbb{R}^+;B^2_{2,1})}+\|\bar u\|_{L^1(\mathbb{R}^+;\dot B^4_{2,1})}+\|\nabla\bar\Pi\|_{L^1(\mathbb{R}^+;B^2_{2,1})}\le C, \tag{5.6}$$

$$\|\bar B\|_{L^\infty(\mathbb{R}^+;L^p)}+\|\bar B\|_{\tilde L^\infty(\mathbb{R}^+;B^2_{2,1})}+\|\bar B\|_{L^1(\mathbb{R}^+;\dot B^4_{2,1})}\le C. \tag{5.7}$$

Proof. Thanks to Proposition 5.1, we get by applying (2.1) to the transport equation in (1.2) that

$$\|\bar a\|_{\tilde L^\infty_T(B^{5/2}_{2,1})}\le\|\bar a_0\|_{B^{5/2}_{2,1}}\exp\left\{C\int_0^T\|\bar u(\tau)\|_{\dot B^{5/2}_{2,1}}\,d\tau\right\}\le C. \tag{5.8}$$

Next, let us turn to the estimates of $\bar u$ and $\bar B$. Indeed, from (4.10), we know that

$$\begin{aligned}\|\bar u\|_{L^\infty_T(L^p)}+\|\bar B\|_{L^\infty_T(L^p)}\lesssim\|(u_0,B_0)\|_{L^p}+\int_0^T\|(u,B)\|_{L^2\cap L^3}\|(\nabla u,\nabla B)\|_{L^2}\,dt'\\ +\|\nabla\Pi\|_{L^1_T(L^p)}.\end{aligned}$$

Then by (4.11), Proposition 4.6, Theorem 1.2 and the above inequality, we deduce that

$$\|\bar u\|_{L^\infty(\mathbb{R}^+;L^p)}+\|\bar B\|_{L^\infty(\mathbb{R}^+;L^p)}\le C. \tag{5.9}$$

On the other hand, applying Proposition 2.3 to the momentum and magnetic field equation of (1.2) ensures that

$$\begin{aligned}&\|\bar u\|_{\tilde L^\infty_T(\dot B^0_{2,1})}+\|\bar u\|_{\tilde L^1_T(\dot B^2_{2,1})}+\|\bar B\|_{\tilde L^\infty_T(\dot B^0_{2,1})}+\|\bar B\|_{\tilde L^1_T(\dot B^2_{2,1})}\\ &\lesssim\exp\left(C\int_0^T\|\bar u\|_{\dot B^{5/2}_{2,1}}+\|\bar B\|_{\dot B^{5/2}_{2,1}}\,dt\right)\Big(\|\bar u_0\|_{\dot B^0_{2,1}}+\|\bar B_0\|_{\dot B^0_{2,1}}+\|\bar a\,\bar B\cdot\nabla\bar B\|_{L^1_T(\dot B^0_{2,1})}\\ &\quad+\|\bar a\|_{\tilde L^\infty_T(\dot H^{3/2})}\|\nabla\bar\Pi\|_{L^1_T(L^2)}+\|\bar a\|_{L^\infty_T(\dot H^2)}\|\bar u\|_{L^1_T(\dot H^{3/2})}\Big).\end{aligned}$$

We know that

$$\begin{aligned}\|\bar a\,\bar B\cdot\nabla\bar B\|_{L^1_T(\dot B^0_{2,1})}&\lesssim\int_0^T\|\bar a\|_{\dot B^1_{2,1}}\|\bar B\cdot\nabla\bar B\|_{\dot B^{1/2}_{2,1}}\,dt\\ &\lesssim\int_0^T\|\bar B\|_{\dot B^{1/2}_{2,1}}\|\nabla\bar B\|_{\dot B^{3/2}_{2,1}}\,dt\le C.\end{aligned}$$

Combining the above estimation, (5.8) and Proposition 4.5, we obtain

$$\|\bar u\|_{\tilde L^\infty_T(\dot B^0_{2,1})}+\|\bar B\|_{\tilde L^\infty_T(\dot B^0_{2,1})}\le C. \tag{5.10}$$

By a minor change of the proof in [1], we obtain the following estimation:

$$\begin{aligned}\|\nabla\bar u\|_{\tilde L^\infty_T(\dot B^{1/2}_{2,1})}+\|\nabla\bar B\|_{\tilde L^\infty_T(\dot B^{1/2}_{2,1})}+\|\nabla\bar u\|_{L^1_T(\dot B^{5/2}_{2,1})}\\ +\|\nabla\bar B\|_{L^1_T(\dot B^{5/2}_{2,1})}+\|\nabla\bar\Pi\|_{L^1_T(\dot B^{3/2}_{2,1})}\le C.\end{aligned} \tag{5.11}$$

Differentiating the momentum equation and magnetic field equation of (1.2) with respect to the spatial variables give rise to

$$
\begin{aligned}
&\partial_t\partial_i\bar{u} + \bar{u}\cdot\nabla\partial_i\bar{u} - \bar{B}\cdot\nabla\partial_i\bar{B} - (1+\bar{a})\Delta\partial_i\bar{u} + (1+\bar{a})\nabla\partial_i\Pi \\
&\qquad\qquad = -\partial_i\bar{u}\cdot\nabla\bar{u} + \partial_i\bar{a}\,\Delta\bar{u} - \partial_i\bar{a}\,\nabla\bar{\Pi} + \partial_i\bar{B}\cdot\nabla\bar{B} \\
&\qquad\qquad\quad + \partial_i\bar{a}\,\bar{B}\cdot\nabla\bar{B} + \bar{a}\,\partial_i\bar{B}\cdot\nabla\bar{B} + \bar{a}\,\bar{B}\cdot\nabla\partial_i\bar{B}, \\
&\partial_t\partial_i\bar{B} - \Delta\partial_i\bar{B} + \bar{u}\cdot\nabla\partial_i\bar{B} - \bar{B}\cdot\nabla\partial_i\bar{u} = \partial_i\bar{B}\cdot\nabla\bar{u} - \partial_i\bar{u}\cdot\nabla\bar{B}.
\end{aligned}
\tag{5.12}
$$

Using Remark 2.6 and Gronwall's inequality, we will obtain

$$
\begin{aligned}
&\|\nabla\bar{u}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{u}\|_{\tilde{L}^1_T(\dot{B}^3_{2,1})} + \|\nabla\bar{B}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{B}\|_{\tilde{L}^1_T(\dot{B}^3_{2,1})} \\
&\lesssim \exp\left\{C\int_0^T \|\bar{u}\|_{\dot{B}^{5/2}_{2,1}} + \|\bar{B}\|_{\dot{B}^{5/2}_{2,1}}\,dt\right\}\Big\{\|\nabla\bar{u}_0\|_{\dot{B}^1_{2,1}} + \|\nabla\bar{B}_0\|_{\dot{B}^1_{2,1}} \\
&\quad + \|\bar{a}\|_{\tilde{L}^\infty_T(\dot{B}^2_{2,1})}\left(\|\bar{u}\|_{L^1_T(\dot{B}^{7/2}_{2,1})} + \|\Pi\|_{L^1_T(\dot{B}^{5/2}_{2,1})} + \|\bar{B}\|_{\tilde{L}^\infty_t(\dot{B}^{3/2}_{2,1})}\|\bar{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})}\right) \\
&\quad + \|\bar{a}\|_{\tilde{L}^\infty_t(\dot{B}^2_{2,1})}\|\bar{B}\|_{\tilde{L}^\infty_t(\dot{B}^{3/2}_{2,1})}\|\bar{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})} + \|\bar{a}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})}\|\bar{B}\|_{\tilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla\bar{B}\|_{L^1_T(\dot{B}^{5/2}_{2,1})}\Big\}.
\end{aligned}
$$

Using Proposition 5.1, (5.8), (5.11) and product laws in Besov space, we get

$$
\begin{aligned}
&\|\nabla\bar{u}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{u}\|_{\tilde{L}^1_T(\dot{B}^3_{2,1})} + \|\nabla\bar{B}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{B}\|_{\tilde{L}^1_T(\dot{B}^3_{2,1})} \\
&\le C + C\left(\|\bar{u}\|_{L^1_T(\dot{B}^{7/2}_{2,1})} + \|\bar{B}\|_{L^1_T(\dot{B}^{7/2}_{2,1})} + \|\bar{\Pi}\|_{L^1_T(\dot{B}^{5/2}_{2,1})}\right) \\
&\le C + C\left(\|\Delta\bar{u}\|_{L^1_T(L^2)} + \|\Delta\bar{B}\|_{L^1_T(L^2)} + \|\nabla\bar{\Pi}\|_{L^1_T(L^2)}\right) \\
&\quad + \eta\left(\|\bar{u}\|_{L^1_T(\dot{B}^4_{2,1})} + \|\bar{B}\|_{L^1_T(\dot{B}^4_{2,1})} + \|\bar{\Pi}\|_{L^1_T(\dot{B}^3_{2,1})}\right).
\end{aligned}
\tag{5.13}
$$

Notice that $\operatorname{div}\bar{u} = 0$, we get by taking div to (5.12) that

$$
\begin{aligned}
\operatorname{div}\left((1+\bar{a})\nabla\partial_i\bar{\Pi}\right) = &-\operatorname{div}\partial_i\left[(\bar{u}\cdot\nabla)\bar{u}\right] + \operatorname{div}\partial_i\left[\bar{a}\,\Delta\bar{u}\right] - \operatorname{div}\left[\partial_i\bar{a}\,\nabla\bar{\Pi}\right] \\
&+ \operatorname{div}\partial_i\left(\bar{B}\cdot\nabla\bar{B}\right) + \operatorname{div}\left(\partial_i\bar{a}\,\bar{B}\cdot\nabla\bar{B}\right) \\
&+ \operatorname{div}\left(\bar{a}\,\partial_i\bar{B}\cdot\nabla\bar{B}\right) + \operatorname{div}\left(\bar{a}\,\bar{B}\cdot\nabla\partial_i\bar{B}\right)
\end{aligned}
$$

From this and (2.2), we deduce that

$$
\begin{aligned}
\|\nabla^2\bar{\Pi}\|_{L^1_T(\dot{B}^1_{2,1})} \lesssim\; &\|\partial_i\left(\bar{u}\cdot\nabla\bar{u}\right)\|_{L^1_T(\dot{B}^1_{2,1})} + \|\partial_i\left(\bar{a}\,\Delta\bar{u}\right)\|_{L^1_T(\dot{B}^1_{2,1})} \\
&+ \|\partial_i\bar{a}\,\nabla\bar{\Pi}\|_{L^1_T(\dot{B}^1_{2,1})} + \|\partial_i\left(\bar{B}\cdot\nabla\bar{B}\right)\|_{L^1_T\dot{B}^1_{2,1}} \\
&+ \|\partial_i\bar{a}\,\bar{B}\cdot\nabla\bar{B}\|_{L^1_T(\dot{B}^1_{2,1})} + \|\bar{a}\,\partial_i\bar{B}\cdot\nabla\bar{B}\|_{L^1_T\dot{B}^1_{2,1}} \\
&+ \|\bar{a}\,\bar{B}\cdot\nabla\partial_i\bar{B}\|_{L^1_T\dot{B}^1_{2,1}} + \|\bar{a}\|_{\tilde{L}^\infty_T(\dot{H}^2)}\|\nabla^2\bar{\Pi}\|_{\tilde{L}^1_T(\dot{H}^{1/2})}.
\end{aligned}
$$

Applying the product laws in Besov space yields that for any $\epsilon > 0$

$$\begin{aligned}
&\|\nabla^2\bar{\Pi}\|_{L^1_T(\dot{B}^1_{2,1})}\\
&\lesssim \|\nabla\bar{u}\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\nabla\bar{u}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} + \|\bar{u}\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla^2\bar{u}\|_{L^1_T(\dot{B}^1_{2,1})}\\
&\quad + \|\nabla\bar{B}\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\nabla\bar{B}\|_{L^1_T(\dot{B}^{3/2}_{2,1})} + \|\bar{B}\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla^2\bar{B}\|_{L^1_T(\dot{B}^1_{2,1})}\\
&\quad + \|\nabla\bar{a}\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\Delta\bar{u}\|_{L^1_T(\dot{B}^1_{2,1})} + \|\bar{a}\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla^3\bar{u}\|_{L^1_T(\dot{B}^1_{2,1})}\\
&\quad + \|\nabla\bar{a}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})}\|\bar{B}\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla\bar{B}\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\\
&\quad + \|\bar{a}\|_{\tilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla\bar{B}\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\nabla\bar{B}\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\\
&\quad + \|\bar{a}\|_{\tilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\bar{B}\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla^2\bar{B}\|_{L^1_T(\dot{B}^1_{2,1})}\\
&\quad + \|\bar{a}\|_{\tilde{L}^\infty_T(\dot{B}^2_{2,1})}\left(\epsilon\|\nabla\bar{\Pi}\|_{L^1_T(\dot{B}^2_{2,1})} + C\,\|\nabla\bar{\Pi}\|_{L^1_T(L^2)}\right).
\end{aligned}\tag{5.14}$$

Combining (5.11), Proposition 5.1 and taking $\epsilon > 0$ in (5.14) small enough, we will have

$$\begin{aligned}
\|\nabla\bar{\Pi}\|_{L^1_T(\dot{B}^2_{2,1})} \le C\big(1 &+ \|\nabla\bar{u}\|_{L^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{B}\|_{L^\infty_T(\dot{B}^1_{2,1})}\\
&+ \|\bar{u}\|_{L^1_T(\dot{B}^4_{2,1})} + \|\bar{B}\|_{L^1_T(\dot{B}^4_{2,1})}\big).
\end{aligned}\tag{5.15}$$

Finally, taking η in (5.13) small enough and substituting (5.15) into (5.13), we obtain

$$\|\nabla\bar{u}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{u}\|_{\tilde{L}^1_T(\dot{B}^3_{2,1})} + \|\nabla\bar{B}\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} + \|\nabla\bar{B}\|_{\tilde{L}^1_T(\dot{B}^3_{2,1})} \le C.$$

This along with (5.9) and (5.10) completes the proof of the proposition. □

6. Stability of the Global Large Solutions

In this section, we will give the proof of Theorem 1.3. Denoting $\tilde{u} := u - \bar{u}$, $\tilde{B} := B - \bar{B}$ and $\tilde{a} := a - \bar{a}$, we have

$$\begin{cases}
\partial_t\tilde{a} + (\bar{u}+\tilde{u})\nabla\tilde{a} = -\tilde{u}\cdot\nabla\bar{a},\\
\partial_t\tilde{u} + \tilde{u}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla\bar{u} + \bar{u}\cdot\nabla\tilde{u} - (1+\bar{a}+\tilde{a})(\Delta\tilde{u} - \nabla\tilde{\Pi})\\
\qquad\qquad -(1+\bar{a}+\tilde{a})(\tilde{B}\cdot\nabla\tilde{B}) - (1+\bar{a}+\tilde{a})(\tilde{B}\cdot\nabla\bar{B} + \bar{B}\cdot\nabla\tilde{B})\\
\qquad\qquad\qquad\qquad\qquad\qquad = \tilde{a}(\Delta\bar{u} - \nabla\bar{\Pi}) + \tilde{a}\,(\bar{B}\cdot\nabla\bar{B}),\\
\partial_t\tilde{B} - \Delta\tilde{B} + \tilde{u}\cdot\nabla\tilde{B} - \tilde{B}\cdot\nabla\tilde{u} = \tilde{B}\cdot\nabla\bar{u} + \bar{B}\nabla\tilde{u} - \tilde{u}\nabla\bar{B} - \bar{u}\cdot\nabla\tilde{B},\\
\operatorname{div}\tilde{u} = \operatorname{div}\tilde{B} = 0,\\
(\tilde{a},\tilde{u},\tilde{B})|_{t=0} = (\tilde{a}_0,\tilde{u}_0,\tilde{B}_0).
\end{cases}\tag{6.1}$$

Then the proof of Theorem 1.3 is equivalent to the proof of the global well posedness of (6.1) with small enough initial data $(\tilde{a}_0,\tilde{u}_0)$. Indeed, according to the coupled parabolic-hyperbolic theory [11], it is standard to prove that there exists a positive time $\tilde{T}^*$ such that (1.2) with initial data $(\bar{a}_0+\tilde{a}_0, \bar{u}_0+\tilde{u}_0, \bar{B}_0+\tilde{B}_0)$ has a unique solution (a,u,B) with

$$\begin{aligned}
&a \in C([0,\tilde{T}^*); B^{7/2}_{2,1}(\mathbb{R}^3)),\\
&u \in C([0,\tilde{T}^*); B^2_{2,1}) \cap L^1_{\mathrm{loc}}((0,\tilde{T}^*); \dot{B}^4_{2,1}(\mathbb{R}^3)),\\
&B \in C([0,\tilde{T}^*); B^2_{2,1}) \cap L^1_{\mathrm{loc}}((0,\tilde{T}^*); \dot{B}^4_{2,1}(\mathbb{R}^3)).
\end{aligned}$$

Then $(\tilde{a}, \tilde{u}, \tilde{B})$ with

$$\begin{aligned}
&\tilde{a} \in C([0, \tilde{T}^*); B^{7/2}_{2,1}(\mathbb{R}^3)),\\
&\tilde{u} \in C([0, \tilde{T}^*); B^2_{2,1}) \cap L^1_{\mathrm{loc}}((0, \tilde{T}^*); \dot{B}^4_{2,1}(\mathbb{R}^3)),\\
&\tilde{B} \in C([0, \tilde{T}^*); B^2_{2,1}) \cap L^1_{\mathrm{loc}}((0, \tilde{T}^*); \dot{B}^4_{2,1}(\mathbb{R}^3)).
\end{aligned}$$

solves (6.1) on $[0, \tilde{T}^*)$. Without loss of generality, we may assume that $\tilde{T}^*$ is the maximal time of the existence to this solution. The aim of what follows is to prove that $\tilde{T}^* = \infty$ and $(\tilde{a}, \tilde{u}, \tilde{B})$ remains small for all $t > 0$.

Lemma 6.1. *Let*

$$V(t) = 2\int_0^t \left(\|\nabla \bar{u}(\tau)\|_{L^\infty} + \|\nabla \bar{B}(\tau)\|_{L^\infty}\right) d\tau.$$

Then under the assumption of Theorem 1.3, we have

$$\begin{aligned}
&\frac{d}{dt}\left[e^{-V(t)}\left(\|(\sqrt{\rho}\tilde{u}(t)\|^2_{L^2} + \|\tilde{B}(t)\|^2_{L^2}\right)\right]\\
&\qquad\qquad + c_0 g^2(t) e^{-V(t)}\left(\|\sqrt{\rho}\tilde{u}(t)\|^2_{L^2} + \|\tilde{B}(t)\|^2_{L^2}\right)\\
&\le C e^{-V(t)}\Big\{ g^2(t)\int_{S_1(t)} e^{-2t|\xi|^2}|\hat{\bar{u}}_0(\xi)|^2\,d\xi + g^2(t)\int_{S_2(t)} e^{-2t|\xi|^2}|\hat{\bar{B}}_0(\xi)|^2\,d\xi\\
&\quad + \|\tilde{\rho}\|^2_{L^3}\|\Delta\bar{u} - \nabla\bar{\Pi}\|^2_{L^2} + \|\tilde{\rho}\|^2_{L^3}\|\bar{B}\cdot\nabla\bar{B}\|^2_{L^2}\\
&\quad + g(t)^7\left(\|\tilde{u}\|^2_{L^2_t(L^2)} + \|\tilde{B}\|^2_{L^2_t(L^2)}\right)\left(\|\bar{u}\|^2_{L^2_t(L^2)} + \|\bar{B}\|^2_{L^2_t(L^2)}\right)\\
&\quad + g(t)^7\left(\|\tilde{B}\|^2_{L^2_t(L^2)} + \|\tilde{u}\|^2_{L^2_t(L^2)}\right)^2\\
&\quad + g(t)^5\|\tilde{B}\|^2_{L^2_t(L^2)}\left(\|\nabla\bar{B}\|^2_{L^2_t(L^2)} + \|\nabla\tilde{B}\|^2_{L^2_t(L^2)}\right)\\
&\quad + g(t)^5\left(\|\Delta\tilde{u}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{\Pi}\|^2_{L^1_t(L^2)} + \|\tilde{\rho}\|^2_{L^\infty_t(L^2)}\|\Delta\bar{u} - \nabla\bar{\Pi}\|^2_{L^1_t(L^2)}\right)\\
&\quad + g(t)^5\|\tilde{\rho}\|^2_{L^\infty_t(L^2)}\|\bar{B}\cdot\nabla\bar{B}\|^2_{L^1_t(L^2)}\Big\},
\end{aligned} \tag{6.2}$$

for $t < \tilde{T}^*$, where the time dependent phase space region $S_1(t)$, $S_2(t)$ is given as in the proof of Proposition 4.3. Here and in what follows, we shall always denote

$$\rho := \frac{1}{1+\bar{a}+\tilde{a}}, \quad \bar{\rho} := \frac{1}{1+\bar{a}}, \quad \tilde{\rho} := \rho - \bar{\rho}.$$

Proof. Thanks to (6.1), $(\rho, \tilde{u}, \tilde{B})$ solves

$$\begin{cases}
\partial_t\rho + \mathrm{div}\,(\rho u) = 0,\\
\rho\partial_t\tilde{u} + \rho u\cdot\nabla\tilde{u} - B\cdot\nabla\tilde{B} + \rho\tilde{u}\cdot\nabla\bar{u} - \tilde{B}\cdot\nabla\bar{B} - \Delta\tilde{u} + \nabla\tilde{\Pi}\\
\qquad\qquad\qquad\qquad = -\frac{\tilde{\rho}}{\bar{\rho}}(\Delta\bar{u} - \nabla\bar{\Pi}) - \frac{\tilde{\rho}}{\bar{\rho}}\bar{B}\cdot\nabla\bar{B},\\
\partial_t\tilde{B} - \Delta\tilde{B} + u\cdot\nabla\tilde{B} - B\cdot\nabla\tilde{u} = \tilde{B}\cdot\nabla\bar{u} - \tilde{u}\cdot\nabla\bar{B},\\
\mathrm{div}\, u = \mathrm{div}\, B = 0.
\end{cases} \tag{6.3}$$

By a standard energy estimate, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(e^{-V(t)}\|(\sqrt{\rho}\tilde{u}(t), \tilde{B}(t))\|^2_{L^2}\right) + e^{-V(t)}\|(\nabla\tilde{u}(t), \nabla\tilde{B}(t))\|^2_{L^2}\\
&\qquad\qquad \le Ce^{-V(t)}\|\tilde{\rho}\|^2_{L^3}\|\Delta\bar{u} - \nabla\bar{\Pi}\|^2_{L^2} + Ce^{-V(t)}\|\tilde{\rho}\|^2_{L^3}\|\bar{B}\cdot\nabla\bar{B}\|^2_{L^2},
\end{aligned} \tag{6.4}$$

which along with a similar derivation of (4.14) ensures

$$\begin{aligned}
&\frac{d}{dt}\left(e^{-V(t)}\left(\|\sqrt{\rho}\tilde{u}(t)\|_{L^2}^2+\|\tilde{B}(t)\|_{L^2}^2\right)\right)\\
&\qquad\qquad +c_0g^2(t)e^{-V(t)}\left(\|\sqrt{\rho}\tilde{u}(t)\|_{L^2}^2+\|\tilde{B}(t)\|_{L^2}^2\right)\\
&\leq Ce^{-V(t)}\|\tilde{\rho}\|_{L^3}^2\left(\|\Delta\bar{u}-\nabla\bar{\Pi}\|_{L^2}^2+\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2\right)\\
&\qquad\qquad +Ce^{-V(t)}g^2(t)\left(\int_{S_1(t)}|\hat{\tilde{u}}(\xi)|^2\,d\xi+\int_{S_2(t)}|\hat{\tilde{B}}(\xi)|^2\,d\xi\right),
\end{aligned}\tag{6.5}$$

with $c_0\leq\min\left(\frac{1}{\rho},1\right)$ and the time dependent phase space region $S_1(t)$, $S_2(t)$ being the same as the one in (4.14).

Now, we need to give the estimate of $\int_{S_1(t)}|\hat{\tilde{u}}(\xi)|^2\,d\xi$ and $\int_{S_2(t)}|\hat{\tilde{B}}(\xi)|^2\,d\xi$. Since the estimates of $\int_{S_1(t)}|\hat{\tilde{u}}(\xi)|^2\,d\xi$ are similar to the estimate (7.6) in [1], we only provide the following necessary new estimates:

$$\begin{aligned}
\int_0^t\left\|\mathcal{F}\left(\frac{1}{\rho}\nabla\left(\tilde{B}\otimes\tilde{B}\right)\right)\right\|_{L_\xi^\infty}d\tau&\leq C\int_0^t\|\frac{1}{\rho}\nabla\cdot\left(\tilde{B}\otimes\tilde{B}\right)\|_{L^1}\,d\tau\\
&\leq C\left(\int_0^t\|\tilde{B}(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|\nabla\tilde{B}(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}},
\end{aligned}$$

$$\int_0^t\left\|\mathcal{F}\left(\frac{1}{\rho}\nabla\left(\tilde{B}\otimes\bar{B}\right)\right)\right\|_{L_\xi^\infty}d\tau\leq C\left(\int_0^t\|\tilde{B}(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|\nabla\bar{B}(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}},$$

$$\int_0^t\left\|\frac{\tilde{\rho}}{\rho\bar{\rho}}\,\bar{B}\,\nabla\bar{B}\right\|_{L^1}d\tau\leq C\,\|\tilde{\rho}\|_{L_t^\infty(L^2)}\int_0^t\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}\,d\tau.$$

Then, we rewrite the third equation in (6.3) as

$$\begin{aligned}
\tilde{B}(t)=e^{t\Delta}\tilde{B}_0+\int_0^t e^{(t-\tau)\Delta}\Big[\nabla\cdot\Big(&\tilde{B}\otimes\bar{u}+\bar{B}\otimes\tilde{u}-\tilde{u}\otimes\bar{B}\\
&-\bar{u}\otimes\tilde{B}+\tilde{B}\otimes\tilde{u}-\tilde{u}\otimes\tilde{B}\Big)\Big]\,d\tau.
\end{aligned}\tag{6.6}$$

Taking the Fourier transformation with respect to the x variables and integrating the resulting equation over $S_2(t)$, we obtain

$$\begin{aligned}
\int_{S_2(t)}|\hat{\tilde{B}}(\xi)|^2\,d\xi\lesssim&\int_{S_2}e^{-2t|\xi|^2}|\hat{\tilde{B}}_0|^2\,d\xi\\
&+g(t)^5\bigg(\int_0^t\|\mathcal{F}(\tilde{B}\otimes\bar{u})\|_{L_\xi^\infty}+\|\mathcal{F}(\bar{B}\otimes\tilde{u})\|_{L_\xi^\infty}\\
&+\|\mathcal{F}(\tilde{u}\otimes\bar{B})\|_{L_\xi^\infty}+\|\mathcal{F}(\bar{u}\otimes\tilde{B})\|_{L_\xi^\infty}\\
&+\|\mathcal{F}(\tilde{B}\otimes\tilde{u})\|_{L_\xi^\infty}+\|\mathcal{F}(\tilde{u}\otimes\tilde{B})\|_{L_\xi^\infty}\bigg)^2.
\end{aligned}\tag{6.7}$$

Using similar estimates employed before (6.6), we obtain

$$
\begin{aligned}
\int_{S_2(t)} |\hat{\tilde{B}}(\xi)|^2 \, d\xi \lesssim \int_{S_2(t)} & e^{-2t|\xi|^2} |\hat{\tilde{B}}_0(\xi)|^2 \, d\xi \\
& + g(t)^5 \left(\|\tilde{u}\|^2_{L^2_t(L^2)} + \|\tilde{B}\|^2_{L^2_t(L^2)} \right) \left(\|\bar{u}\|^2_{L^2_t(L^2)} + \|\bar{B}\|^2_{L^2_t(L^2)} \right) \\
& + g(t)^5 \left(\|\tilde{B}\|^2_{L^2_t(L^2)} + \|\tilde{u}\|^2_{L^2_t(L^2)} \right)^2
\end{aligned} \tag{6.8}
$$

Based on the above considerations, the detailed proof can be easily reconstructed. □

Lemma 6.2. *Let*

$$
\begin{aligned}
U(t) := C \int_0^t \Big(& \|\Delta \bar{u}(\tau)\|_{L^2} \|\nabla \bar{u}(\tau)\|_{L^2} + \|\bar{u}(\tau)\|^2_{L^\infty} \\
& + \|\Delta \bar{B}(\tau)\|_{L^2} \|\nabla \bar{B}(\tau)\|_{L^2} + \|\bar{B}(\tau)\|^2_{L^\infty} \Big) \, d\tau.
\end{aligned}
$$

If

$$
\sup_{t \in [0, \bar{T})} \left[\|\tilde{u}(t)\|_{L^2} \|\nabla \tilde{u}(t)\|_{L^2} + \|\tilde{B}(t)\|_{L^2} \|\nabla \tilde{B}(t)\|_{L^2} \right] \le \nu \tag{6.9}
$$

for some $\bar{T} \le \tilde{T}^*$ and some sufficiently small positive constant ν, then under the assumptions of Theorem 1.3, we have

$$
\begin{aligned}
& \frac{d}{dt} \left(e^{-U(t)} \|(\nabla \tilde{u}, \nabla \tilde{B})\|^2_{L^2} \right) + e^{-U(t)} \|(\sqrt{\rho} \partial_t \tilde{u}, \partial_t \tilde{B})\|^2_{L^2} \\
& \qquad\qquad + c_0 e^{-U(t)} \|(\Delta \tilde{u}, \Delta \tilde{B})\|^2_{L^2} \\
& \le C \, e^{-U(t)} \left(\|\tilde{\rho}\|^2_{L^\infty} \|\Delta \bar{u} - \nabla \bar{\Pi}\|^2_{L^2} + \|\tilde{\rho}\|^2_{L^\infty} \|\bar{B} \cdot \nabla \bar{B}\|^2_{L^2} \right) \qquad \text{for } t < \bar{T}.
\end{aligned} \tag{6.10}
$$

Proof. For the velocity equation, following the procedure of the proof of Lemma 7.2 in [1], we obtain

$$
\begin{aligned}
& \frac{d}{dt} \|\nabla \tilde{u}\|^2_{L^2} + \|\sqrt{\rho} \partial_t \tilde{u}\|^2_{L^2} + (c_0 - C \|\tilde{u}\|_{L^2} \|\nabla \tilde{u}\|_{L^2}) \|\Delta \tilde{u}\|^2_{L^2} \\
& \lesssim \|\nabla \tilde{u}\|^2_{L^2} (\|\Delta \bar{u}\|_{L^2} \|\nabla \bar{u}\|_{L^2} + \|\bar{u}\|^2_{L^\infty}) + \|\tilde{\rho}\|^2_{L^\infty} \|\Delta \bar{u} - \nabla \bar{\Pi}\|^2_{L^2} \\
& \quad + \|\nabla \tilde{B}\|^2_{L^2} (\|\Delta \bar{B}\|_{L^2} \|\nabla \bar{B}\|_{L^2} + \|\bar{B}\|^2_{L^\infty}) + \|\tilde{\rho}\|^2_{L^\infty} \|\bar{B} \cdot \nabla \bar{B}\|^2_{L^2} \\
& \quad + \|\tilde{B}\|_{L^2} \|\nabla \tilde{B}\|_{L^2} \|\Delta \tilde{B}\|^2_{L^2}
\end{aligned} \tag{6.11}
$$

for $t < \tilde{T}^*$.

Then, we consider the estimates of the magnetic field. Taking the L^2 inner product between the third equation of (6.3) and $\partial_t \tilde{B}$, we have

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{B}\|^2_{L^2} + \|\partial_t \tilde{B}\|^2_{L^2} \lesssim & \|\tilde{u} \cdot \nabla \tilde{B}\|^2_{L^2} + \|\bar{u} \cdot \nabla \tilde{B}\|^2_{L^2} \\
& + \|\bar{B} \cdot \nabla \tilde{u}\|^2_{L^2} + \|\tilde{B} \cdot \nabla \tilde{u}\|^2_{L^2} + \|\tilde{B} \cdot \nabla \bar{u}\|^2_{L^2} + \|\tilde{u} \cdot \nabla \bar{B}\|^2_{L^2}.
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&\|\tilde u\cdot\nabla\tilde B\|_{L^2}+\|\bar u\cdot\nabla\tilde B\|_{L^2}+\|\tilde u\cdot\nabla\bar B\|_{L^2}\\
&\qquad\qquad+\|\bar B\cdot\nabla\tilde u\|_{L^2}+\|\tilde B\cdot\nabla\tilde u\|_{L^2}+\|\tilde B\cdot\nabla\bar u\|_{L^2}\\
\lesssim{}&\|\tilde u\|_{L^2}^{\frac12}\|\nabla\tilde u\|_{L^2}^{\frac12}\|\Delta\tilde B\|_{L^2}+\|\nabla\tilde u\|_{L^2}\|\nabla\bar B\|_{L^2}^{\frac12}\|\Delta\bar B\|_{L^2}^{\frac12}\\
&+\|\tilde B\|_{L^2}^{\frac12}\|\nabla\tilde B\|_{L^2}^{\frac12}\|\Delta\tilde u\|_{L^2}+\|\nabla\tilde B\|_{L^2}\|\nabla\bar u\|_{L^2}^{\frac12}\|\Delta\bar u\|_{L^2}^{\frac12}\\
&+\|\nabla\tilde B\|_{L^2}\|\bar u\|_{L^\infty}+\|\nabla\tilde u\|_{L^2}\|\bar B\|_{L^\infty},
\end{aligned}
$$

which ensures

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\tilde B\|_{L^2}^2+\|\partial_t\tilde B\|_{L^2}^2\lesssim\|\tilde u\|_{L^2}\|\nabla\tilde u\|_{L^2}\|\Delta\tilde B\|_{L^2}^2\\
&\quad+\|\nabla\tilde u\|_{L^2}^2\|\nabla\bar B\|_{L^2}\|\Delta\bar B\|_{L^2}+\|\tilde B\|_{L^2}\|\nabla\tilde B\|_{L^2}\|\Delta\tilde u\|_{L^2}^2\\
&\quad+\|\nabla\tilde B\|_{L^2}^2\|\nabla\bar u\|_{L^2}\|\Delta\bar u\|_{L^2}+\|\nabla\tilde B\|_{L^2}^2\|\bar u\|_{L^\infty}^2+\|\nabla\tilde u\|_{L^2}^2\|\bar B\|_{L^\infty}^2.
\end{aligned}
\tag{6.12}
$$

By taking the L^2 inner product between (6.3) and $\Delta\tilde B$, we get

$$
\begin{aligned}
\frac12\frac{d}{dt}\|\nabla\tilde B\|_{L^2}^2+\|\Delta\tilde B\|_{L^2}^2\le\|\Delta\tilde B\|_{L^2}\Big(&\|\tilde u\cdot\nabla\tilde B\|_{L^2}+\|\bar u\cdot\nabla\tilde B\|_{L^2}\\
&+\|\tilde B\cdot\nabla\tilde u\|_{L^2}+\|\bar B\cdot\nabla\tilde u\|_{L^2}\\
&+\|\tilde B\cdot\nabla\bar u\|_{L^2}+\|\tilde u\cdot\nabla\bar B\|_{L^2}\Big).
\end{aligned}
$$

Combining this with (6.12), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\tilde B\|_{L^2}^2+\|\partial_t\tilde B\|_{L^2}^2+(1-C\|\tilde u\|_{L^2}\|\nabla\tilde u\|_{L^2})\|\Delta\tilde B\|_{L^2}^2\\
\lesssim{}&\|\nabla\tilde u\|_{L^2}^2\|\nabla\bar B\|_{L^2}\|\Delta\bar B\|_{L^2}+\|\nabla\tilde B\|_{L^2}^2\|\bar u\|_{L^\infty}^2+\|\nabla\tilde u\|_{L^2}^2\|\bar B\|_{L^\infty}^2\\
&+\|\tilde B\|_{L^2}\|\nabla\tilde B\|_{L^2}\|\Delta\tilde u\|_{L^2}^2+\|\nabla\tilde B\|_{L^2}^2\|\nabla\bar u\|_{L^2}\|\Delta\bar u\|_{L^2}.
\end{aligned}
\tag{6.13}
$$

Hence, by (6.11) and (6.13), we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla\tilde u\|_{L^2}^2+\|\nabla\tilde B\|_{L^2}^2\Big)+\|\sqrt{\rho}\partial_t\tilde u\|_{L^2}^2+\|\partial_t\tilde B\|_{L^2}^2\\
&\qquad+(c_0-C\|\tilde u\|_{L^2}\|\nabla\tilde u\|_{L^2}-C\|\tilde B\|_{L^2}\|\nabla\tilde B\|_{L^2})\|\Delta\tilde u\|_{L^2}^2\\
&\qquad+(1-C\|\tilde u\|_{L^2}\|\nabla\tilde u\|_{L^2}-C\|\tilde B\|_{L^2}\|\nabla\tilde B\|_{L^2})\|\Delta\tilde B\|_{L^2}^2\\
\lesssim{}&\Big(\|\nabla\tilde u\|_{L^2}^2+\|\nabla\tilde B\|_{L^2}^2\Big)\Big(\|\Delta\bar u\|_{L^2}\|\nabla\bar u\|_{L^2}+\|\Delta\bar B\|_{L^2}\|\nabla\bar B\|_{L^2}\\
&+\|\bar u\|_{L^\infty}^2+\|\bar B\|_{L^\infty}^2\Big)+\|\tilde\rho\|_{L^\infty}^2\|\Delta\bar u-\nabla\bar\Pi\|_{L^2}^2+\|\tilde\rho\|_{L^\infty}^2\|\bar B\cdot\nabla\bar B\|_{L^2}^2
\end{aligned}
$$

for $t<\tilde T^*$. Choosing ν small enough, then simple calculations lead to

$$
\begin{aligned}
\frac{d}{dt}\left(e^{-U(t)}\left(\|\nabla\tilde u\|_{L^2}^2+\|\nabla\tilde B\|_{L^2}^2\right)\right)+e^{-U(t)}\left(\|\sqrt{\rho}\partial_t\tilde u\|_{L^2}^2+\|\partial_t\tilde B\|_{L^2}^2\right)&\\
+e^{-U(t)}\left(c_0\|\Delta\tilde u\|_{L^2}^2+\|\Delta\tilde B\|_{L^2}^2\right)&\\
\le C\,e^{-U(t)}\|\tilde\rho\|_{L^\infty}^2\left(\|\Delta\bar u-\nabla\bar\Pi\|_{L^2}^2+\|\bar B\cdot\nabla\bar B\|_{L^2}^2\right)&
\end{aligned}
$$

for $t<\bar T$. □

Proposition 6.3. Under the assumptions in Theorem 1.3, there exist constants C and c so that if

$$\delta_0 := \|\tilde{u}_0\|_{H^1} + \|\tilde{u}_0\|_{L^p} + \|\tilde{B}_0\|_{H^1} + \|\tilde{B}_0\|_{L^p} + \|\tilde{\rho}_0\|_{L^2} + \|\tilde{\rho}_0\|_{L^\infty} < c,$$

then there hold

$$\begin{aligned} &\|\tilde{u}(t)\|_{L^2} + \|\tilde{B}(t)\|_{L^2} \le C\delta_0 \langle t\rangle^{-\beta(p)}, \\ &\|\nabla\tilde{u}\|_{L^2} + \|\nabla\tilde{B}\|_{L^2} \le C\delta_0\langle t\rangle^{-\frac{1}{2}-\beta(p)} \quad \text{for } t < \tilde{T}^*, \\ &\int_0^{\tilde{T}^*} \|\Delta\tilde{u}\|_{L^2} + \|\Delta\tilde{B}\|_{L^2} + \|\nabla\tilde{\Pi}\|_{L^2}\, dt \le C\delta_0, \\ &\int_0^{\tilde{T}^*} \|\tilde{u}\|_{L^\infty} + \|\tilde{B}\|_{L^\infty}\, dt \le C\delta_0, \end{aligned} \tag{6.14}$$

and

$$\int_0^{\tilde{T}^*} \left(\|(\partial_t\tilde{u}, \partial_t\tilde{B})\|_{L^2}^2 + \|(\Delta\tilde{u}, \Delta\tilde{B})\|_{L^2}^2 + \|\nabla\tilde{\Pi}\|_{L^2}^2\right)\langle t\rangle^{(1+2\beta(p))^-}\, d\tau \le C\delta_0 \tag{6.15}$$

for $\beta(p) = \frac{3}{4}\left(\frac{2}{p} - 1\right)$.

Proof. Let $\xi(t) := \sup_{t'\in[0,t]}\left(\|\tilde{\rho}(t)\|_{L^2} + \|\tilde{\rho}(t)\|_{L^\infty}\right)$. Integrating (6.4) and (6.10) over $[0,t]$ for $t \le \bar{T}$ that

$$\begin{aligned} e^{-V(t)}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{B}\|_{L^2}^2\right) \lesssim\ & \|\tilde{u}_0\|_{L^2}^2 + \|\tilde{B}_0\|_{L^2}^2 \\ &+ \int_0^t e^{-V(t')}\|\tilde{\rho}\|_{L^3}^2\|\Delta\bar{u} - \nabla\bar{\Pi}\|_{L^2}^2\, dt' \\ &+ \int_0^t e^{-V(t')}\|\tilde{\rho}\|_{L^3}^2\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2\, dt', \end{aligned}$$

and

$$\begin{aligned} &e^{-U(t)}\|(\nabla\tilde{u}, \nabla\tilde{B})\|_{L^2}^2 + \int_0^t e^{-U(t')}\|(\sqrt{\rho}\partial_t\tilde{u}, \partial_t\tilde{B})\|_{L^2}^2 + c_0 e^{-U(t')}\|(\Delta\tilde{u}, \Delta\tilde{B})\|_{L^2}^2\, dt' \\ &\lesssim \|(\nabla\tilde{u}_0, \nabla\tilde{B}_0)\|_{L^2}^2 + \int_0^t e^{-V(t')}\|\Delta\bar{u} - \nabla\bar{\Pi}\|_{L^2}^2\, dt' + \int_0^t e^{-V(t')}\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2\, dt', \end{aligned}$$

where $V(t)$ and $U(t)$ defined as in Lemma 6.1 and Lemma 6.2 respectively. From the decay properties of the reference solution, we have

$$\begin{aligned} &\|\tilde{u}(t)\|_{L^2} + \|\tilde{B}(t)\|_{L^2} + \|\nabla\tilde{u}\|_{L^2_t(L^2)} + \|\nabla\tilde{B}\|_{L^2_t(L^2)} \\ &\lesssim \|(\tilde{u}_0, \tilde{B}_0)\|_{L^2} + \sup_{t'\in[0,t]}\|\rho(t')\|_{L^3} \le C\left(\delta_0 + \xi(t)\right), \\ &\|\nabla\tilde{u}(t)\|_{L^2} + \|\nabla\tilde{B}(t)\|_{L^2} + \|\sqrt{\rho}\partial_t\tilde{u}\|_{L^2_t(L^2)} + \|\partial_t\tilde{B}\|_{L^2_t(L^2)} \\ &\quad + \|\Delta\tilde{u}\|_{L^2_t(L^2)} + \|\Delta\tilde{B}\|_{L^2_t(L^2)} \le C\left(\delta_0 + \xi(t)\right). \end{aligned} \tag{6.16}$$

Now, we use Schonbek's strategy in [23] to prove (6.14) and (6.15).

Step 1: Rough decay estimates of $\|\tilde{u}(t)\|_{L^2}$, $\|\nabla\tilde{u}(t)\|_{L^2}$ and $\|\tilde{B}(t)\|_{L^2}$, $\|\nabla\tilde{B}(t)\|_{L^2}$. Let $S_1(t) := \left\{\xi : |\xi| \le \sqrt{\frac{2}{c_0}}g(t)\right\}$, $S_2(t) := \left\{\xi : |\xi| \le \sqrt{2}g(t)\right\}$ with $g(t)$ satisfying

$g(t) \lesssim \langle t\rangle^{-\frac{1}{2}}$. Then, following the Step 1 of the proof of Proposition 7.3 in [1], we can prove that

$$\|\tilde{u}(t)\|_{L^2} + \|\tilde{B}(t)\|_{L^2} \leq C(\delta_0 + \xi(t))\langle t\rangle^{-\frac{1}{4}}, \tag{6.17}$$

$$\begin{aligned}
&\langle t\rangle^{\alpha}\left(\|\nabla\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{B}\|_{L^2}^2\right) + \int_0^t \langle t\rangle^{\alpha}\left(\|\partial_t\tilde{u}\|_{L^2}^2 + \|\partial_t\tilde{B}\|_{L^2}^2\right) dt' \\
&\quad \lesssim \delta_0^2 + \xi(t)^2 \int_0^t \langle t'\rangle^{\alpha}\|\Delta\bar{u} - \nabla\bar{\Pi}\|_{L^2}^2\, dt' \\
&\qquad + \xi(t)^2 \int_0^t \langle t'\rangle^{\alpha}\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2\, dt' + \Big(\delta_0 + \xi(t)\Big)^2 \int_0^t \langle t'\rangle^{\alpha-\frac{5}{2}}\, dt',
\end{aligned} \tag{6.18}$$

and

$$\|\nabla\tilde{u}(t)\|_{L^2} + \|\nabla\tilde{B}(t)\|_{L^2} \lesssim (\delta_0 + \xi(t))\,\langle t\rangle^{-\frac{3}{4}}, \tag{6.19}$$

where the magnetic field B can be estimated similar to the velocity u.

Step 2 : Improved decay estimates of $\|\tilde{u}(t)\|_{L^2}$, $\|\nabla\tilde{u}(t)\|_{L^2}$ and $\|\tilde{B}(t)\|_{L^2}$, $\|\nabla\tilde{B}(t)\|_{L^2}$. We provide an estimate as follows:

$$\begin{aligned}
&\int_0^t \|\tilde{B}\cdot\nabla\tilde{B}\|_{L^2} + \|\tilde{B}\cdot\nabla\bar{B}\|_{L^2} + \|\bar{B}\cdot\nabla\tilde{B}\|_{L^2}\, dt' \\
\lesssim &\int_0^t \|\tilde{B}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{B}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{B}\|_{L^2} + \|\tilde{B}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{B}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{u}\|_{L^2} \\
&\qquad\qquad + \|\bar{B}\|_{L^2}^{\frac{1}{2}}\|\nabla\bar{B}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{B}\|_{L^2}\, dt' \\
\lesssim &\Big(\delta_0 + \xi(t)\Big)\ln\langle t\rangle.
\end{aligned}$$

With this estimate, following the proof of the Step 2 of Proposition 7.3 in [1], we arrive at

$$\left(\int_0^t \|\Delta\tilde{u}\|_{L^2} + \|\nabla\tilde{\Pi}\|_{L^2}\, dt'\right)^2 \leq C\left(\delta_0 + \xi(t)\right)^2 \ln^2\langle t\rangle, \tag{6.20}$$

$$\int_0^t \|\tilde{u}\|_{L^2}^2 + \|\tilde{B}\|_{L^2}^2\, dt' \lesssim \Big(\delta_0 + \xi(t)\Big)^2 \langle t\rangle^{\frac{1}{2}}, \tag{6.21}$$

$$\|\nabla\tilde{u}(t)\|_{L^2} + \|\nabla\tilde{B}(t)\|_{L^2} \lesssim \Big(\delta_0 + \xi(t)\Big)\langle t\rangle^{-\frac{1}{2}-\beta(p)} \tag{6.22}$$

for $t < \bar{T}$,

$$\int_0^t \langle t'\rangle^{(1+2\beta(p))^-}\left(\|\partial_t\tilde{u}\|_{L^2}^2 + \|\partial_t\tilde{B}\|_{L^2}^2\right) dt' \lesssim (\delta_0 + \xi(t))^2 \quad \text{for } t \leq \bar{T}, \tag{6.23}$$

and

$$\|\tilde{u}(t)\|_{L^2} + \|\tilde{B}(t)\|_{L^2} \lesssim \Big(\delta_0 + \xi(t)\Big)\langle t\rangle^{-\beta(p)}. \tag{6.24}$$

Step 3 : Time integral estimates of $\|\nabla\bar{\Pi}\|_{L^2}$, $\|\Delta\tilde{u}(t)\|_{L^2}$ and $\|\Delta\tilde{B}(t)\|_{L^2}$. From the equation of $\tilde{B}$, we have

$$\begin{aligned}\|\Delta\tilde{B}\|_{L^2} \lesssim\ & \|\partial_t\tilde{B}\|_{L^2} + \|\nabla\tilde{B}\|_{L^2}\|\bar{u}\|_{L^\infty} + \|\nabla\tilde{u}\|_{L^2}\|\bar{B}\|_{L^\infty} \\ & + \|\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{B}\|_{L^2} + \|\tilde{B}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{B}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{u}\|_{L^2} \\ & + \|\nabla\tilde{u}\|_{L^2}\|\nabla\bar{B}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{B}\|_{L^2}^{\frac{1}{2}} + \|\nabla\tilde{B}\|_{L^2}\|\nabla\bar{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{u}\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

Then, we notice that the magnetic field B can be estimated similar to u. Hence, we just need to repeat the proof of the Step 3 of Proposition 7.3 in [1] to obtain

$$\int_0^t \langle t'\rangle^{(1+2\beta(p))^-}\left(\|\Delta\tilde{u}\|_{L^2}^2 + \|\Delta\tilde{B}\|_{L^2}^2 + \|\nabla\tilde{\Pi}\|_{L^2}^2\right) dt' \le C\left(\delta_0 + \xi(t)\right)^2, \tag{6.25}$$

which leads to

$$\int_0^t \|\Delta\tilde{u}\|_{L^2} + \|\Delta\tilde{B}\|_{L^2} + \|\nabla\tilde{\Pi}\|_{L^2}\, dt' \le C\left(\delta_0 + \xi(t)\right) \tag{6.26}$$

for $t \le \bar{T}$.

Step 4 : Estimate of $\int_0^t \|\tilde{u}(t')\|\, dt'$ and $\int_0^t \|\tilde{B}(t')\|_{L^\infty}\, dt'$. In this part, the estimates of $\tilde{B}$ are all similar to the estimates of $\tilde{u}$, which already illustrated in [1]. Hence, we omit all the details of this step. □

Proposition 6.4. Under the assumptions of Theorem 1.3, there exist constants C and c so that if

$$A_0 := \|\tilde{u}_0\|_{H^1} + \|\tilde{u}_0\|_{L^p} + \|\tilde{B}_0\|_{H^1} + \|\tilde{B}_0\|_{L^p} + \|\tilde{a}_0\|_{B^{3/2}_{2,1}} \le c,$$

then we have

$$\begin{aligned}&\|\tilde{a}\|_{\tilde{L}^\infty_t(B^{3/2}_{2,1})} + \|\tilde{u}\|_{\tilde{L}^\infty_t(L^p)} + \|\tilde{u}\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|\tilde{u}\|_{L^1_t(\dot{B}^{5/2}_{2,1})} \\ &\qquad + \|\tilde{B}\|_{\tilde{L}^\infty_t(L^p)} + \|\tilde{B}\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|\tilde{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})} \le CA_0\end{aligned} \tag{6.27}$$

for all $t < \tilde{T}^*$.

Proof. Firstly, we need to prove the following facts:

$$\begin{aligned}&\|\tilde{a}\|_{\tilde{L}^\infty_t(B^{3/2}_{2,1})} + \|\tilde{u}\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|\tilde{u}\|_{L^1_t(\dot{B}^{5/2}_{2,1})} \\ &\qquad + \|\tilde{B}\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|\tilde{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})} \le CA_0\end{aligned} \tag{6.28}$$

for any $t < \tilde{T}^*$. This estimate can be deduced by using similar methods of the proof of Proposition 7.4 in [1], so we only list some estimates used to estimate terms with

$\bar{B}$ and $\tilde{B}$ as follows:

$$\begin{aligned}
&\|(1+a)\tilde{B}\cdot\nabla\tilde{B}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim (1+\|a\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})})\|\tilde{B}\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}\|\tilde{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})},\\
&\|\tilde{B}\cdot\nabla\tilde{u}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim \|\tilde{B}\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}\|\tilde{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})},\\
&\|\tilde{u}\cdot\nabla\tilde{B}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim \|\tilde{u}\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}\|\tilde{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})},\\
&\|a\,\bar{B}\cdot\nabla\tilde{B}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim \|a\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})}\|\bar{B}\|_{L^\infty_t(\dot{B}^{1}_{2,1})}\|\tilde{B}\|_{L^1_t(\dot{B}^{2}_{2,1})},\\
&\|(1+a)\tilde{B}\cdot\nabla\bar{B}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim (1+\|a\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})})\|\tilde{B}\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}\|\bar{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})},\\
&\|\tilde{a}\,\bar{B}\cdot\nabla\bar{B}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim \|\tilde{a}\|_{L^1_t(\dot{B}^{3/2}_{2,1})}\|\bar{B}\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}\|\bar{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})},\\
&\|\tilde{u}\cdot\nabla\bar{B}\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \lesssim \|\tilde{u}\|_{L^\infty_t(\dot{B}^{1/2}_{2,1})}\|\bar{B}\|_{L^1_t(\dot{B}^{5/2}_{2,1})}.
\end{aligned}$$

Secondly, we need to prove that

$$\|\tilde{u}\|_{L^\infty_t(L^p)} + \|\tilde{B}\|_{L^\infty_t(L^p)} \lesssim A_0. \tag{6.29}$$

This along with (6.28) complete the proof. Inequality (6.29) can also be obtained by following the proof of Proposition 7.4 in [1], so as before we only provide some estimates of the magnetic field as follows:

$$\begin{aligned}
&\|\tilde{B}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} + \|\tilde{B}\cdot\nabla\bar{B}\|_{L^1_t(L^p)} \lesssim A_0,\\
&\|\tilde{u}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} + \|\tilde{u}\cdot\nabla\bar{B}\|_{L^1_t(L^p)} \lesssim A_0,\\
&\|\tilde{B}\cdot\nabla\tilde{u}\|_{L^1_t(L^p)} + \|\tilde{B}\cdot\nabla\bar{u}\|_{L^1_t(L^p)} \lesssim A_0,
\end{aligned}$$

$$\begin{aligned}
&\|(1+\bar{a}+\tilde{a})\tilde{B}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} + \|(1+\bar{a}+\tilde{a})\tilde{B}\cdot\nabla\bar{B}\|_{L^1_t(L^p)}\\
\lesssim\;& \left(1+\|\bar{a}\|_{L^\infty_t(L^\infty)}+\|\tilde{a}\|_{L^\infty_t(L^\infty)}\right)\left(\|\tilde{B}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} + \|\tilde{B}\cdot\nabla\bar{B}\|_{L^1_t(L^p)}\right) \lesssim A_0,
\end{aligned}$$

and

$$\begin{aligned}
&\|\bar{B}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} + \|\bar{B}\cdot\nabla\tilde{u}\|_{L^1_t(L^p)} + \|\bar{u}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} \lesssim A_0,\\
&\|(1+\bar{a}+\tilde{a})\bar{B}\cdot\nabla\tilde{B}\|_{L^1_t(L^p)} \lesssim (1+\|\bar{a}\|_{L^\infty_t(L^\infty)}\\
&\qquad\qquad + \|\tilde{a}\|_{L^\infty_t(L^\infty)})\|\bar{B}\|_{L^\infty_t(H^{1/2})}\|\nabla\tilde{B}\|_{L^1_t(L^2)} \lesssim A_0.
\end{aligned}$$

With these estimates, the complete proof can be easily reconstructed. $\square$

Proposition 6.5. Under the assumption of Theorem 1.3, there holds

$$\|\tilde{a}(t)\|_{L^\infty} \le CA_0 \tag{6.30}$$

and

$$\|\nabla\tilde{a}(t)\|_{L^\infty} \le C\|\nabla\tilde{a}_0\|_{L^\infty}, \tag{6.31}$$

where A_0 defined as before.

Proof. Firstly, let us give the equation of $\tilde{a}$

$$\partial_t\tilde{a} + u\cdot\nabla\tilde{a} = -\tilde{u}\cdot\nabla\bar{a}.$$

Using basic estimates about transport equations, we have

$$\|\tilde{a}(t)\|_{L^\infty} \le Ce^{C\int_0^t\|\nabla u\|_{L^\infty}\,dt'}\left(\|\tilde{a}_0\|_{L^\infty} + \int_0^t\|\tilde{u}\cdot\nabla\bar{a}\|_{L^\infty}\,dt'\right)$$

Moreover, we obtain

$$\|\tilde{a}(t)\|_{L^\infty} \lesssim \|\tilde{a}_0\|_{L^\infty} + \|\nabla \bar{a}\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})} A_0.$$

By the definition of A_0 and Proposition 5.2, we get

$$\|\tilde{a}(t)\|_{L^\infty} \le C A_0.$$

Similarly, we give the equation of $\nabla \tilde{a}$ as follows:

$$\partial_t \nabla \tilde{a} + (u \cdot \nabla)\nabla \tilde{a} = -(\nabla u \cdot \nabla)\tilde{a}.$$

Then, classical estimates about transport equations yield

$$\|\nabla \tilde{a}(t)\|_{L^\infty} \le C \exp\left(C \int_0^t \|\nabla u\|_{L^\infty} dt' \right) \left(\|\nabla \tilde{a}_0\|_{L^\infty} + \int_0^t \|\nabla u\|_{L^\infty} \|\nabla \tilde{a}\|_{L^\infty} dt' \right)$$

Using Proposition 4.5 and Gronwall's inequality, we obtain

$$\|\nabla \tilde{a}(t)\|_{L^\infty} \le C \|\nabla \tilde{a}_0\|_{L^\infty}.$$

□

Proposition 6.6. Under the assumptions of Theorem 1.3, there holds

$$\begin{aligned}\sup_{t \ge t_0} \left(\|\nabla^2 \tilde{u}(t)\|_{L^2}^2 + \|\nabla^2 \tilde{B}(t)\|_{L^2}^2 \right) + \int_{t_0}^\infty \|\partial_t \nabla \tilde{u}(t)\|_{L^2}^2 + \|\partial_t \nabla \tilde{B}(t)\|_{L^2}^2 \, dt \\ + \int_{t_0}^\infty \|\nabla^3 \tilde{u}(t)\|_{L^2}^2 + \|\nabla^3 \tilde{B}(t)\|_{L^2}^2 \, dt \le C,\end{aligned}$$

where C depends on the initial data.

Proof. Taking derivative to the second and third equation of (6.3), we obtain

$$\begin{aligned}\rho \partial_t \partial_j \tilde{u}^i - \Delta \partial_j \tilde{u}^i + \partial_j \partial_i \tilde{\Pi} = &-\partial_j \rho \, \partial_t \tilde{u}^i - \partial_j \rho \, u \cdot \nabla \tilde{u}^i - \rho \, \partial_j u \cdot \nabla \tilde{u}^i \\ &- \rho \, u \cdot \nabla \partial_j \tilde{u}^i + \partial_j B \cdot \nabla \tilde{B}^i + B \cdot \nabla \partial_j \tilde{B}^i - \partial_j \rho \, \tilde{u} \cdot \nabla \bar{u}^i - \rho \, \partial_j \tilde{u} \cdot \nabla \bar{u}^i \\ &- \rho \, \tilde{u} \cdot \nabla \partial_j \bar{u}^i + \partial_j \tilde{B} \cdot \nabla \bar{B}^i + \tilde{B} \cdot \nabla \partial_j \bar{B}^i - \frac{\tilde{\rho}}{\bar{\rho}} (\Delta \partial_j \bar{u}^i - \partial_j \partial_i \tilde{\Pi}) \\ &- \partial_j \left(\frac{\tilde{\rho}}{\bar{\rho}} \right) (\Delta \bar{u}^i - \partial_i \tilde{\Pi}) - \frac{\tilde{\rho}}{\bar{\rho}} \partial_j \bar{B} \cdot \nabla \bar{B}^i - \frac{\tilde{\rho}}{\bar{\rho}} \bar{B} \cdot \nabla \partial_j \bar{B}^i \\ &- \partial_j \left(\frac{\tilde{\rho}}{\bar{\rho}} \right) \bar{B} \cdot \nabla \bar{B}^i\end{aligned} \tag{6.32}$$

and

$$\begin{aligned}\partial_t \partial_j \tilde{B}^i - \Delta \partial_i \tilde{B}^i = &-\partial_j u \cdot \nabla \tilde{B}^i - u \cdot \nabla \partial_j \tilde{B}^i + \partial_j B \cdot \nabla \tilde{u}^i + B \cdot \nabla \partial_j \tilde{u}^i \\ &- \partial_j \tilde{B} \cdot \nabla \bar{u}^i - \tilde{B} \cdot \nabla \partial_j \bar{u}^i + \partial_j \tilde{u} \cdot \nabla \bar{B}^i + \tilde{u} \cdot \nabla \partial_j \bar{B}^i.\end{aligned} \tag{6.33}$$

Multiplying (6.32) with $\partial_t \partial_j \tilde{u}^i$ and integrating over $\mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\|\sqrt{\rho}\partial_t \nabla \tilde{u}(t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla^2 \tilde{u}(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla\rho\, \partial_t \tilde{u}\, \partial_t \nabla \tilde{u}\, dx \\
&\quad - \int_{\mathbb{R}^3} \nabla\rho\, \tilde{u}\cdot\nabla\tilde{u}\, \partial_t\nabla\tilde{u}\, dx - \int_{\mathbb{R}^3} \nabla\rho\, \bar{u}\cdot\nabla\tilde{u}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad - \int_{\mathbb{R}^3} \rho\, \nabla\tilde{u}\cdot\nabla\tilde{u}\, \partial_t\nabla\tilde{u}\, dx - \int_{\mathbb{R}^3} \rho\, \nabla\bar{u}\cdot\nabla\tilde{u}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad - \int \mathbb{R}^3 \rho\, u\cdot\nabla\nabla\tilde{u}\, \partial_t\nabla\tilde{u}\, dx + \int_{\mathbb{R}^3} \nabla\tilde{B}\cdot\nabla\tilde{B}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad + \int_{\mathbb{R}^3} \nabla\bar{B}\cdot\nabla\tilde{B}\, \partial_t\nabla\tilde{u}\, dx + \int_{\mathbb{R}^3} \tilde{B}\cdot\nabla\nabla\tilde{B}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad + \int_{\mathbb{R}^3} \bar{B}\cdot\nabla\nabla\tilde{B}\, \partial_t\nabla\tilde{u}\, dx - \int_{\mathbb{R}^3} \nabla\rho\, \tilde{u}\cdot\nabla\bar{u}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad - \int_{\mathbb{R}^3} \rho\, \nabla\tilde{u}\cdot\nabla\bar{u}\, \partial_t\nabla\tilde{u}\, dx - \int_{\mathbb{R}^3} \rho\, \tilde{u}\cdot\nabla\nabla\bar{u}\, dx \\
&\quad + \int_{\mathbb{R}^3} \nabla\tilde{B}\cdot\nabla\bar{B}\, \partial_t\nabla\tilde{u}\, dx + \int_{\mathbb{R}^3} \tilde{B}\cdot\nabla\nabla\bar{B}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad - \int_{\mathbb{R}^3} \frac{\tilde{\rho}}{\bar{\rho}}(\Delta\partial_j\bar{u}^i - \partial_j\partial_i\tilde{\Pi})\, \partial_t\nabla\tilde{u}\, dx - \int_{\mathbb{R}^3} \partial_j\left(\frac{\tilde{\rho}}{\bar{\rho}}\right)(\Delta\bar{u}^i - \partial_i\tilde{\Pi})\, \partial_t\nabla\tilde{u}\, dx \\
&\quad - \int_{\mathbb{R}^3} \frac{\tilde{\rho}}{\bar{\rho}}\nabla\bar{B}\cdot\nabla\bar{B}\, dx - \int_{\mathbb{R}^3} \frac{\tilde{\rho}}{\bar{\rho}}\bar{B}\cdot\nabla\nabla\bar{B}\, \partial_t\nabla\tilde{u}\, dx \\
&\quad - \int_{\mathbb{R}^3} \partial_j\left(\frac{\tilde{\rho}}{\bar{\rho}}\right)\bar{B}\cdot\nabla\bar{B}\, dx.
\end{aligned} \tag{6.34}
$$

For $\tilde{B}$, using same methods, we have

$$
\begin{aligned}
\|\partial_t\nabla\tilde{B}(t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla^2\tilde{B}(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \nabla u\cdot\nabla\tilde{B}\, \partial_t\tilde{B}\, dx \\
&\quad - \int_{\mathbb{R}^3} u\cdot\nabla\nabla\tilde{B}\, \partial_t\nabla\tilde{B}\, dx + \int_{\mathbb{R}^3} \nabla B\cdot\nabla\tilde{u}\, \partial_t\nabla\tilde{B}\, dx \\
&\quad + \int_{\mathbb{R}^3} B\cdot\nabla\nabla\tilde{u}\, \partial_t\nabla\tilde{B}\, dx - \int_{\mathbb{R}^3} \nabla\tilde{B}\cdot\nabla\bar{u}\, \partial_t\nabla\tilde{B}\, dx \\
&\quad - \int_{\mathbb{R}^3} \tilde{B}\cdot\nabla\nabla\bar{u}\, \partial_t\nabla\tilde{B}\, dx + \int_{\mathbb{R}^3} \nabla\tilde{u}\cdot\nabla\bar{B}\, \partial_t\nabla\tilde{B}\, dx \\
&\quad + \int_{\mathbb{R}^3} \tilde{u}\cdot\nabla\nabla\bar{B}\, \partial_t\nabla\tilde{B}\, dx.
\end{aligned} \tag{6.35}
$$

Combining (6.34) and (6.35), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2\tilde{u}(t)\|_{L^2}^2 + \|\nabla^2\tilde{B}(t)\|_{L^2}^2\right) + \|\sqrt{\rho}\,\partial_t\nabla\tilde{u}\|_{L^2}^2 + \|\partial_t\nabla\tilde{B}\|_{L^2}^2 \\
&\le \|\partial_t\tilde{u}\|_{L^2}^2 + (\|\tilde{u}\cdot\nabla\tilde{u}\|_{L^2} + \|\bar{u}\cdot\nabla\tilde{u}\|_{L^2} + \|\tilde{u}\cdot\nabla\bar{u}\|_{L^2})^2 \\
&\quad + \|\nabla\tilde{u}\|_{L^2}^2\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2} + \|\nabla\bar{u}\|_{L^2}^2\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2} \\
&\quad + \|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}^2 + \|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}\|\nabla^2\tilde{u}\|_{L^2}^2 \\
&\quad + \|\nabla\tilde{B}\|_{L^2}^2\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2} + \|\nabla\bar{B}\|_{L^2}^2\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}^2+\|\nabla\bar{B}\|_{L^2}\|\nabla^2\bar{B}\|_{L^2}\|\nabla^2\tilde{B}\|_{L^2}^2\\
&+\|\nabla\bar{u}\|_{L^2}^2\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\bar{u}\|_{L^2}^2\\
&+\|\nabla\bar{B}\|_{L^2}^2\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\bar{B}\|_{L^2}^2\\
&+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla(\Delta\bar{u}-\nabla\bar{\Pi})\|_{L^2}^2+\|\nabla\tilde{a}\|_{L^\infty}^2\|\Delta\bar{u}-\nabla\bar{\Pi}\|_{L^2}^2\\
&+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2+\|\tilde{\rho}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\nabla\bar{B}\|_{L^2}^2\\
&+\|\nabla\tilde{a}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}^2\\
&+\|\nabla\bar{u}\|_{L^2}^2\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}^2\\
&+\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}\|\nabla^2\tilde{B}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}^2\\
&+\|\nabla\bar{B}\|_{L^2}^2\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}^2\\
&+\|\nabla\bar{B}\|_{L^2}\|\nabla^2\bar{B}\|_{L^2}\|\nabla^2\tilde{u}\|_{L^2}^2+\|\nabla\bar{u}\|_{L^2}^2\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\\
&+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\bar{u}\|_{L^2}^2+\|\nabla\bar{B}\|_{L^2}^2\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\\
&+\|\tilde{u}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\bar{B}\|_{L^2}^2.
\end{aligned}
$$

Multiplying $\frac{1}{\rho}\Delta\tilde{u}$ and $\Delta\tilde{u}$ to (6.32) and (6.33) separately and doing some basic energy estimates, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2\tilde{u}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}^2\right)+\|\rho\|_{L^\infty}^{-1}\|\nabla^3\tilde{u}\|_{L^2}^2+\|\nabla^3\tilde{B}\|_{L^2}^2\\
\le\;&\|\partial_t\nabla\tilde{u}\|_{L^2}^2+\left(\|\tilde{u}\cdot\nabla\tilde{u}\|_{L^2}+\|\bar{u}\cdot\nabla\tilde{u}\|_{L^2}+\|\tilde{u}\cdot\nabla\bar{u}\|_{L^2}\right)^2\\
&+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}^2+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla\bar{u}\|_{L^2}^2\\
&+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}^2+\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}\|\nabla^2\tilde{u}\|_{L^2}^2\\
&+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\bar{B}\|_{L^2}^2\\
&+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}^2+\|\nabla\bar{B}\|_{L^2}\|\nabla^2\bar{B}\|_{L^2}\|\nabla^2\tilde{B}\|_{L^2}^2\\
&+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^3}\|\nabla\bar{u}\|_{L^2}^2+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\bar{u}\|_{L^2}^2\\
&+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\bar{B}\|_{L^2}^2+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\bar{B}\|_{L^2}^2\\
&+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla(\Delta\bar{u}-\nabla\bar{\Pi})\|_{L^2}^2+\|\nabla\tilde{a}\|_{L^\infty}^2\|\Delta\bar{u}-\nabla\bar{\Pi}\|_{L^2}^2\\
&+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2+\|\tilde{\rho}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\nabla\bar{B}\|_{L^2}^2\\
&+\|\nabla\tilde{a}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}^2\\
&+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\bar{u}\|_{L^2}^2+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}^2\\
&+\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}\|\nabla^2\tilde{B}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}^2\\
&+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla\bar{B}\|_{L^2}^2+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}^2\\
&+\|\nabla\bar{B}\|_{L^2}\|\nabla^2\bar{B}\|_{L^2}\|\nabla^2\tilde{B}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}\|\nabla^3\tilde{B}\|_{L^2}\|\nabla\bar{u}\|_{L^2}^2\\
&+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\|\nabla^3\bar{u}\|_{L^2}^2+\|\nabla^2\tilde{u}\|_{L^2}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla\bar{B}\|_{L^2}^2\\
&+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^3\bar{B}\|_{L^2}^2.
\end{aligned}
$$

At this point, we got two completed inequalities. The second inequality times a small number then plus the first inequality yieds

$$
\frac{d}{dt}\left(\|\nabla^2\tilde{u}(t)\|_{L^2}^2+\|\nabla^2\tilde{B}(t)\|_{L^2}^2\right)+\|\sqrt{\rho}\partial_t\nabla\tilde{u}\|_{L^2}^2+\|\partial_t\nabla\tilde{B}\|_{L^2}^2
$$

$$
\begin{aligned}
&+\left(\frac{c_0}{2}-C\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}-C\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\right)\left(\|\nabla^3\tilde{u}\|_{L^2}^2+\|\nabla^3\tilde{B}\|_{L^2}^2\right)\\
\le&\left(\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}+\|\nabla\bar{B}\|_{L^2}\|\nabla^2\bar{B}\|_{L^2}\right)\left(\|\nabla^2\tilde{u}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}^2\right)\\
&+\left(\|\nabla\tilde{u}\|_{L^2}^4+\|\nabla\tilde{B}\|_{L^2}^4+\|\nabla\bar{u}\|_{L^2}^4+\|\nabla\bar{B}\|_{L^2}^4\right)\Big(\|\nabla^2\tilde{u}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}^2\\
&+\|\nabla^2\bar{u}\|_{L^2}^2+\|\nabla^2\bar{B}\|_{L^2}^2\Big)+\|\partial_t\tilde{u}\|_{L^2}^2+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^2\tilde{u}\|_{L^2}^2\\
&+\|\nabla\tilde{u}\|_{L^2}^2\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}+\left(\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\right)\Big(\|\nabla^3\bar{u}\|_{L^2}^2\\
&+\|\nabla^3\bar{B}\|_{L^2}^2\Big)+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla(\Delta\bar{u}-\nabla\bar{\Pi})\|_{L^2}^2+\|\nabla\tilde{a}\|_{L^\infty}^2\|\Delta\bar{u}-\nabla\bar{\Pi}\|_{L^2}^2\\
&+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2+\|\tilde{\rho}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\nabla\bar{B}\|_{L^2}^2+\|\nabla\tilde{a}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2.
\end{aligned}
$$

Taking c in Theorem 1.3 small enough, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla^2\tilde{u}(t)\|_{L^2}^2+\|\nabla^2\tilde{B}(t)\|_{L^2}^2\right)+\|\sqrt{\rho}\partial_t\nabla\tilde{u}\|_{L^2}^2+\|\partial_t\nabla\tilde{B}\|_{L^2}^2\\
&\quad+\|\nabla^3\tilde{u}\|_{L^2}^2+\|\nabla^3\tilde{B}\|_{L^2}^2\\
\le&\left(\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}+\|\nabla\bar{B}\|_{L^2}\|\nabla^2\bar{B}\|_{L^2}\right)\left(\|\nabla^2\tilde{u}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}^2\right)\\
&+\left(\|\nabla\tilde{u}\|_{L^2}^4+\|\nabla\tilde{B}\|_{L^2}^4+\|\nabla\bar{u}\|_{L^2}^4+\|\nabla\bar{B}\|_{L^2}^4\right)\Big(\|\nabla^2\tilde{u}\|_{L^2}^2+\|\nabla^2\tilde{B}\|_{L^2}^2\\
&+\|\nabla^2\bar{u}\|_{L^2}^2+\|\nabla^2\bar{B}\|_{L^2}^2\Big)+\|\partial_t\tilde{u}\|_{L^2}^2+\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla^2\tilde{u}\|_{L^2}^2\\
&+\|\nabla\tilde{u}\|_{L^2}^2\|\nabla\bar{u}\|_{L^2}\|\nabla^2\bar{u}\|_{L^2}+\left(\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}+\|\tilde{B}\|_{L^2}\|\nabla\tilde{B}\|_{L^2}\right)\Big(\|\nabla^3\bar{u}\|_{L^2}^2\\
&+\|\nabla^3\bar{B}\|_{L^2}^2\Big)+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla(\Delta\bar{u}-\nabla\bar{\Pi})\|_{L^2}^2+\|\nabla\tilde{a}\|_{L^\infty}^2\|\Delta\bar{u}-\nabla\bar{\Pi}\|_{L^2}^2\\
&+\|\tilde{\rho}\|_{L^\infty}^2\|\nabla\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2+\|\tilde{\rho}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\nabla\bar{B}\|_{L^2}^2+\|\nabla\tilde{a}\|_{L^\infty}^2\|\bar{B}\cdot\nabla\bar{B}\|_{L^2}^2.
\end{aligned}
$$

Integrating the above inequality, using decay estimates about reference solution and perturbed solution, we obtain

$$
\begin{aligned}
\sup_{t\ge t_0}\left(\|\nabla^2\tilde{u}(t)\|_{L^2}^2+\|\nabla^2\tilde{B}(t)\|_{L^2}^2\right)+\int_{t_0}^\infty\|\sqrt{\rho}\,\partial_t\nabla\tilde{u}\|_{L^2}^2+\|\partial_t\nabla\tilde{B}\|_{L^2}^2\,dt&\\
+\int_{t_0}^\infty\|\nabla^3\tilde{u}\|_{L^2}^2+\|\nabla^3\tilde{B}\|_{L^2}^2\,dt\le C.&
\end{aligned}
$$

Hence, the proof of Proposition 6.6 is completed. □

Now, we can complete the proof of Theorem 1.3 as following.

Proof. According to the statement at the beginning of this section, given initial data $(\bar{a}_0+\tilde{a}_0,\bar{u}_0+\tilde{u}_0,\bar{B}_0+\tilde{B}_0)$, (1.2) has a unique solution (a,u,B) on $[0,\tilde{T}^*)$ such that

$$
\begin{aligned}
&a\in C([0,\tilde{T}^*);B_{2,1}^{7/2}(\mathbb{R}^3)),\\
&u\in C([0,\tilde{T}^*);B_{2,1}^2)\cap L_{\mathrm{loc}}^1((0,\tilde{T}^*);\dot{B}_{2,1}^4(\mathbb{R}^3)),\\
&B\in C([0,\tilde{T}^*);B_{2,1}^2)\cap L_{\mathrm{loc}}^1((0,\tilde{T}^*);\dot{B}_{2,1}^4(\mathbb{R}^3)).
\end{aligned}
$$

We need only prove the maximal existence time $\tilde{T}^*=\infty$. Indeed, according to all the decay estimates for reference solution and perturbed solution, we repeat the argument used in the proof of Proposition 5.1 and Proposition 5.2 to prove that

$\tilde{T}^* = \infty$. Then a standard interpolation argument gives (1.10) and (1.11). This completes the proof of Theorem 1.3. □

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