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Global smooth solutions in $\mathbb{R}^3$ to short wave-long wave interactions in magnetohydrodynamics

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Abstract

We consider a Benney-type system modeling short wave-long wave interactions in compressible viscous fluids under the influence of a magnetic field. Accordingly, this large system now consists of the compressible MHD equations coupled with a nonlinear Schrödinger equation along particle paths. We study the global existence of smooth solutions to the Cauchy problem in $\mathbb{R}^3$ when the initial data are small smooth perturbations of an equilibrium state. An important point here is that, instead of the simpler case having zero as the equilibrium state for the magnetic field, we consider an arbitrary non-zero equilibrium state $\bar{B}$ for the magnetic field. This is motivated by applications, e.g., Earth's magnetic field, and the lack of invariance of the MHD system with respect to either translations or rotations of the magnetic field. The usual time decay investigation through spectral analysis in this non-zero equilibrium case meets serious difficulties, for the eigenvalues in the frequency space are no longer spherically symmetric. Instead, we employ a recently developed technique of energy estimates involving evolution in negative Besov spaces, and combine it with the particular interplay here between Eulerian and Lagrangian coordinates.

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Keywords: Compressible MHD system; Nonlinear Schrödinger equations; Time decay rate

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1. Introduction and main results

We consider the following Benney-type system, modeling short wave-long wave interactions for compressible viscous magnetohydrodynamic fluids,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) = \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u \\ \qquad\qquad + (\nabla \times H) \times H - \nabla P(\rho, \theta) + \alpha \nabla (g'(\frac{1}{\rho}) h(|w \circ Y|^2)), \\ \theta_t + u \cdot \nabla \theta + \dfrac{\theta P_\theta}{c_\vartheta \rho} \operatorname{div} u = \dfrac{1}{c_\vartheta \rho}(\kappa \Delta \theta + \Psi^*), \\ H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \\ i w_t + \Delta_y w = |w|^2 w + \tilde{\alpha} g(\frac{1}{\rho}) h'(|w|^2) w, \\ \operatorname{div} H = 0, \end{cases} \tag{1.1}$$

whose terms will be explained subsequently, and the main purpose of this paper is to solve the Cauchy problem in $\mathbb{R}^3$, so $(t, x) \in [0, \infty) \times \mathbb{R}^3$, for prescribed initial data

$$(\rho(0, x), u(0, x), \theta(0, x), H(0, x), w(0, y)) = (\rho_0(x), u_0(x), \theta_0(x), H_0(x), w_0(y)), \tag{1.2}$$

which are small and smooth perturbations of constant states, say, $\rho = \bar{\rho}_0 > 0$, $u = 0$, $\theta = \bar{\theta}_0$, $H = \bar{B}$, $w = 0$, where $\bar{B} \in \mathbb{R}^3$ is arbitrary and we are interested in the case where $\bar{B} \neq 0$. The latter is motivated by real applications, such as, the Earth's magnetic field in the ionosphere where radiative electrons, issued from solar explosions, get trapped. Without loss of generality, throughout this paper, we assume that $\bar{\rho}_0 = \bar{\theta}_0 = 1$.

Let us recall the compressible MHD system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) = \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u \\ \qquad\qquad + (\nabla \times H) \times H - \nabla P(\rho, \theta) + \rho F, \\ \theta_t + u \cdot \nabla \theta + \dfrac{\theta P_\theta}{c_\vartheta \rho} \operatorname{div} u = \dfrac{1}{c_\vartheta \rho}(\kappa \Delta \theta + \Psi^*), \\ H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \\ \operatorname{div} H = 0, \end{cases} \tag{1.3}$$

where $\rho, u, \theta, H, P(\rho, \theta)$ and F are, respectively, density, velocity field, temperature, magnetic field, pressure and external force. Further, μ is the first viscosity coefficient, λ is the second viscosity coefficient, with $\mu > 0$, $\lambda + \mu > 0$. $\nu > 0$ is the magnetic diffusivity constant. As usual, we assume that the pressure function $P(\rho, \theta)$ satisfies $P_\rho(\rho, \theta) > 0$, $P_\theta(\rho, \theta) > 0$. Also, $\kappa > 0$ is the heat conduction coefficient. c_ϑ is the specific heat at constant volume, which, in general, is a positive function of (ρ, θ). Finally,

$$\Psi^* = \mu |\nabla u|^2 + (\lambda + \mu)|\operatorname{div} u|^2 + \nu |\nabla H|^2,$$

where, as usual, for a d-vector $V=(v_1,\cdots,v_d)$, $|V|^2=v_1^2+\cdots+v_d^2$, and for a $d\times d$-matrix $A=(a_{ij})$, $i,j=1,\dots,d$, we denote $|A|^2:=\sum_{i,j=1}^{d} a_{ij}^2$. Here we will always assume the space dimension $d=3$.

Next, we recall the nonlinear Schrödinger equation describing the propagation of the short waves, referred to an observer with the group velocity. As in [4] and [5], the latter is taken to be equal to the fluid velocity u, in accordance to Benney's general prescription in [1]. The equation then reads

$$iw_t+\Delta_y w=|w|^2w+Gw, \tag{1.4}$$

where w is the complex-valued wave function, G is the potential due to the interaction with the fluid, and y denotes the Lagrangian coordinate. For the reader's convenience, we recall the definition of the Lagrangian coordinate (*cf.* [5]).

For $(t,x)\in[0,\infty)\times\mathbb{R}^d$, let $\Phi(t,x)$ be the solution of the initial value problem

$$\begin{cases}\frac{d}{dt}\Phi(t,x)=u(t,\Phi(t,x)),\\ \Phi(0,x)=x.\end{cases} \tag{1.5}$$

The Jacobian $J_\Phi(t,x)=\det(\partial_x\Phi(t,x))$ of the transformation $x\mapsto\Phi(t,x)$ satisfies

$$\begin{cases}\frac{d}{dt}J_\Phi(t,x)=\operatorname{div}u(t,\Phi(t,x))J_\Phi(t,x),\\ J_\Phi(0,x)=1.\end{cases} \tag{1.6}$$

We define the Lagrange transformation $Y(t,z)=(t,y(t,z))$ by the relation

$$y(t,\Phi(t,x))=y_0(x), \tag{1.7}$$

for some given (diffeomorphic) transformation $y_0:\mathbb{R}^d\to\mathbb{R}^d$, and we choose

$$y_0(x):=\left(x_1,\cdots,x_{d-1},\int_0^{x_d}\rho(0,x_1,\cdots,x_{d-1},s)ds\right). \tag{1.8}$$

We observe, from the relations (1.6), (1.7), (1.8), that the Jacobian $J_y(t):=\det(\partial_z y(t,\Phi(t,x)))$ satisfies

$$\begin{cases}\frac{d}{dt}J_y(t)=-\operatorname{div}u(t,\Phi(t,x))J_y(t),\\ J_y(0)=\rho(0,x).\end{cases} \tag{1.9}$$

Therefore, we have $\frac{d}{dt}\frac{\rho(t,\Phi(t,x))}{J_y(t)}=0$, and, since $J_y(0)=\rho(0,x)$, we conclude that

$$\begin{aligned}&\det(\partial_z y(t,\Phi(t,x)))=J_y(t)=\rho(t,\Phi(t,x)),\\ &\quad\text{i.e.,}\quad \det(\partial_z y(t,z))=\rho(t,z),\quad \text{for all } (t,z)\in[0,\infty)\times\mathbb{R}^d.\end{aligned} \tag{1.10}$$

Using the same justification first proposed in [4], based on the energy identity, for which there is an analogue here, when the only force acting on the fluid is the one due to the interaction with the short wave, the model is completed taking F and G having the form

$$F = \frac{\alpha_1}{\rho}\nabla(g'(v)h(|w \circ Y|^2)), \quad G = \tilde{\alpha}_1 g(v(y,t))h'(|w|^2), \tag{1.11}$$

where α_1, $\tilde{\alpha}_1$ are positive constants, $Y(t,x) = (t, y(t,x))$ is the Lagrangian transformation described above, $v(t,y)$ is the specific volume defined by the relation

$$v(t, y(t,x)) = \frac{1}{\rho(t,x)}, \tag{1.12}$$

and $g, h : [0,\infty) \to [0,\infty)$ are nonnegative smooth functions with $h(0) = h'(0) = 0$ (*cf.* [5]).

We are going to rewrite system (1.1) in a more convenient form. We first recall the following calculus identities:

$$\begin{aligned}
\nabla(|H|^2) &= 2(H\cdot\nabla)H + 2H \times \nabla \times H,\\
\nabla \times \nabla \times H &= \nabla \mathrm{div} H - \Delta H,\\
\nabla \times (u \times H) &= u(\mathrm{div}H) - H(\mathrm{div}u) + (H\cdot\nabla)u - (u\cdot\nabla)H.
\end{aligned}$$

Using these identities we may write the momentum and the magnetic field equations in the form

$$\begin{aligned}
&(\rho u)_t + \mathrm{div}(\rho u \otimes u) = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u\\
&\qquad\qquad + (H\cdot\nabla)H - \frac{1}{2}\nabla(|H|^2) - \nabla P(\rho,\theta),\\
&H_t + (\mathrm{div}u)H + (H\cdot\nabla)H - (H\cdot\nabla)u = \nu\Delta H.
\end{aligned} \tag{1.13}$$

We next write system (1.1) setting the linear part around $(\rho, u, \theta, H, w) = (1, 0, 1, \bar{B}, 0)$ on the left-hand side. We denote $a = \rho - 1$, $\tilde{\theta} = \theta - 1$, $B = H - \bar{B}$,

$$\mathfrak{f}(a,\tilde{\theta}) = \frac{P_\rho(1+a, 1+\tilde{\theta})}{1+a} - 1, \quad \mathfrak{g}(a,\tilde{\theta}) = \frac{P_\theta(1+a, 1+\tilde{\theta})}{1+a} - 1,$$
$$\mathfrak{h}(a) = \frac{a}{1+a},$$

and, without loss of generality, we assume

$$P_\rho(1,1) = P_\theta(1,1) = c_\vartheta(1,1) = 1.$$

Then the Cauchy problem (1.1)–(1.2) can be rewritten as

$$
\begin{cases}
a_t + \mathrm{div} u = F_1, \\
u_t - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla a + \nabla\theta + \nabla(\bar{B}\cdot B) - (\bar{B}\cdot\nabla)B = F_2 + \tilde{G}, \\
\tilde{\theta}_t - \kappa\Delta\tilde{\theta} + \mathrm{div} u = F_3, \\
B_t - \nu\Delta B + (\mathrm{div} u)\bar{B} - (\bar{B}\cdot\nabla)u = F_4, \\
i w_t + \Delta_y w = |w|^2 w + \tilde{\alpha_1} g(v) h'(|w|^2) w, \\
\mathrm{div} B = 0, \\
(a(x,0), u(x,0), \tilde{\theta}(x,0), B(x,0), w(y,0)) = (a_0(x), u_0(x), \tilde{\theta}_0(x), B_0(x), w_0(y)),
\end{cases}
\tag{1.14}
$$

where $a_0(x) = \rho_0(x) - 1$, $B_0(x) = H_0(x) - \bar{B}$, and

$$
\begin{aligned}
F_1 &= -a\,\mathrm{div} u - u\cdot\nabla a, \\
F_2 &= -\frac{1}{2}(\mathfrak{h}(a)-1)\nabla(|B|^2) + \mathfrak{h}(a)\nabla(\bar{B}\cdot B) - \mathfrak{h}(a)\bar{B}\cdot\nabla B + (\mathfrak{h}(a)-1)B\cdot\nabla B \\
&\quad - \mathfrak{f}(a,\tilde{\theta})\nabla a - \mathfrak{g}(a,\tilde{\theta})\nabla\tilde{\theta} - \mathfrak{h}(a)(\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div} u) - u\cdot\nabla u, \\
\tilde{G} &= \frac{\alpha_1}{1+a}\nabla(g'(\frac{1}{1+a})h(|w\circ Y|^2)), \\
F_3 &= -u\cdot\nabla\theta - \left(\frac{\theta P_\theta}{c_\vartheta\rho} - 1\right)\mathrm{div} u + \left(\frac{1}{c_\vartheta\rho} - 1\right)\kappa\Delta\theta + \frac{1}{c_\vartheta\rho}\Psi^* \\
F_4 &= -u\cdot\nabla B - (\mathrm{div} u)B + B\cdot\nabla u.
\end{aligned}
\tag{1.15}
$$

Our main theorem concerning the problem (1.1) is the following.

Theorem 1.1. *Assume that*

$$
(a_0, u_0, \tilde{\theta}_0, B_0) \in H^3(\mathbb{R}^3) \cap B^{-s}_{2,\infty}, \quad w_0 \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3),
$$

with $s \in (1, \frac{3}{2}]$. *For* $0 < \epsilon < 1$ *sufficiently small, if*

$$
\delta_0 = \|(a_0, u_0, \tilde{\theta}_0, B_0, w_0)\|_{H^3} + \|(a_0, u_0, \tilde{\theta}_0, B_0)\|_{B^{-s}_{2,\infty}} + \|w_0\|_{L^1} < \epsilon,
$$

then there exists a unique global smooth solution (a, u, B, w) *to the Cauchy problem* (1.14) *such that for all* $t > 0$,

$$
\begin{aligned}
&\|(a, u, \tilde{\theta}, B, w)(t)\|^2_{H^3} + \int_0^t \|\nabla a(\tau)\|^2_{H^2} + \|(\nabla u, \nabla\tilde{\theta}, \nabla B)(\tau)\|^2_{H^3}\, d\tau \\
&\quad \le C\|(a_0, u_0, \tilde{\theta}_0, B_0, w_0)\|^2_{H^3},
\end{aligned}
\tag{1.16}
$$

for some constant $C > 0$ *independent of* t. *Moreover, we have the following decay estimates*

$$
\begin{aligned}
&\|w(t)\|_{L^\infty} \le C\delta_0(1+t)^{-\frac{3}{2}}, \\
&\|(a, u, \tilde{\theta}, B)\|_{L^2} \le C\delta_0(1+t)^{-\frac{s}{2}}, \\
&\|\nabla(a, u, \tilde{\theta}, B)\|_{H^2} \le C\delta_0(1+t)^{-\frac{1+s}{2}}.
\end{aligned}
\tag{1.17}
$$

We briefly explain some important difficulties that had to be overcome in the proof of Theorem 1.1. Concerning the MHD system, the most interesting aspect of the problem we address here is the fact that our equilibrium magnetic field is non-zero. A consequence of this fact is that the corresponding linear MHD system, that is,

$$\begin{cases} a_t + \mathrm{div} u = 0, \\ u_t - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla a + \nabla\tilde{\theta} + \nabla(\bar{B}\cdot B) - (\bar{B}\cdot\nabla)B = 0, \\ \tilde{\theta}_t - \kappa\Delta\tilde{\theta} + \mathrm{div} u = 0, \\ B_t - \nu\Delta B + (\mathrm{div} u)\bar{B} - (\bar{B}\cdot\nabla)u = 0, \end{cases} \tag{1.18}$$

no longer admits a treatable spectral analysis. The reason is that the Fourier transform in the space variables of (1.18) yields a system whose eigenvalues are not spherically symmetric with respect to the Fourier frequencies $\xi = (\xi_1, \xi_2, \xi_3)$, that is, the eigenvalues are not functions only of $|\xi|$. This is connected to the lack of rotational invariance of the corresponding matrix. Indeed, as pointed out in [14], the Navier–Stokes system is rotational invariant but MHD system, with non-zero equilibrium magnetic field, is not. Namely, if we write (1.18) in matrix form as

$$U_t + \mathbf{A}(\partial_x)U = 0,$$

the corresponding system obtained through Fourier transform in the space variables has the form

$$\hat{U}_t + \mathbf{A}(\xi)\hat{U} = 0,$$

where the eigenvalues of matrix $\mathbf{A}(\xi)$ are no longer functions of $|\xi|$, as opposed to the Navier–Stokes case (*cf.* [9,10]) or the case where $\bar{B} = 0$, which allows a trivial decoupling essentially reducing the analysis to the Navier–Stokes case (*cf.* [16,8,12,3]).

To exemplify, we consider the matrix and eigenvalues obtained when $\bar{B} = (0, 1, 0)$. It suffices to consider the matrix corresponding to the minor obtained eliminating the line and column corresponding to the variable $\tilde{\theta}$, that is, the non-heat conductive case. Indeed, the eliminated line and column are exactly equal to the ones corresponding to the variable a, except that, instead of 0, the diagonal element is $\kappa|\xi|^2$, which then must be counted also as an eigenvalue of the whole system. So, let us consider the reduced matrix corresponding to the non-heat conductive case. In this case, we have $\mathbf{A}(\xi)$ given by

$$\begin{pmatrix} 0 & i\xi_1 & i\xi_2 & i\xi_3 & 0 & 0 & 0 \\ i\xi_1 & \mu|\xi|^2 + (\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_1\xi_3 & -i\xi_2 & i\xi_1 & 0 \\ i\xi_2 & (\lambda+\mu)\xi_2\xi_1 & \mu|\xi|^2 + (\lambda+\mu)\xi_2^2 & (\lambda+\mu)\xi_2\xi_3 & 0 & 0 & 0 \\ i\xi_3 & (\lambda+\mu)\xi_3\xi_1 & (\lambda+\mu)\xi_3\xi_2 & \mu|\xi|^2 + (\lambda+\mu)\xi_3^2 & 0 & i\xi_3 & -i\xi_2 \\ 0 & -i\xi_2 & 0 & 0 & \nu|\xi|^2 & 0 & 0 \\ 0 & i\xi_1 & 0 & i\xi_3 & 0 & \nu|\xi|^2 & 0 \\ 0 & 0 & 0 & -i\xi_2 & 0 & 0 & \nu|\xi|^2 \end{pmatrix}.$$

We may calculate the eigenvalues of this matrix obtaining that

$$\lambda_1 = \nu|\xi|^2,$$
$$\lambda_2 = \frac{1}{2}(\mu+\nu)|\xi|^2 - \frac{1}{2}\sqrt{(\mu-\nu)^2|\xi|^4 - 4\xi_2^2},$$
$$\lambda_3 = \frac{1}{2}(\mu+\nu)|\xi|^2 + \frac{1}{2}\sqrt{(\mu-\nu)^2|\xi|^4 - 4\xi_2^2},$$

and the remaining eigenvalues λ_4 to λ_7 are the solutions of the following polynomial equation

$$x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0, \tag{1.19}$$

where

$$c_3 = -(3\mu+\lambda+\nu)|\xi|^2,$$
$$c_2 = 2|\xi|^2 + (\lambda\nu + 2\mu^2 + \lambda\mu + 3\mu\nu)|\xi|^4,$$
$$c_1 = -(\lambda+2\mu)\mu\nu|\xi|^6 - (2\mu+\nu)|\xi|^4 - 4\lambda|\xi|^3 - \mu|\xi|^2\xi_2^2,$$
$$c_0 = \mu\nu|\xi|^6 + |\xi|^2\xi_2^2.$$

By these formulas, it becomes evident that the eigenvalues are not only functions of $|\xi|$ which basically renders inviable the analysis of the decay of the solutions by the method of spectral analysis: even if one could find a suitable partition of the space of frequencies for a particular value of $\bar{B}$, say, $\bar{B} = (0, 1, 0)$, as above, this partition, in general, would not be adequate for the matrix obtained with some other value of $\bar{B}$, say, $\bar{B} = (0, 0, 1)$, due to the lack of rotational invariance with respect to $\bar{B}$. For example, when $\bar{B} = (0, 0, 1)$, as just mentioned, the first three eigenvalues read

$$\lambda_1^2 = \nu|\xi|^2,$$
$$\lambda_2^2 = \frac{1}{2}(\mu+\nu)|\xi|^2 - \frac{1}{2}\sqrt{(\mu-\nu)^2|\xi|^4 - 4\xi_3^2},$$
$$\lambda_3^2 = \frac{1}{2}(\mu+\nu)|\xi|^2 + \frac{1}{2}\sqrt{(\mu-\nu)^2|\xi|^4 - 4\xi_3^2},$$

while for the coefficients of polynomial equation (1.19) in the case where $\bar{B} = (0, 0, 1)$ we have

$$c_3^2 = -(3\mu+\lambda+\nu)|\xi|^2,$$
$$c_2^2 = 2|\xi|^2 + (\lambda\nu + 2\mu^2 + \lambda\mu + 3\mu\nu)|\xi|^4,$$
$$c_1^2 = -(\lambda+2\mu)\mu\nu|\xi|^6 - (2\mu+\nu)|\xi|^4 - 4\lambda|\xi|^3 - \mu|\xi|^2\xi_3^2,$$
$$c_0^2 = \mu\nu|\xi|^6 + |\xi|^2\xi_3^2.$$

Therefore, to overcome this difficulty, to get the desired optimal time decay estimates for MHD systems, instead of going into tedious and complicated spectrum analysis, we adopt the newly developed pure energy method with the help of negative Besov space estimates by Y. Guo and Y.J. Wang [6], see also V. Sohinger and R. Strain [11]. The gist of the method is illustrated in [6] by the fundamental example of the heat equation where the estimate

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\ell}u(t)\|_{L^2}+\|\Lambda^{\ell+1}u\|_{L^2}=0,$$

trivially holds for any $\ell\in\mathbb{R}$, where $\Lambda^{\ell}u:=\mathcal{F}^{-1}(|\xi|^{\ell}\mathcal{F}(u))$, from which one immediately obtains

$$\|\Lambda^{\ell}u(t)\|_{L^2}^2\le\|\Lambda^{\ell}u_0\|_{L^2}^2,$$

for any $\ell\in\mathbb{R}$. Now, assuming that $\|\Lambda^{-s}u_0\|_{L^2}<\infty$, for a real number $s\ge 0$, the key trick is the use of the Gagliardo–Nirenberg interpolation inequality

$$\|\Lambda^{\ell}u(t)\|_{L^2}\le\|\Lambda^{-s}u(t)\|_{L^2}^{\frac{1}{\ell+1+s}}\|\Lambda^{\ell+1}u(t)\|_{L^2}^{\frac{\ell+s}{\ell+1+s}},$$

and then come back to the original inequality to deduce that

$$\frac{d}{dt}\|\Lambda^{\ell}u\|_{L^2}^2+C_0\left(\|\Lambda^{\ell}u\|_{L^2}^2\right)^{1+\frac{1}{\ell+s}}\le 0,$$

from which the optimal time decay estimate follows.

In the present case, including short wave-long wave interaction effect, application of this pure energy method, with initial data in negative order Besov space, is still possible and indeed the only feasible way as it seems. Nevertheless, due to the strong coupling, we need, not only to assume the initial data to be bounded in negative order Besov space, but we also need smallness of the Besov norm of the initial data. This reflects one of the new difficulties of our problem.

Concerning the coupling of the MHD system with the nonlinear Schrödinger equation in the short wave-long wave interaction model (1.1), we observe that the MHD equations are written in Eulerian coordinates while the nonlinear Schrödinger equation is based on the Lagrangian coordinates induced by the fluid motion, as a consequence of the assumption that the group velocity of the short wave coincides with the fluid velocity. No large data theory for global smooth solutions is available as yet for the MHD equations, even without the coupling with the nonlinear Schrödinger equation. The only existence theories available assume initial data close to a constant state, and so the whole point is to show that the solution never leaves a small neighborhood of the equilibrium state. Because of the presence of both coordinate systems in (1.1), in order to get all estimates in just one of them, say, Eulerian coordinates, an additional concern is the increase of the Sobolev norms of the Jacobian of the Lagrangian transformation. The general estimate for the increase in time of the Sobolev norms of the Jacobian of the Lagrangian transformation involves an exponential of the time integral of the corresponding Sobolev norm of the gradient of the fluid velocity (see Lemma 2.5 below). Therefore, in order to ensure global boundedness of the Sobolev norms in switching between these two coordinate systems, we need to obtain the optimal time decay estimates of the gradients of u and B in the MHD system. In [5], without the influence of magnetic field, this was carried out through the spectrum analysis technique and usual energy estimates for the compressible Navier–Stokes system. Now, for the MHD system, in the presence of a magnetic field around a non-zero equilibrium state, the mathematical setting changes significantly, which led us to the assumption that the initial data are small also in certain Besov type space of negative order, as explained above.

Concerning the correlated literature, we recall that in [7] the global small smooth solution for the MHD system is obtained by means of time decay estimates for all components, and in particular for the velocity and magnetic fields. However, these decay estimates for the fields themselves do not provide a suitable estimate for the decay of the gradients of the velocity and magnetic fields. Recently, in [16,8,12,3], under the assumption of zero equilibrium magnetic field, optimal time decay estimates for the gradients of velocity and magnetic fields have been obtained by simply treating the magnetic field equation as a perturbed heat equation, and so, with left-hand side totally uncoupled from the compressible Navier–Stokes equations. Basically, this reduces the analysis to the case of the latter, avoiding the difficulties caused by the cumbersome formulas for the eigenvalues of the coupled linearized system. That is possible because, in the case of a zero equilibrium state, the terms containing the magnetic field in the momentum equation are all of order greater than one in the unknowns and their derivatives.

We further remark that, if we set $w \equiv 0$, the analysis in this paper establishes optimal decay estimates for the Cauchy problem of the compressible MHD (1.3), which is actually new because of the non-zero equilibrium magnetic field. In this case the Cauchy problem (1.14) becomes

$$\begin{cases} a_t + \mathrm{div} u = F_1, \\ u_t - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div} u + \nabla a + (\nabla B)\cdot \bar{B} - (\bar{B}\cdot\nabla)B = F_2, \\ \tilde{\theta}_t - \kappa \Delta \tilde{\theta} + \mathrm{div} u = F_3, \\ B_t - \nu \Delta B + (\mathrm{div} u)\bar{B} - (\bar{B}\cdot\nabla)u = F_4, \\ \mathrm{div} B = 0, \\ (a(x,0), u(x,0), \tilde{\theta}(x,0), B(x,0)) = (a_0(x), u_0(x), \tilde{\theta}_0(x), B_0(x)), \end{cases} \tag{1.20}$$

where F_1, F_2, F_3, F_4 are as given in (1.15).

For this simpler problem, some difficulties caused by the coordinate systems and the interaction source terms disappear, and the following result holds.

Theorem 1.2. *Assume that*

$$(a_0, u_0, \tilde{\theta}_0, B_0) \in H^3(\mathbb{R}^3) \cap B^{-s}_{2,\infty},$$

for any $s \in (0, \frac{3}{2}]$. For $0 < \epsilon < 1$ sufficiently small, if

$$\delta_0 = \|(a_0, u_0, \tilde{\theta}_0, B_0)\|_{H^3} \le \epsilon,$$

then there exists a unique global smooth solution $(a, u, \tilde{\theta}, B)$ to the Cauchy problem (1.20), such that for all $t > 0$,

$$\begin{aligned} &\|(a, u, \tilde{\theta}, B)(t)\|^2_{H^3} + \int_0^t \|\nabla a(\tau)\|^2_{H^2} + \|(\nabla u, \nabla\tilde{\theta}, \nabla B)(\tau)\|^2_{H^3}\, d\tau \\ &\quad \le C\|(a_0, u_0, \tilde{\theta}_0, B_0)\|^2_{H^3}, \end{aligned} \tag{1.21}$$

for some constant $C > 0$ independent of t. Moreover, we have the following decay estimates

$$\begin{aligned} &\|(a, u, \tilde{\theta}, B)(\cdot, t)\|_{L^2} \le C(1+t)^{-\frac{s}{2}}, \\ &\|\nabla(a, u, \tilde{\theta}, B)(\cdot, t)\|_{H^2} \le C(1+t)^{-\frac{1+s}{2}}. \end{aligned} \tag{1.22}$$

Before closing this section, we would like to give a very sketchy but useful idea of the proof of both Theorem 1.1 and Theorem 1.2. We first comment on the latter. As already said, we are going to use the pure energy method through interpolation with Besov space technique introduced in [6]. The first part of this technique consists of standard energy estimates of the type that goes back to [10], which explore the fact that, for $U=(a,u,\tilde{\theta},B)$, (1.20) has the structure

$$U_t+\mathbf{A}(\partial_x)U=\mathbf{F}, \tag{1.23}$$

where $\mathbf{A}(\partial_x)$ is a 8×8 symmetric matrix of linear differential operators, which may be written as $\mathbf{A}(\partial_x)=\mathbf{A}_0(\partial_x)+\mathbf{A}_1(\partial_x)$, where $\mathbf{A}_0(\partial_x)$ is a diagonal matrix with all diagonal entries containing a Laplacian-like dissipative operator except for the first diagonal entry which is null; $\mathbf{A}_1(\partial_x)$ is the symmetric off-diagonal part, containing first order differential operators; and $\mathbf{F}=(F_1,F_2,F_3,F_4)$ as in (1.20). However, before we go further into the exposition of the process, we remark that also in this part of the energy method in [6], there is an important new idea, which we will point out as we get there.

Denoting by $(V|W)$ the scalar product in $L^2(\mathbb{R}^3)$, observe that

$$(\mathbf{A}(\partial_x)D^\alpha U|D^\alpha U)=(\mathbf{A}_0(\partial_x)D^\alpha U|D^\alpha U)\simeq\|\nabla D^\alpha(u,\theta,B)\|_{L^2}^2, \tag{1.24}$$

since $(\mathbf{A}_1(\partial_x)W|W)=0$, for any smooth W, say, with compact support, where we use the multi-index notation $D^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, with $|\alpha|=\alpha_1+\alpha_2+\alpha_3\le 3$, and for positive numbers X,Y, $X\simeq Y$ means $cY\le X\le CY$, for certain positive constants c,C. On the other hand, using standard inequalities, such as Hölder, Young, and Gagliardo–Nirenberg, using also the smallness of the H^3 norm of the solution, assumed a priori, denoting $\nabla^k=(D^\alpha)_{|\alpha|=k}$, we can easily obtain the estimates

$$(\nabla^k\mathbf{F}|\nabla^k U)\le\delta\|\nabla^{k+1}U\|_{L^2}^2,\quad \text{for } k=0,1,2, \tag{1.25}$$

and

$$(\nabla^{k+1}\mathbf{F}|\nabla^{k+1}U)\le\delta\left(\|\nabla^{k+1}U\|_{L^2}^2+\|\nabla^{k+2}(u,\tilde{\theta},B)\|_{L^2}^2\right),\quad \text{for } k=0,1,2. \tag{1.26}$$

From (1.25) and (1.26) it follows the inequality

$$\frac{d}{dt}\sum_{l\le k\le 3}\|\nabla^k U\|_{L^2}^2+C\sum_{l\le k\le 3}\|\nabla^{k+1}(u,\tilde{\theta},B)\|_{L^2}^2\lesssim\delta\sum_{l+1\le k\le 3}\|\nabla^k a\|^2, \tag{1.27}$$

$l=0,1,2$, where by $X\lesssim Y$ we mean $X\le CY$, for some positive constant C. The suitable use of a combination of both (1.25) and (1.26), is the important new ingredient here. The point is to be able to get separate estimates for the $(3-l)$-Sobolev norm of $\nabla^l U$ for each $l=0,1,2$.

Now, from the second equation in (1.20), it follows

$$\frac{d}{dt}(\nabla\nabla^k a\,|\,\nabla^k u)+C\|\nabla^{k+1}a\|^2\lesssim\|\nabla^{k+1}(u,\tilde{\theta},B)\|_{L^2}^2+\|\nabla^{k+2}(u,\tilde{\theta},B)\|_{L^2}^2. \tag{1.28}$$

Combining (1.25), (1.26), (1.27) and (1.28), we arrive at inequalities of the form

$$\frac{d}{dt}\mathcal{E}_k^3 + \|\nabla^{k+1} a\|_{H^{3-k-1}}^2 + \|\nabla^{k+1}(u, \tilde{\theta}, B)\|_{H^{3-k}}^2 \le 0, \quad \text{for } k = 0, 1, 2, \tag{1.29}$$

where $\mathcal{E}_k^3(t)$ is equivalent to $\|\nabla^k U(t)\|_{H^{3-k}}^2$, $k = 0, 1, 2$, from which (1.21) follows.

Now, for Δ_q as in (2.2) and Λ^{-s} as above, again we have

$$\begin{aligned}(\mathbf{A}(\partial_x)\Lambda^{-s}\Delta_q U \mid \Lambda^{-s}\Delta_q U) &= (\mathbf{A}_0(\partial_x)\Lambda^{-s}\Delta_q U \mid \Lambda^{-s}\Delta_q U)\\ &\simeq \|\nabla\Lambda^{-s}\Delta_q(u, \tilde{\theta}, B)\|_{L^2}^2.\end{aligned} \tag{1.30}$$

Now, we reach the crucial point of the application of the method in [6], which are the estimates

$$(\Lambda^{-s}\Delta_q \mathbf{F} \mid \Lambda^{-s}\Delta_q U) \lesssim \|\nabla U\|_{H^1}^2 \|\Lambda^{-s}\Delta_q U\|_{L^2}, \quad \text{for } s \in (0, 1/2], \tag{1.31}$$

and

$$(\Lambda^{-s}\Delta_q \mathbf{F} \mid \Lambda^{-s}\Delta_q U) \lesssim \|U\|_{L^2}^{s-1/2} \|\nabla U\|_{H^1}^{5/2-s} \|\Lambda^{-s}\Delta_q U\|_{L^2}, \quad \text{for } s \in (1/2, 3/2). \tag{1.32}$$

For $s \in (0, 1/2]$, (1.31), together with (1.30) and (1.21) leads to

$$\|U(t)\|_{B_{2,\infty}^{-s}} \le C_0, \tag{1.33}$$

where the Besov norm $\|\cdot\|_{B_{2,\infty}^{-s}}$ is defined in Definition 2.2. We then use (1.33) together with Gagliardo–Nirenberg type interpolation inequality, as in the example of the heat equation, to get the desired decay for $s \in (0, 1/2]$. On the other hand, for $s \in (1/2, 3/2)$, by interpolation we have that $U_0 \in B_{2,\infty}^{-1/2}$, and so we can apply the decay obtained for $s = -1/2$ in order to bound the time integral of the factor multiplying $\|\Lambda^{-s}\Delta_q U\|_{L^2}$ in the right-hand side of (1.32) to conclude that (1.33) holds also for $s \in (1/2, 3/2)$, and get the desired decay as before.

Concerning Theorem 1.1, for the short wave-long wave interaction system (1.14), we combine the procedures just described, with a time decay estimate for the Schrödinger component, obtained by first assuming a priori the time decay estimate for the gradient of the velocity. The a priori assumption will be justified, by a bootstrap argument, once we get the desired decay, with a small factor depending on the initial data, as in (1.17).

Since there is essentially no difference between the proofs for the non-heat conductive case and the heat conductive one, we will present the proof of Theorem 1.1 only for the non-heat conductive case. Therefore, for the sake of reference, we set below the non-heat conductive system corresponding to (1.14),

$$\begin{cases} a_t + \operatorname{div} u = F_1,\\ u_t - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div} u + \nabla a + \nabla(\bar{B}\cdot B) - (\bar{B}\cdot\nabla)B = F_2 + \tilde{G},\\ B_t - \nu\Delta B + (\operatorname{div} u)\bar{B} - (\bar{B}\cdot\nabla)u = F_4,\\ i w_t + \Delta_y w = |w|^2 w + \tilde{\alpha_1} g(v) h'(|w|^2) w,\\ \operatorname{div} B = 0,\\ (a(x,0), u(x,0), B(x,0), w(y,0)) = (a_0(x), u_0(x), B_0(x), w_0(y)), \end{cases} \tag{1.34}$$

where $a_0(x) = \rho_0(x) - 1$, $B_0(x) = H_0(x) - \bar{B}$, and

$$
\begin{aligned}
&F_1 = -a\mathrm{div}u - u\cdot\nabla a,\\
&F_2 = -\frac{1}{2}(\mathfrak{h}(a)-1)\nabla(|B|^2) + \mathfrak{h}(a)\nabla(\bar{B}\cdot B) - \mathfrak{h}(a)\bar{B}\cdot\nabla B + (\mathfrak{h}(a)-1)B\cdot\nabla B\\
&\qquad - \mathfrak{f}(a)\nabla a - \mathfrak{h}(a)(\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u) - u\cdot\nabla u, \qquad (1.35)\\
&\tilde{G} = \frac{\alpha_1}{1+a}\nabla(g'(\frac{1}{1+a})h(|w\circ Y|^2)),\\
&F_4 = -u\cdot\nabla B - (\mathrm{div}u)B + B\cdot\nabla u.
\end{aligned}
$$

The present paper is organized as follows. In Section 2, we introduce some notations and give the proof of the local existence. In Section 3, we give the proof of three lemmas which are important energy estimates for our global existence. In Section 4, we establish the evolution of the solution in the Besov type space with negative order. Using the results in the previous two sections, we then prove the Theorem 1.1, in Section 5. For the convenience of the reader, we include the Appendix with some useful lemmas.

2. Preliminaries and local solutions

We will use, throughout this paper, some standard notations. $H^k(\mathbb{R}^3)$ is the k-th order Sobolev space based on $L^2(\mathbb{R}^3)$, and if we need to emphasize the space coordinates, whether Eulerian or Lagrangian, we denote $H^k_x(\mathbb{R}^3)$ or $H^k_y(\mathbb{R}^3)$, respectively. We use any of the notations D_i, D_{x_i}, ∂_i, ∂_{x_i} to denote the partial derivative with respect to the i-th spatial coordinate. We also use the usual multi-index notation, so, $D^\alpha_x = D^{\alpha_1}_{x_1}D^{\alpha_2}_{x_2}D^{\alpha_3}_{x_3}$. The subscript x is used for Eulerian coordinates, y, instead, is used for Lagrangian coordinates, and they may be omitted when there is no risk of confusion. We denote $\nabla^k_x u$ the collection $(D^\alpha_x u)_{|\alpha|=k}$ of all k-th order partial derivatives with respect to x, and when $k=1$ we omit the superscript, as usual. For the estimates in this paper, we will use C for a general constant depending only on the data, and sometimes we may use $C(a,b,\cdots)$ to emphasize the dependence of C on $a, b, \cdots$. The notation $A \lesssim B$ as an equivalent of $A \lesssim B$. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

We recall some definitions, notations and few basic facts concerning homogeneous Besov type spaces, based on [2]. The homogeneous Littlewood–Paley decomposition relies upon a dyadic partition of unity. We can use for instance any $\phi \in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C} := \{\xi \in \mathbb{R}^N, 3/4 \le |\xi| \le 8/3\}$ such that

$$
\sum_{q\in\mathbb{Z}} \phi(2^{-q}\xi) = 1 \quad \text{if} \quad \xi \neq 0. \tag{2.1}
$$

Denote $h = \mathcal{F}^{-1}\phi$, where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^N$. We then define the dyadic blocks as follows

$$
\Delta_q u := \phi(2^{-q}D)u = 2^{qN}\int_{\mathbb{R}^N} h(2^q y)u(x-y)\,dy, \quad \text{and} \quad S_q u = \sum_{k\le q-1}\Delta_k u. \tag{2.2}
$$

The formal decomposition

$$
u = \sum_{q\in\mathbb{Z}} \Delta_q u \tag{2.3}
$$

is called homogeneous Littlewood–Paley decomposition. The above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of ϕ, we have

$$\Delta_k \Delta_q u = 0 \quad \text{if} \quad |k-q| \geq 2, \tag{2.4}$$

$$\Delta_k (S_{q-1} u \Delta_q u) = 0 \quad \text{if} \quad |k-q| \geq 5. \tag{2.5}$$

Let us now introduce the homogeneous Besov space.

Definition 2.1. [2] We denote by $\mathcal{S}'_h$ the space of tempered distributions u such that

$$\lim_{j \to -\infty} S_j u = 0 \quad \text{in} \quad \mathcal{S}'. \tag{2.6}$$

Definition 2.2. [2] Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $B^s_{p,r}$ consists of distributions u in $\mathcal{S}'_h$ such that

$$\|u\|_{B^s_{p,r}} := \left(\sum_{j \in \mathbb{Z}} 2^{rjs} \|\Delta_j u\|^r_{L^p} \right)^{1/r} < +\infty, \quad \text{for } 1 \leq r < \infty, \tag{2.7}$$

$$\|u\|_{B^s_{p,\infty}} := \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p} < +\infty. \tag{2.8}$$

We also define the operator Λ^{-s} by $\Lambda^{-s}\varphi := \mathcal{F}^{-1}(|\xi|^{-s}\mathcal{F}\varphi)$, for any $\varphi \in \mathcal{D}$.

With the help of the notion of Besov spaces, the classical Gagliardo–Nirenberg inequality can be further generalized, see [6,11,15]. We will use these interpolation extensively. Some useful lemmas were listed in the Appendix as Lemmas 6.1–6.5.

The following three lemmas were proved in [5], which form the basis for the analysis in this paper.

Lemma 2.3. *[5] Let $x = x(t,y)$ be the inverse Lagrange transformation in $\mathbb{R}^3$ induced by velocity field u, $F(t,y) = \nabla_y x(t,y)$ be the deformation gradient matrix at time t. For any function $\varphi \in W^{1,p}(\mathbb{R}^3)$ and $\mathbb{I}$ the identity matrix, if $\|F(t) - \mathbb{I}\|_{L^\infty} \leq \frac{1}{18}$, for $t > 0$, then for any $1 \leq p \leq \infty$, it holds that*

$$\frac{1}{2}\|\nabla_x \varphi\|_{L^p_x} \leq \|\nabla_y \varphi(x(y))\|_{L^p_x} \leq 2\|\nabla_x \varphi\|_{L^p_x}.$$

If for two positive constants ρ_1, ρ_2, $0 < \rho_1 \leq \rho \leq \rho_2$, where ρ satisfies the continuity equation $\rho_t + \operatorname{div}(\rho u) = 0$, then

$$\frac{1}{C(\rho_1,\rho_2)}\|\varphi\|_{L^p_x} \leq \|\varphi(x(y))\|_{L^p_y} \lesssim C(\rho_1,\rho_2)\|\varphi\|_{L^p_x}.$$

Furthermore, under all the above conditions, there exists a constant $C(p)$ independent of ϕ, such that

$$\frac{1}{C(p)}\|\varphi\|_{W^{1,p}_x} \leq \|\varphi(x(y))\|_{W^{1,p}_y} \lesssim C(p)\|\varphi\|_{W^{1,p}_x}.$$

Lemma 2.4. *[5] Let $F(t, y)$ be, as above, the deformation matrix associated with the Lagrange transformation in $\mathbb{R}^3$. For $\delta > 0$ small enough, if $\|F(t) - \mathbb{I}\|_{H^2_x} \le \delta$, and $0 < \rho_1 \le \rho(t, x) \le \rho_2$, for $t > 0$, then for $k = 1, 2, 3$, it holds that*

$$\frac{1}{C(\rho_1, \rho_2)} \|\varphi\|_{H^k_x} \le \|\varphi(x(y))\|_{H^k_y} \lesssim C(\rho_1, \rho_2) \|\varphi\|_{H^k_x}.$$

Denote $E(t) = F(t) - F_0$, where F is the deformation gradient, $F_0 = \nabla_y y_0^{-1}$ is the initial deformation gradient, and let $E_{i,j}$ be the (i, j) entry of E. Then we have the following Lemma.

Lemma 2.5. *[5] For some $C = C(\epsilon) > 0$, and for $k = 2, 3, \cdots$, the matrix E satisfies*

$$\|E(t)\|^2_{H^k} \le (\int_0^t \|\nabla u(\tau)\|_{H^k} d\tau) \exp\left\{ C \int_0^t \|\nabla u(\tau)\|_{H^k} d\tau \right\}.$$

Now, we establish the local existence for problem (1.34). Let us recall the space $X^l(t_1, t_2; E)$ used in [5,9], where, for some $E > 0$, $0 \le t_1 < t_2 \le \infty$, we have

$$\begin{aligned} X^l(t_1, t_2; E) := \Big\{ &(a, u, B) : a(t, x) \in C^0(t_1, t_2; H^l) \cap C^1(t_1, t_2; H^{l-1}), \\ &u^i(t, x), B(t, x) \in C^0(t_1, t_2; H^l) \cap C^1(t_1, t_2; H^{l-2}) \cap L^2(t_1, t_2; H^{l+1}) \\ &i = 1, 2, 3, \quad \text{and} \quad \|(a, u, B)\|^2_{X^l} \\ &:= \sup_{t_1 \le t \le t_2} \|(a, u, B)(t)\|^2_l + \int_{t_1}^{t_2} \|a(s)\|^2_l + \|(u, B)(s)\|^2_{l+1} ds \le E^2 \Big\}. \end{aligned} \tag{2.9}$$

We also define

$$\begin{aligned} X^l_w(t_1, t_2; E) := \Big\{ w \in C(t_1, t_2; H^l(\mathbb{R}^3_y; \mathbb{C})) \cap C^1(t_1, t_2; H^{l-2}(\mathbb{R}^2_y; \mathbb{C})) : \\ \|w\|_{X^l_w} := \sup_{t_1 \le t \le t_2} \|w(t)\|_l \le E \Big\}. \end{aligned} \tag{2.10}$$

Theorem 2.6. *Consider system (1.34) and take $l = 3$. Given the initial data at a time $t_1 \ge 0$,*

$$(a, u, B)(t_1) \in H^l(\mathbb{R}^3), \quad w(t_1) \in H^l(\mathbb{R}^3_y; \mathbb{C}), \tag{2.11}$$

where the subscript y is used to indicate that we use the Lagrangian coordinates induced by $u(t)$, $0 \le t \le t_1$, at $t = t_1$, in $\mathbb{R}^3$, and we recall that w is complex valued, which we emphasize by putting $\mathbb{C}$ as the image of w, which will be frequently omitted. Then, there exist three constants $\delta_1 > 0$, $\epsilon > 0$ ($\delta_1 < \epsilon$) and $\tau > 0$, such that if

$$\|(a, u, B)(t_1)\|_{H^l} < \delta_1, \quad \|w(t_1)\|_{H^l} < \delta_1,$$

then the initial value problem for (1.34), with initial data (2.11) prescribed at $t = t_1$, has the unique solution

$$(a, u, B)(t) \in X^l(t_1, t_1+\tau; \epsilon), \quad w \in X^l_w(t_1, t_1+\tau; \epsilon),$$

where δ_1, ϵ, τ do not depend on t_1.

Proof. We consider the following linear operator

$$\begin{aligned}
&L^0_{\tilde{u}}(a) := a_t + \tilde{u}\cdot\nabla a + (\operatorname{div}\tilde{u})a,\\
&\mathbf{L}(u) := u_t - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div}u,\\
&L^4(B) := B_t - \nu\Delta B,\\
&L^5_{\tilde{u}}(w) := i w_t + \Delta_y w.
\end{aligned}$$

Let $\bar{X}^3(t_1, t_1+\tau; \epsilon)$ and $\bar{X}^3_w(t_1, t_1+\tau; \epsilon)$ be the completions of $X^3(t_1, t_1+\tau; \epsilon)$ and $X^3_w(t_1, t_1+\tau; \epsilon)$ with respect to the norms $\|\cdot\|_{X^3}$ and $\|\cdot\|_{X^3_w}$, respectively. Given $(\tilde{a}, \tilde{u}, \tilde{B}) \in \bar{X}^3(t_1, t_1+\tau; \epsilon)$ and $\tilde{w} \in \bar{X}^3_w(t_1, t_1+\tau; \epsilon)$, let $\tilde{U} := (\tilde{a}, \tilde{u}, \tilde{B})$, $\tilde{V} := (\tilde{U}, \tilde{w})$, and let us consider the solution $V = (U, w) = (a, u, B, w) = \mathcal{L}(\tilde{V})$ of the initial value problem for the linear system

$$\begin{cases}
L^0_{\tilde{u}}(a) = -\operatorname{div}\tilde{u},\\
\mathbf{L}(u) = \mathbf{R}(\tilde{a}, \tilde{u}, \tilde{B}, \tilde{w}),\\
L^4(B) = R^4(\tilde{a}, \tilde{u}, \tilde{B}),\\
L^5_{\tilde{u}}(w) = R^5(\tilde{a}, \tilde{u}, \tilde{w}),
\end{cases} \tag{2.12}$$

with initial data (2.11), where

$$\begin{aligned}
\mathbf{R}(\tilde{a}, \tilde{u}, \tilde{B}, \tilde{w}) = &-\nabla\tilde{a} - \nabla(\bar{B}\cdot\tilde{B}) + (\bar{B}\nabla)\tilde{B} - \frac{1}{2}(\mathfrak{h}(\tilde{a})-1)\nabla(|\tilde{B}|^2) + \mathfrak{h}(\tilde{a})\nabla(\bar{B}\cdot\tilde{B})\\
&+ (\mathfrak{h}(\tilde{a})-1)\tilde{B}\cdot\nabla\tilde{B} - \mathfrak{f}(\tilde{a})\nabla\tilde{a} - \mathfrak{h}(\tilde{a})(\mu\Delta\tilde{u} + (\lambda+\mu)\nabla\operatorname{div}\tilde{u})\\
&- \tilde{u}\cdot\nabla\tilde{u} + \alpha_1\nabla(g'(\frac{1}{1+\tilde{a}})h(|\tilde{w}\circ Y|^2)),\\
R^4(\tilde{a}, \tilde{u}, \tilde{B}) = &-(\operatorname{div}\tilde{u})\bar{B} + (\bar{B}\nabla)\tilde{u} - \tilde{u}\cdot\nabla\tilde{B} + (\operatorname{div}\tilde{u})\tilde{B} - \tilde{B}\cdot\nabla\tilde{u},\\
R^5(\tilde{a}, \tilde{u}, \tilde{w}) = &|\tilde{w}|^2\tilde{w} + \tilde{\alpha_1}g(\frac{1}{1+\tilde{a}})h'(|\tilde{w}|^2)\tilde{w}.
\end{aligned}$$

Our goal is to prove the existence of a fixed point of $\mathcal{L} : (X^3 \times X^3_w)(t_1, t_1+\tau; \epsilon) \to (X^3 \times X^3_w)(t_1, t_1+\tau; \epsilon)$, for τ and ϵ sufficiently small.

Let $\mathbf{S}(t) = (S^1(t), S^2(t), S^3(t))$ be the semigroup associated with $\mathbf{L}(u)$. $S^4(t)$ and $S^5(t)$ be the semigroup associated with $L^4(B)$ and $L^5_{\tilde{u}}(w)$ respectively. Let $\tilde{\Phi}^s(t, x)$ be defined exactly as in [5] and the following proof are similar to the local part in [5] so we only give a sketch of the proof.

Through Duhamel's principle and estimates about semigroup, we could obtain

$$\begin{aligned}
\sup_{t_1\le s\le t}\|D^\alpha_x(u, B)(s)\|_{L^2} \le &\|D^\alpha_x(u, B)(t_1)\|_{L^2}\\
&+ C(t-t_1)\sup_{t_1\le s\le t}\left\{\|(\tilde{a}, \tilde{B})(s)\|_{H^{|\alpha|+1}} + \|\tilde{u}(s)\|_{H^{|\alpha|+2}} + \|\tilde{w}(s)\|_{H^{|\alpha|}}\right\},
\end{aligned} \tag{2.13}$$

for $|\alpha| = 0, 1$ and

$$\begin{aligned}\sup_{t_1 \le s \le t} \|D_y^\alpha w(s)\|_{L^2} &\le \|D_y^\alpha w(t_1)\|_{L^2} \\ &\quad + C(t-t_1) \sup_{t_1 \le s \le t} \Big\{ \|\tilde{a}(s)\|_{H^{|\alpha|}} + \|\tilde{w}(s)\|_{H^{|\alpha|}} \Big\},\end{aligned} \tag{2.14}$$

for $|\alpha| = 0, 1, 2, 3$. Using the same arguments as in [5], we have the estimates

$$\begin{aligned}\|a(t)\|_{L^2} &\le \Big(\|a(t_1)\|_{L^2} (1 + C(t-t_1)^{\frac{1}{2}}) \\ &\quad + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^1} \Big) \Big(1 + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big), \\ \|a(t)\|_{L^\infty} &\le \Big(\|a(t_1)\|_{L^2} (1 + C(t-t_1)^{\frac{1}{2}}) \\ &\quad + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big) \Big(1 + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big),\end{aligned} \tag{2.15}$$

for $|\alpha| = 1$,

$$\begin{aligned}\|D_x^\alpha a\|_{L^2} &\le \Big(\|D^\alpha a(t_1)\|_{L^2} (1 + C(t-t_1)^{\frac{1}{2}}) \\ &\quad + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big) \Big(1 + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big), \\ \|D_x^\alpha a\|_{L^\infty} &\le \Big(\|D^\alpha a(t_1)\|_{L^\infty} (1 + C(t-t_1)^{\frac{1}{2}}) + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \\ &\quad + C(t-t_1)^{\frac{1}{2}} \big(\int_{t_1}^{t} \|\tilde{u}(s)\|_{H^4}^2 \, ds \big)^{\frac{1}{2}} \Big) \Big(1 + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big),\end{aligned} \tag{2.16}$$

and, for $|\alpha| = 2, 3$, we have

$$\begin{aligned}\|D_x^\alpha a\|_{L^2} &\le \Big(\|a(t_1)\|_{H^3} (1 + C(t-t_1)^{\frac{1}{2}}) + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \\ &\quad + C(t-t_1)^{\frac{1}{2}} \big(\int_{t_1}^{t} \|\tilde{u}(s)\|_{H^4}^2 \big)^{\frac{1}{2}} \Big) \Big(1 + C(t-t_1) \sup_{t_1 \le s \le t} \|\tilde{u}\|_{H^3} \Big).\end{aligned} \tag{2.17}$$

To estimate $\|D_x^\alpha (u, B)\|_{L^2}$ for $|\alpha| = 2, 3$, we proceed as follows. We integrate by parts the equation

$$D_x^\alpha (\mathbf{L}(u) - \mathbf{R}(\tilde{a}, \tilde{u}, \tilde{B}, \tilde{w})) \cdot D_x^\alpha u + D_x^\alpha (L^4(B) - R^4(\tilde{a}, \tilde{u}, \tilde{B})) \cdot D_x^\alpha B = 0,$$

from which we obtain

$$\begin{aligned}&\frac{d}{dt}\|D_x^\alpha(u,B)(t)\|_{L^2}^2+c_0\|\nabla_x D_x^\alpha(u,B)(t)\|_{L^2}^2\\&\quad\lesssim\|D_x^\alpha(u,B)(t)\|_{L^2}^2+C\sum_{|\beta|<|\alpha|}\Big(\|D_x^\beta(\tilde a,\tilde w)(t)\|_{L^2}^2+\|\nabla_x D_x^\beta(\tilde u,\tilde B)(t)\|_{L^2}^2\Big),\end{aligned}$$

where c_0 is a positive small number. Applying Gronwall's inequality, we deduce that

$$\begin{aligned}&\sup_{t_1\le s\le t}\|D_x^\alpha(u,B)(s)\|_{L^2}+\Big(\int_{t_1}^{t}\|\nabla_x D_x^\alpha(u,B)(s)\|_{L^2}^2\,ds\Big)^{\frac12}\\&\qquad\lesssim\|D_x^\alpha(u,B)(t_1)\|_{L^2}+C\int_{t_1}^{t}\|(\tilde u,\tilde B)(s)\|_{H^4}^2\,ds\\&\qquad+C(t-t_1)\sup_{t_1\le s\le t}\Big\{\|\tilde a(s)\|_{H^{|\alpha|}}+\|(\tilde u,\tilde B)(s)\|_{H^3}+\|\tilde w(s)\|_{H^{|\alpha|}}\Big\},\end{aligned}\tag{2.18}$$

holds for $|\alpha|=2,3$. Using (2.13) to (2.18), we can complete the proof as in [5]. □

In the rest of this paper, we will assume that

$$\sup_{0\le t\le T}(1+t)^{\frac{1+s}{2}}\|\nabla u(t)\|_{H^2}\le\sigma_0,\tag{2.19}$$

where $s\in(1,\frac32]$, $0<\sigma_0<1$ is chosen so that, in view of Lemma 2.5, the hypothesis of Lemma 2.3 and Lemma 2.4 are satisfied. The justification for the assumption (2.19) is obtained as a result of our decay estimates for the MHD equations and is one of the key points in the proof of Theorem 1.1.

Using the same arguments as in [5], the following lemma is due to classical results for non-linear Schrödinger equation.

Lemma 2.7. *Let $(a,u,B,w)\in X^3(0,T;\epsilon)\times X_w^3(0,T;\epsilon)$ be a local solution of (1.34). There exists $C=C(\epsilon)>0$ such that, for all $t\in[0,T]$, where the solution is defined and $(a,u,B,w)\in C([0,T];H^3(\mathbb{R}^3))$,*

$$\|w(t)\|_{L^\infty}\le\frac{C\delta_0}{(1+t)^{\frac32}},\qquad\|w(t)\|_{H^3}\lesssim\|w_0\|_{H^3}.$$

3. Proof of the Theorem 1.1: energy estimates

In this section, we will derive the a priori energy estimates for the local solution (a,u,B,w) to the system (1.34), on the time interval where it exists as a classical solution. Hence we assume a priori that for sufficiently small $\delta>0$ (to be specified later), it holds that

$$\sqrt{\mathcal{E}_0^3(t)}=\|a(t)\|_{H^3}+\|u(t)\|_{H^3}+\|B(t)\|_{H^3}+\|w(t)\|_{H^3}\le\delta.\tag{3.1}$$

First of all, by (3.1) and Sobolev's inequality, we obtain

$$\frac{1}{2} \le a + 1 \le 2. \tag{3.2}$$

Hence, we immediately have

$$|\mathfrak{h}(a)|, |\mathfrak{f}(a)| \lesssim |a| \quad \text{and} \quad |\mathfrak{h}^{(k)}(a)|, |\mathfrak{f}^{(k)}(a)| \le C \quad \text{for any } k \ge 1. \tag{3.3}$$

In other to systematize our estimation procedure, let us write the MHD part of (1.34), that is, the first 3 equations in (1.34), for $V = (a, u, B)$, in the form

$$V_t + \mathbf{A}(\partial_x)V = \mathbf{H}, \tag{3.4}$$

where $\mathbf{A}(\partial_x)$ is the matrix of linear differential operators which we may write as $\mathbf{A}(\partial_x) = \mathbf{A}_0(\partial_x) + \mathbf{A}_1(\partial_x)$, where $\mathbf{A}_0(\partial_x)$ is the diagonal dissipative second order part, and $\mathbf{A}_1(\partial_x)$ is the formal symmetric off diagonal matrix of first order differential operators. Also, $\mathbf{H} = (F_1, F_2 + \tilde{G}, F_4)$, and $F_1, F_2, \tilde{G}, F_4$ are defined just after (1.34).

We just need to know the “qualitative” form of the terms composing the components of $\mathbf{H}$. So we use the following self-explanatory notation which seeks to display formally only the dependent variables, functions of them and their derivatives, involved in each term, completely omitting the indices which specify vector components:

$$F_1 \sim a\partial_i u + u\partial_i a, \tag{3.5}$$

$$F_2 \sim B\partial_i B + \mathfrak{h}(a)B\partial_i B + \mathfrak{h}(a)\bar{B}\partial_j B + \mathfrak{f}(a)\partial_i a + \mathfrak{h}(a)\partial_{ij}^2 u + u\partial_j u, \tag{3.6}$$

$$\tilde{G} \sim \left(\frac{1}{1+a}\right)\partial_i \left(g'(\frac{1}{1+a})\right) h(|w \circ Y|^2) + \left(\frac{1}{1+a}\right) g'(\frac{1}{1+a})\partial_i \left(h(|w \circ Y|^2)\right), \tag{3.7}$$

$$F_4 \sim u\partial_i B + B\partial_i u. \tag{3.8}$$

So, for example, in the right-hand sides of the above expressions a term like $a\partial_i u$ means any expression formed as the product of a by a first order derivative of a component of u, while, say, $u\partial_i B$ means any expression formed as the product of a component of u by a first order derivative of a component of B. The left-hand side means that any component of the respective vector, $F_1, F_2, \cdots$, has the form given by the right-hand side.

Denoting by $(W_1 \mid W_2)$ the scalar product in L^2, we also notice that, as in (1.24),

$$\begin{aligned}
&(\nabla^k V_t + \mathbf{A}(\partial_x)\nabla^k V \mid \nabla^k V) = (\nabla^k V_t + \mathbf{A}_0(\partial_x)\nabla^k V \mid \nabla^k V) \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla^k a|^2 + |\nabla^k u|^2 + |\nabla^k B|^2\, dx + \int_{\mathbb{R}^3} \mu|\nabla^{k+1} u|^2\, dx \\
&\quad + \int_{\mathbb{R}^3} (\lambda + \mu)|\nabla^k \mathrm{div} u|^2 + \nu|\nabla^{k+1} B|^2\, dx \\
&\simeq \frac{d}{dt}\int_{\mathbb{R}^3} (|\nabla^k a|^2 + |\nabla^k u|^2 + |\nabla^k B|^2)\, dx + C_0\left(\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2\right)
\end{aligned} \tag{3.9}$$

We first derive the following energy estimates which contain the dissipation estimate for u and B.

Lemma 3.1. *If $\sqrt{\mathcal{E}_0^3} \le \delta$, $\|w \circ Y\|_{L^\infty} \simeq \|w\|_{L^\infty}$ and $\|w \circ Y\|_{H^l} \simeq \|w\|_{H^l}$ with $l = k-1, k$, then for $k = 1, 2, 3$ we have*

$$\frac{d}{dt}\int_{\mathbb{R}^3} (|\nabla^k a|^2 + |\nabla^k u|^2 + |\nabla^k B|^2)\,dx + \frac{1}{2}\left(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}B\|_{L^2}^2\right)$$
$$\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^k(a,u,B)\|_{L^2}^2 + \|\nabla^{k+1}(u,B)\|_{L^2}^2 + (1+t)^{-6}\|(\nabla^{k-1}w, \nabla^k w)\|_{L^2}^2\right).$$

Proof. Applying ∇^k to (1.34) and multiplying the resulting identities by $\nabla^k a$, $\nabla^k u$, $\nabla^k B$ respectively, summing up them and then integrating over $\mathbb{R}^3$ by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla^k a|^2 + |\nabla^k u|^2 + |\nabla^k B|^2\,dx + \int_{\mathbb{R}^3}\mu|\nabla^{k+1}u|^2\,dx\\ &\quad + \int_{\mathbb{R}^3}(\lambda+\mu)|\nabla^k \mathrm{div} u|^2 + \nu|\nabla^{k+1}B|^2\,dx = I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{3.10}$$

where we used (1.24), and

$$I_1 := (\nabla^k F_1 \,|\, \nabla^k a), \quad I_2 := (\nabla^k F_2 \,|\, \nabla^k u)$$
$$I_3 := (\nabla^k F_4 \,|\, \nabla^k B), \quad I_4 := (\nabla^k \tilde{G} \,|\, \nabla^k u).$$

We then decompose each I_i according to its terms obtaining

$$\begin{aligned}
&I_1 = J_1 + J_2,\\
&J_1 \simeq (\nabla^k(a\partial_i u) \,|\, \nabla^k a), \quad J_2 \simeq (\nabla^k(u\partial_i a) \,|\, \nabla^k a),\\
&I_2 := J_3 + J_4 + J_5 + J_6 + J_7 + J_8,\\
&J_3 \simeq (\nabla^k(B\partial_i B) \,|\, \nabla^k u), \quad J_4 \simeq (\nabla^k(\mathfrak{h}(a)B\partial_i B) \,|\, \nabla^k u),\\
&J_5 \simeq (\nabla^k(\mathfrak{h}(a)\bar{B}\partial_i B) \,|\, \nabla^k u), \quad J_6 \simeq (\nabla^k(\mathfrak{f}(a)\partial_i a) \,|\, \nabla^k u),\\
&J_7 \simeq (\nabla^k(\mathfrak{h}(a)\partial_{ij}^2 u) \,|\, \nabla^k u), \quad J_8 \simeq (\nabla^k(u\partial_i u) \,|\, \nabla^k u),\\
&I_3 := J_9 + J_{10},\\
&J_9 \simeq (\nabla^k(u\partial_i B) \,|\, \nabla^k B), \quad J_{10} \simeq (\nabla^k(B\partial_i u) \,|\, \nabla^k B),\\
&I_4 := J_{11} + J_{12},\\
&J_{11} \simeq \left(\nabla^k\left(\left(\frac{1}{1+a}\right)\partial_i\left(g'(\frac{1}{1+a})\right)h(|w\circ Y|^2)\right) \,\middle|\, \nabla^k u\right),\\
&J_{12} \simeq \left(\nabla^k\left(\left(\frac{1}{1+a}\right)g'(\frac{1}{1+a})\partial_i\left(h(|w\circ Y|^2)\right)\right) \,\middle|\, \nabla^k u\right).
\end{aligned}$$

We first proceed the analysis of the terms J_1, J_2, J_6, J_7, J_8 which follow by the same computation as in [6,13]. We start with J_1. For $k=1$,

$$J_1 \simeq \big((\nabla a)\partial_i u + a\nabla\partial_i u \,\big|\, \nabla a\big),$$

and so

$$\begin{aligned} J_1 &\lesssim \|\nabla u\|_{L^\infty}\|\nabla a\|_{L^2}^2 + \|a\|_{L^\infty}\|\nabla a\|_{L^2}\|\nabla^2 u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla a\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2). \end{aligned}$$

For $k=2$, we have

$$J_1 \simeq \big((\nabla^2 a)\partial_i u + \nabla a\nabla\partial_i u + a\nabla^2\partial_i u \,\big|\, \nabla^2 a\big),$$

so

$$\begin{aligned} J_1 &\lesssim \|\nabla u\|_{L^\infty}\|\nabla^2 a\|_{L^2}^2 + \|\nabla a\|_{L^\infty}\|\nabla^2 u\|_{L^2}\|\nabla^2 a\|_{L^2} + \|a\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^2 a\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 a\|^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2). \end{aligned}$$

For $k=3$,

$$J_1 \simeq \big((\nabla^3 a)\partial_i u + \nabla^2 a\nabla\partial_i u + \nabla a\nabla^2\partial_i u + a\nabla^3\partial_i u \,\big|\, \nabla^3 a\big),$$

so

$$\begin{aligned} J_1 &\lesssim \|\nabla u\|_{L^\infty}\|\nabla^3 a\|_{L^2}^2 + \|\nabla^2 a\nabla^2 u\|_{L^2}\|\nabla^3 a\|_{L^2} + \|\nabla a\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^3 a\|_{L^2} \\ &\quad + \|a\|_{L^\infty}\|\nabla^4 u\|_{L^2}\|\nabla^3\|_{L^2} \\ &\lesssim \|\nabla u\|_{L^\infty}\|\nabla^3 a\|_{L^2}^2 + \|\nabla a\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^3 a\|_{L^2} + \|a\|_{L^\infty}\|\nabla^4 u\|_{L^2}\|\nabla^3 a\|_{L^2} \\ &\quad + \|\nabla^2 u\|_{L^3}\|\nabla^2 a\|_{L^6}\|\nabla^3 a\|_{L^2}, \end{aligned}$$

now we use Gagliardo–Nirenberg inequalities (see (6.1), where $\alpha=2, m=0, l=3,\ p=3,$ $q,r=2$ for the second inequality)

$$\|\nabla^2 a\|_{L^6} \le C\|\nabla^3 a\|_{L^2}, \quad \|\nabla^2 u\|_{L^3} \le C\|u\|_{L^2}^{1/6}\|\nabla^3 u\|_{L^2}^{5/6} \lesssim \sqrt{\mathcal{E}_0^3},$$

to get

$$J_1 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2).$$

Concerning J_2, an entirely similar analysis gives

$$J_2 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^k a\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2),$$

for $k=1,2,3$.

Let us now analyze the terms J_6, noticing that $J_6 \simeq -(\nabla^{k-1}(\mathfrak{f}(a)\partial_i a) \,|\, \nabla^{k+1}u)$. For $k=1$, we trivially have

$$J_6 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla a\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).$$

For $k=2$, we have

$$J_6 \simeq (\mathfrak{f}'(a)\nabla a \partial_i a + \mathfrak{f}(a)\nabla\partial_i a \,|\, \nabla^3 u),$$

and so, using Gagliardo–Nirenberg inequalities (6.1) with $p=3$, $q,r=2$, $\alpha=1, m=0, l=2$, and the standard case $p=6$, $q,r=2$, $\alpha=0, m=1, l=1$,

$$\begin{aligned} J_6 &\lesssim \|\nabla a\|_{L^3}\|\nabla a\|_{L^6}\|\nabla^3 u\|_{L^2} + \|a\|_{L^\infty}\|\nabla^2 a\|_{L^2}\|\nabla^3 u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 a\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2). \end{aligned}$$

For $k=3$, we have

$$J_6 \simeq ((\mathfrak{f}''(a)(\nabla a)^2 + \mathfrak{f}'(a)\nabla^2 a)\partial_i a + \mathfrak{f}'(a)\nabla a \nabla\partial_i a + \mathfrak{f}(a)\nabla^2\partial_i a \,|\, \nabla^4 u),$$

and so

$$\begin{aligned} J_6 &\lesssim \Big(\|(\nabla a)^3\|_{L^2} + \|(\nabla^2 a)\partial_i a\|_{L^2} + \|(\nabla a)(\nabla\partial_i a)\|_{L^2} + \|a\|_{L^\infty}\|\nabla^2\partial_i a\|_{L^2}\Big)\|\nabla^4 u\|_{L^2} \\ &\lesssim \Big(\|\nabla a\|_{L^6}^{3/2}\|\nabla a\|_{L^6}^{3/2} + \|\nabla a\|_{L^3}\|\nabla^2 a\|_{L^6} + \|a\|_{L^\infty}\|\nabla^3 a\|_{L^2}\Big)\|\nabla^4 u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2), \end{aligned}$$

where we again use (6.2) with $p=3$, $q,r=2$, $\alpha=1, m=0, l=2$, the standard case $p=6$, $q,r=2$, $\alpha=0, m=1, l=1$, and also the following other particular case of (6.2)

$$\|\nabla f\|_{L^6} \lesssim \|f\|_{L^2}^{1/3}\|\nabla^3 f\|_{L^2}^{2/3}.$$

As to J_7, J_8, we first write

$$J_7 \simeq -(\nabla^{k-1}(\mathfrak{h}(a)\partial_{ij}^2 u) \,|\, \nabla^{k+1}u), \quad J_8 \simeq -(\nabla^{k-1}(u\partial_i u) \,|\, \nabla^{k+1}u).$$

So, for $k=1$, we have

$$J_7 \simeq (\mathfrak{h}(a)\partial_{ij}^2 u \,|\, \nabla^2 u), \quad J_8 \simeq (u\partial_i u \,|\, \nabla^2 u),$$

then, trivially,

$$\begin{aligned} J_7 &\lesssim \|a\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla^2 u\|_{L^2}, \\ J_8 &\lesssim \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2). \end{aligned}$$

For $k=2$, we have

$$J_7 \simeq (\mathfrak{h}'(a)\nabla a\partial_{ij}^2 u + \mathfrak{h}(a)\nabla\partial_{ij}^2 u \,|\, \nabla^3 u), \quad J_8 \simeq (\nabla u\partial_i u + u\nabla\partial_i u \,|\, \nabla^3 u),$$

and, so,

$$J_7 \lesssim (\|\nabla a\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \|a\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^3 u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2),$$
$$J_8 \lesssim (\|\nabla u\|_{L^3}\|\nabla u\|_{L^6} + \|u\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^3 u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2),$$

where besides the estimates already used, we use

$$\|\nabla f\|_{L^3} \le \|f\|_{L^2}^{1/2}\|\nabla^2 f\|_{L^2}^{1/2}$$

always as application of Gagliardo–Nirenberg inequalities (6.2).

For $k=3$, we have

$$J_7 \simeq ((\mathfrak{h}''(a)(\nabla a)^2 + \mathfrak{h}'(a)\nabla^2 a)\partial_{ij}^2 u \,|\, \nabla^4 u), \quad J_8 \simeq (\nabla^2 u\partial_i u + \nabla u\nabla\partial_i u \,|\, \nabla^4 u),$$
$$J_7 \lesssim (\|\nabla a\|_{L^6}^2\|\nabla^2 u\|_{L^6} + \|\nabla^2 a\|_{L^3}\|\nabla^2 u\|_{L^6})\|\nabla^4 u\|_{L^2}$$
$$\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2),$$
$$J_8 \lesssim \|\nabla u\|_{L^3}\|\nabla^2 u\|_{L^6}\|\nabla^4 u\|_{L^2}$$
$$\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2),$$

using inequalities already used above by application on (6.2).

We now analyze the terms $J_3, J_4, J_5, J_9, J_{10}$ which together with those already analyzed complete the list of the terms not involving w. They are treated in a manner very similar to the one used for the terms already considered, so, it will suffice to consider the representative case of J_4. So, let us consider $J_4 \simeq -(\nabla^{k-1}(\mathfrak{h}(a)B\partial_i B \,|\, \nabla^{k+1}u)$. For $k=1$, we have

$$J_4 \simeq (\mathfrak{h}(a)B\partial_i B \,|\, \nabla^2 u) \lesssim \|a\|_{L^6}\|B\|_{L^6}\|\nabla B\|_{L^6}\|\nabla^2 u\|_{L^2}$$
$$\simeq \sqrt{\mathcal{E}_0^3}(\|\nabla^2 B\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).$$

For $k=2$, we have

$$J_4 \simeq (\mathfrak{h}'(a)\nabla a B\partial_i B + \mathfrak{h}(a)\nabla B\partial_i B + \mathfrak{h}(a)B\nabla\partial_i B \,|\, \nabla^3 u)$$
$$\lesssim \Big(\|\nabla a\|_{L^6}\|B\|_{L^6}\|\nabla B\|_{L^6} + \|a\|_{L^6}\|\nabla B\|_{L^6}\|\nabla B\|_{L^6} + \|a\|_{L^\infty}\|B\|_{L^\infty}\|\nabla^2 B\|_{L^2}\Big)\|\nabla^3 u\|_{L^2}$$
$$\lesssim \Big(\|\nabla^2 a\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2} + \|\nabla a\|_{L^2}\|\nabla^2 B\|_{L^2}^{1/2}\|B\|_{L^2}^{1/2}\|\nabla^3 B\|_{L^2}$$
$$+ \|a\|_{L^\infty}\|B\|_{L^\infty}\|\nabla^2 B\|_{L^2}\Big)\|\nabla^3 u\|_{L^2}$$
$$\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 B\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2).$$

For $k=3$, we have

$$
\begin{aligned}
J_4 &\simeq \Big((\mathfrak{h}''(a)(\nabla a)^2+\mathfrak{h}'(a)\nabla^2 a)B\partial_i B+\mathfrak{h}(a)(\nabla^2 B\partial_i B+\nabla B\nabla\partial_i B+B\nabla^2\partial_i B)\,|\,\nabla^4 u\Big)\\
&\lesssim \Big(\|\nabla a\|_{L^6}^{1/2}\|\nabla a\|_{L^6}^{3/2}\|B\|_{L^\infty}\|\nabla B\|_{L^6}+\|\nabla^2 a\|_{L^6}\|B\|_{L^6}\|\nabla B\|_{L^6}\\
&\quad+\|a\|_{L^6}\|\nabla^2 B\|_{L^6}\|\nabla B\|_{L^6}+\|a\|_{L^6}\|B\|_{L^6}\|\nabla^3 B\|_{L^6}\Big)\|\nabla^4 u\|_{L^2}\\
&\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 a\|_{L^2}^2+\|\nabla^3 B\|_{L^2}^2+\|\nabla^4 B\|_{L^2}^2+\|\nabla^4 u\|_{L^2}^2).
\end{aligned}
$$

Now we consider the terms involving w, i.e., J_{11}, J_{12}. We first notice that

$$
\begin{aligned}
&\frac{1}{1+a}\partial_j(g'(\frac{1}{1+a}))h(|w|^2)\sim a(\partial_j a)|w|^4+(\partial_j a)|w|^4,\\
&\frac{1}{1+a}g'(\frac{1}{1+a})\partial_i(h(|w|^2))\sim a|w|^2\partial_j(|w|^2)+|w|^2\partial_j(|w|^2),
\end{aligned}
$$

where $j=1,2,3$. We then have

$$
J_{11}\lesssim L_1+L_2,\quad J_{12}\lesssim L_3+L_4,
$$

where

$$
\begin{aligned}
&L_1\simeq|\Big(\nabla^k(a\cdot\nabla a|w|^4)\,|\,\nabla^k u\Big)|,\quad L_2\simeq|\Big(\nabla^k(\nabla a|w|^4)\,|\,\nabla^k u\Big)|,\\
&L_3\simeq|\Big(\nabla^k(a|w|^2\nabla(|w|^2))\,|\,\nabla^k u\Big)|,\quad L_4=|\Big(\nabla^k(|w|^2\nabla(|w|^2))\,|\,\nabla^k u\Big)|.
\end{aligned}
$$

We first consider L_1,

$$
L_1\simeq|\Big(\nabla^{k-1}(a\nabla a|w|^4)\,|\,\nabla^{k+1}u\Big)|\lesssim\|\nabla^{k-1}(a\nabla a|w|^4)\|_{L^2}\|\nabla^{k+1}u\|_{L^2}. \tag{3.11}
$$

For $k=1$, we have

$$
\begin{aligned}
L_1&\lesssim\|(a\nabla a|w|^4)\|_{L^2}\|\nabla^2 u\|_{L^2}\\
&\lesssim\|w\|_{L^\infty}^4\|a\|_{L^\infty}\|\nabla a\|_{L^2}\|\nabla^2 u\|_{L^2}\\
&\lesssim\sqrt{\mathcal{E}_0^3}\,\Big(\|\nabla a\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\Big).
\end{aligned}\tag{3.12}
$$

For $k=2$,

$$
\begin{aligned}
L_1&\lesssim\Big(\|\nabla(a\nabla a)\|_{L^2}\|w\|_{L^\infty}^4+\|(a\nabla a)\cdot\nabla(|w|^4)\|_{L^2}\Big)\|\nabla^3 u\|_{L^2}\\
&\lesssim\|w\|_{L^\infty}^4(\|\nabla a\|_{L^3}\|\nabla a\|_{L^6}+\|a\|_{L^\infty}\|\nabla^2 a\|_{L^2})\|\nabla^3 u\|_{L^2}\\
&\qquad+\|w\|_{L^\infty}^3\|(a\nabla a)\|_{L^\infty}\|\nabla w\|_{L^2}\|\nabla^3 u\|_{L^2}\\
&\lesssim\sqrt{\mathcal{E}_0^3}\,\|w\|_{L^\infty}^4\|\nabla^2 a\|_{L^2}\|\nabla^3 u\|_{L^2}+\sqrt{\mathcal{E}_0^3}\|w\|_{L^\infty}^3\|\nabla w\|_{L^2}\|\nabla^3 u\|_{L^2}\\
&\lesssim\sqrt{\mathcal{E}_0^3}\,\Big(\|\nabla^2 a\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2+(1+t)^{-9}\|\nabla w\|_{L^2}^2\Big).
\end{aligned}\tag{3.13}
$$

For $k=3$, we need to deal with

$$\|\nabla^2(a\nabla a|w|^4)\|_{L^2}\|\nabla^4 u\|_{L^2} \lesssim (\|\nabla^3(a^2)\|_{L^2}\|w\|_{L^\infty}^4 + \|a\nabla a\|_{L^\infty}\|\nabla^2(|w|^4)\|_{L^2})\|\nabla^4 u\|_{L^2}.$$

Now,

$$\begin{aligned}
\|\nabla^3(a^2)\|_{L^2} &\lesssim \|a\nabla^3 a\|_{L^2} + \|(\nabla a)(\nabla^2 a)\|_{L^2} \\
&\lesssim \|a\|_{L^\infty}\|\nabla^3 a\|_{L^2} + \|\nabla a\|_{L^3}\|\nabla^2 a\|_{L^6} \\
&\lesssim (\|a\|_{L^\infty} + \|\nabla a\|_{L^3})\|\nabla^3 a\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla^2(|w|^4)\|_{L^2} &\lesssim \|\nabla(|w|^3\nabla w)\|_{L^2} \\
&\lesssim \||w|^2|\nabla w|^2\|_{L^2} + \||w|^3\nabla^2 w\|_{L^2} \\
&\lesssim \|w\|_{L^\infty}^2\|\nabla w\|_{L^3}\|\nabla w\|_{L^6} + \|w\|_{L^\infty}^2\|w\|_{L^3}\|\nabla^2 w\|_{L^6} \\
&\lesssim \|w\|_{L^\infty}^2\big(\|\nabla w\|_{L^3} + \|w\|_{L^3}\big)\|\nabla^2 w\|_{H^1}.
\end{aligned}$$

Therefore, for $k=3$,

$$L_1 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2 + (1+t)^{-6}\|\nabla^2 w\|_{H^1}^2\right).$$

Hence, we have that for $k=1,2,3$,

$$L_1 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^k a\|_{H^1}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + (1+t)^{-6}\|\nabla^{k-1} w\|_{H^1}^2\right).$$

For the term L_2, we have

$$\begin{aligned}
L_2 &\simeq |\left(\nabla^{k-1}(\nabla a|w|^4)\,|\,\nabla^{k+1}u\right)| \\
&\lesssim \left(\|\nabla a\|_{L^\infty}\|\nabla^{k-1}(|w|^4)\|_{L^2} + \|w\|_{L^\infty}^4\|\nabla^k a\|_{L^2}\right)\|\nabla^{k+1}u\|_{L^2} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^k a\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + (1+t)^{-9}\|\nabla^{k-1}w\|_{L^2}^2\right).
\end{aligned}$$

For the term L_3, we have

$$\begin{aligned}
L_3 &\simeq |\left(\nabla^k(a\nabla(|w|^4))\,|\,\nabla^k u\right)| \\
&\lesssim |\left(\nabla^k(\nabla a|w|^4)\,|\,\nabla^k u\right)| + |\left(\nabla^k(a|w|^4)\,|\,\nabla^{k+1}u\right)| \\
&:= L_{31} + L_{32}.
\end{aligned}$$

The term L_{31} is a constant multiple of I_2, while L_{32} can be estimated as follows

$$L_{32} \leq \left(\|a\|_{L^\infty} \|\nabla^k (|w|^4)\|_{L^2} + \|w\|_{L^\infty}^4 \|\nabla^k a\|_{L^2} \right) \|\nabla^{k+1} u\|_{L^2}$$
$$\lesssim \sqrt{\mathcal{E}_0^3} \left(\|\nabla^k a\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + (1+t)^{-9} \|\nabla^k w\|_{L^2}^2 \right).$$

So at this point, we have the following estimates about L_3

$$L_3 \lesssim \sqrt{\mathcal{E}_0^3} \left(\|\nabla^k a\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + (1+t)^{-9} (\|\nabla^{k-1} w\|_{L^2}^2 + \|\nabla^k w\|_{L^2}^2) \right).$$

For the term L_4, we have

$$L_4 \lesssim \|w\|_{L^\infty}^3 \|\nabla^k w\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3} \left(\|\nabla^{k+1} u\|_{L^2}^2 + (1+t)^{-6} \|\nabla^k w\|_{L^2}^2 \right).$$

Combining all the estimates for J_1 through J_{10} and L_1 through L_4 for $J_{11} + J_{12}$, we finally complete the proof of this lemma. □

The next lemma offers estimates that complement those gave by Lemma 3.1.

Lemma 3.2. *If $\sqrt{\mathcal{E}_0^3} \leq \delta$, $\|w \circ Y\|_{L^\infty} \simeq \|w\|_{L^\infty}$ and $\|w \circ Y\|_{H^l} \simeq \|w\|_{H^l}$ with $l = k-1 (k \geq 1)$, $k, k+1$, then for $k = 0, 1, 2$ we have*

$$\frac{d}{dt} \left(\|\nabla^k a\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k B\|_{L^2}^2 \right) + \frac{1}{2} \left(\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right)$$
$$\lesssim \sqrt{\mathcal{E}_0^3} \left(\|\nabla^{k+1}(a, u, B)\|_{L^2}^2 + (1+t)^{-6} \|(k\nabla^{k-1} w, \nabla^k w)\|_{L^2}^2 \right),$$

where $k\nabla^{k-1} w$ is used to confirm it is zero when $k = 0$.

Proof. We first recall (3.10) in the proof of Lemma 3.1,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^k a|^2 + |\nabla^k u|^2 + |\nabla^k B|^2 \, dx + \int_{\mathbb{R}^3} \mu |\nabla^{k+1} u|^2 \, dx \\ &\quad + \int_{\mathbb{R}^3} (\lambda + \mu) |\nabla^k \mathrm{div} u|^2 + \nu |\nabla^{k+1} B|^2 \, dx = J_1 + \cdots + J_{12}, \end{aligned} \tag{3.14}$$

however, here we need to give different estimates. We start with J_1. If $k = 0$, we have

$$J_1 \lesssim \|a\|_{L^3} \|\nabla u\|_{L^2} \|a\|_{L^6} \leq \sqrt{\mathcal{E}_0^3} \left(\|\nabla u\|_{L^2}^2 + \|\nabla a\|_{L^2}^2 \right),$$

if $k \geq 1$, we have

$$\begin{aligned} J_1 &\lesssim \sum_{0 \leq l \leq k-1} \|\nabla^l a \nabla^{k-l-1} \partial_i u\|_{L^2} \|\nabla^{k+1} a\|_{L^2} \\ &\lesssim \sum_{0 \leq l \leq k-1} T(J_1, k, l). \end{aligned}$$

$$
\begin{aligned}
T(J_1,1,0) &\simeq \|a(\partial_i u)\|_{L^2}\|\nabla^2 a\|_{L^2} \lesssim \|a\|_{L^3}\|\nabla u\|_{L^6}\|\nabla^2 a\|_{L^2} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\,(\|\nabla^2 a\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).
\end{aligned}
$$

$$
\begin{aligned}
T(J_1,2,0) &\simeq \|a\nabla\partial_i u\|_{L^2}\|\nabla^3 a\|_{L^2} \\
&\lesssim \|a\|_{L^3}\|\nabla^2 u\|_{L^6}\|\nabla^3 a\|_{L^2} \\
&\lesssim \|a\|_{L^3}\|\nabla^3 u\|_{L^2}\|\nabla^3 a\|_{L^2} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\,\left(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2\right),
\end{aligned}
$$

$$
\begin{aligned}
T(J_1,2,1) &\simeq \|\nabla a\partial_i u\|_{L^2}\|\nabla^3 a\|_{L^2} \\
&\lesssim \|\nabla a\|_{L^4}\|\nabla u\|_{L^4}\|\nabla^3 a\|_{L^2} \\
&\lesssim \|a\|_{L^3}^{\frac{1}{2}}\|\nabla^3 a\|_{L^2}^{\frac{1}{2}}\|u\|_{L^3}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 a\|_{L^2} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\,\left(\|\nabla^3 a\|_{L^2} + \|\nabla^3 u\|_{L^2}\right)\|\nabla^3 a\|_{L^2} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2\right).
\end{aligned}
$$

Therefore, we summarize the above estimates for J_1 as

$$
J_1 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1} a\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2\right). \tag{3.15}
$$

Similarly, we can obtain the following estimates on J_2, J_6, J_7, J_8,

$$
J_2, J_6 \lesssim \sqrt{\mathcal{E}_0^3}\,\left(\|\nabla^{k+1} a\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2\right), \quad J_7, J_8 \lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla^{k+1} u\|_{L^2}^2. \tag{3.16}
$$

Analogously, we obtain

$$
J_3, J_4, J_5, J_9, J_{10} \lesssim \sqrt{\mathcal{E}_0^3}\,\left(\|\nabla^{k+1} a\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2\right). \tag{3.17}
$$

Since by now the estimates are routine matter, we just consider the case of J_3. For $k = 0$, we have

$$
\begin{aligned}
J_3 &\lesssim \|\mathfrak{h}(a)\|_{L^\infty}\|u\|_{L^3}\|B\|_{L^6}\|\nabla B\|_{L^2} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla B\|_{L^2}^2.
\end{aligned} \tag{3.18}
$$

If $k \geq 1$, we have

$$
\begin{aligned}
J_3 &\lesssim \left(\nabla^{k-1}(\mathfrak{h}(a)B\partial_i B) \mid \nabla^{k+1} u\right) \\
&\lesssim \sum_{0\leq l\leq k-1} \|\nabla^l \mathfrak{h}(a)\nabla^{k-l-1}(B\partial_i B)\|_{L^2}\|\nabla^{k+1} u\|_{L^2} \\
&:= \sum_{0\leq l\leq k-1} T(J_{31},k,l).
\end{aligned} \tag{3.19}
$$

These terms has been carried out in Lemma 3.1. We can simply utilize those estimates except for $T(J_{31}, 2, 1)$, because of the mismatch of the order therein. So, we do as follows

$$\begin{aligned} T(J_{31}, 2, 1) &\lesssim \|\nabla(\mathfrak{h}(a))(B\partial_i B)\|_{L^2}\|\nabla^3 u\|_{L^2} \\ &\lesssim \|(\nabla a)(B\partial_i B)\|_{L^2}\|\nabla^3 u\|_{L^2} \\ &\lesssim \|B\|_{L^\infty}\|\nabla a\|_{L^4}\|\nabla B\|_{L^4}\|\nabla^3 u\|_{L^2} \\ &\lesssim \|B\|_{L^\infty}(\|B\|_{L^3}^{\frac{1}{2}}\|\nabla^3 B\|_{L^2}^{\frac{1}{2}}\|a\|_{L^3}^{\frac{1}{2}}\|\nabla^3 a\|_{L^2}^{\frac{1}{2}})\|\nabla^3 u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 B\|_{L^2} + \|\nabla^3 a\|_{L^2})\|\nabla^3 u\|_{L^2}^2 \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2). \end{aligned} \tag{3.20}$$

Therefore, we conclude from (3.18)–(3.20) that, for $k = 0, 1, 2$, it holds

$$J_3 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1} a\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2\right). \tag{3.21}$$

We finally consider the terms involving w, i.e., J_{11}, J_{12}. As in the proof of Lemma 3.1, we decompose $J_{11} + J_{12} \lesssim L_1 + L_2 + L_3 + L_4$. For L_1, since those estimates carried out in (3.12)–(3.13) have different orders, we are going to work it through in different ways. For $k = 0$, we know that

$$L_1 \lesssim \|a\|_{L^3}\|w\|_{L^\infty}^4\|\nabla a\|_{L^2}\|u\|_{L^6} \tag{3.22}$$

$$\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla a\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right). \tag{3.23}$$

For $k = 1$, using integration by parts, we have

$$\begin{aligned} L_1 &\lesssim \|(a\nabla a)|w|^4\|_{L^2}\|\nabla^2 u\|_{L^2} \\ &\lesssim \|w\|_{L^\infty}^4\|a\|_{L^3}\|\nabla a\|_{L^6}\|\nabla^2 u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^2 a\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right). \end{aligned} \tag{3.24}$$

For $k = 2$, we have

$$\begin{aligned} L_1 &\lesssim \left(\|\nabla(a\nabla a)\|_{L^2}\|w\|_{L^\infty}^4 + \|(a\nabla a)\cdot\nabla(|w|^4)\|_{L^2}\right)\|\nabla^3 u\|_{L^2} \\ &\lesssim \|w\|_{L^\infty}^4(\|\nabla a\|_{L^4}^2 + \|a\nabla^2 a\|_{L^2})\|\nabla^3 u\|_{L^2} \\ &\qquad + \|w\|_{L^\infty}^3\|(a\nabla a)\|_{L^\infty}\|\nabla w\|_{L^2}\|\nabla^3 u\|_{L^2} \\ &\lesssim \|w\|_{L^\infty}^4\|a\|_{L^3}\|\nabla^3 a\|_{L^2}\|\nabla^3 u\|_{L^2} + \sqrt{\mathcal{E}_0^3}\|w\|_{L^\infty}^3\|\nabla w\|_{L^2}\|\nabla^3 u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^3 a\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + (1+t)^{-9}\|\nabla w\|_{L^2}^2\right). \end{aligned} \tag{3.25}$$

Therefore, we have for all $k = 0, 1, 2$, the following estimate on L_1,

$$L_1 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+(1+t)^{-9}\|k\nabla^{k-1}w\|_{L^2}^2\right). \tag{3.26}$$

For the term L_2, we have when $k=0$ that

$$L_2 \lesssim \|u\|_{L^6}\|\nabla a\|_{L^2}\|w\|_{L^\infty}^3\|w\|_{L^3} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla a\|_{L^2}^2+\|\nabla u\|_{L^2}^2),$$

when $k=1$, using integration by parts,

$$\begin{aligned}L_2 &\lesssim \|w\|_{L^\infty}^3\|w\|_{L^3}\|\nabla a\|_{L^6}\|\|\nabla^2 u\|_{L^2}\\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 a\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2),\end{aligned}$$

and when $k=2$, using integration by parts,

$$\begin{aligned}L_2 &\le \|\nabla(\nabla a|w|^4)\|_{L^2}\|\nabla^3 u\|_{L^2}\\ &\lesssim \|w\|_{L^\infty}^3(\|\nabla a\|_{L^\infty}\|\nabla w\|_{L^2}+\|\nabla^2 a\|_{L^6}\|w\|_{L^3})\|\nabla^3 u\|_{L^2},\end{aligned}$$

therefore, we obtain for each $k=0,1,2$, the following estimates on I_2,

$$L_2 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+(1+t)^{-9}\|k\nabla^{k-1}w\|_2^2\right).$$

For the term L_3, like in the proof of Lemma 3.1, it can be bounded by L_{31} and L_{32}, where L_{31} is a constant multiple of L_2 and L_{32} is defined by

$$L_{32} \simeq \left(\nabla^k(a|w|^4)\,|\,\nabla^{k+1}u\right).$$

It is easy to see, for $k=0$, that

$$L_{32} \lesssim \|a\|_{L^6}\|w\|_{L^\infty}^3\|w\|_{L^3}\|\nabla u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla a\|_{L^2}^2+\|\nabla u\|_{L^2}^2),$$

and for $k=1$ that

$$\begin{aligned}L_{32} &\lesssim \|w\|_{L^\infty}^3(\|\nabla a\|_{L^6}\|w\|_{L^3}+\|a\|_{L^\infty}\|\nabla w\|_{L^2})\|\nabla^2 u\|_{L^2}\\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^2 a\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+(1+t)^{-9}\|\nabla w\|_{L^2}^2).\end{aligned}$$

For $k=2$, direct calculation shows

$$\begin{aligned}L_{32} &\lesssim \|w\|_{L^\infty}^3(\|\nabla^2 a\|_{L^6}\|w\|_{L^3}+\|\nabla a\|_{L^3}\|\nabla w\|_{L^6})\|\nabla^3 u\|_{L^2}\\ &\quad+\|a\|_{L^\infty}(\|w\|_{L^\infty}^3+\|w\|_{L^\infty}^2\|\nabla w\|_{L^\infty})(\|\nabla^2 w\|_{L^2}+\|\nabla w\|_{L^2})\|\nabla^3 u\|_{L^2}\\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 a\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2+(1+t)^{-6}\|(\nabla w,\nabla^2 w)\|_{L^2}^2).\end{aligned}$$

So we can get the estimates about I_3 as follows

$$L_3 \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + (1+t)^{-6}\|(k\nabla^{k-1}w, \nabla^k w\|_{L^2}^2\right).$$

Since L_4 is similar in form to a term contained in L_3, we have

$$\begin{aligned} L_4 &\lesssim \|\nabla^k(|w|^4)\|_{L^2}\|\nabla^{k+1}u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}u\|_{L^2}^2 + (1+t)^{-6}\|(k\nabla^{k-1}w, \nabla^k w\|_{L^2}^2\right). \end{aligned}$$

Combining the estimates about L_1 to L_4, we have

$$J_{11} + J_{12} \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + (1+t)^{-6}\|(k\nabla^{k-1}w, \nabla^k w)\|_{L^2}^2\right).$$

Combining all the estimates on J_1 to J_{12}, we finally complete the proof. □

Lemma 3.3. *If* $\sqrt{\mathcal{E}_0^3} \le \delta$, $\|w \circ Y\|_{L^\infty} \simeq \|w\|_{L^\infty}$ *and* $\|w \circ Y\|_{H^l} \simeq \|w\|_{H^l}$ *with* $l = k, k+1$, *then for* $k = 0, 1, 2$,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u \nabla\nabla^k a\, dx + \frac{1}{2}\|\nabla^{k+1}a\|_{L^2}^2 \\ &\lesssim \|\nabla^{k+1}(u, B)\|_{H^1}^2 + C\sqrt{\mathcal{E}_0^3}(1+t)^{-6}\|\nabla^k w\|_{H^1}^2. \end{aligned}$$

Proof. For $k = 0, 1, 2$, from the momentum equation of (1.34), we know that

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla^{k+1}a|^2\, dx \le &-\int_{\mathbb{R}^3} \nabla^k u_t \nabla\nabla^k a\, dx + C\|\nabla^{k+2}u\|_{L^2}\|\nabla^{k+1}a\|_{L^2} \\ &+ C\|\nabla^{k+1}B\|_{L^2}\|\nabla^{k+1}a\|_{L^2} + (J_u + J_w)\|\nabla^{k+1}a\|_{L^2}, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} J_u \simeq\ &\|\nabla^k(\mathfrak{h}(a)(\bar{B}\cdot\nabla)B\|_{L^2} + \|\nabla^k(\mathfrak{f}(a)\nabla a)\|_{L^2} + \|\nabla^k(u\cdot\nabla u)\|_{L^2} \\ &+ \|\nabla^k(\mathfrak{h}(a)(\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u))\|_{L^2} + \|\nabla^k((\mathfrak{h}(a)-1)B\cdot\nabla B)\|_{L^2} \\ :=\ &J_{u1} + J_{u2} + J_{u3} + J_{u4} + J_{u5}, \end{aligned}$$

and

$$\begin{aligned} J_w \simeq\ &\|\nabla^k(a\nabla a|w|^4)\|_{L^2} + \|\nabla^k(\nabla a|w|^4)\|_{L^2} \\ &+ \|\nabla^k(a\nabla(|w|^4))\|_{L^2} + \|\nabla^{k+1}(|w|^4)\|_{L^2}, \\ :=\ &J_{w1} + J_{w2} + J_{w3} + J_{w4}. \end{aligned}$$

The term $-\int_{\mathbb{R}^3} \nabla^k u_t \nabla\nabla^k a\, dx$ in (3.27) appears in compressible Navier–Stokes equations, for which we have

$$\begin{aligned}
-\int\limits_{\mathbb{R}^3} \nabla^k u_t \nabla\nabla^k a\,dx &= -\frac{d}{dt}\int\limits_{\mathbb{R}^3} \nabla^k u\nabla\nabla^k a\,dx - \int\limits_{\mathbb{R}^3} \nabla^k \mathrm{div}u\nabla^k a_t\,dx \\
&= -\frac{d}{dt}\int\limits_{\mathbb{R}^3} \nabla^k u\nabla\nabla^k a\,dx + \|\nabla^k \mathrm{div}u\|_{L^2}^2 \\
&\qquad + \int\limits_{\mathbb{R}^3} \nabla^k \mathrm{div}u\nabla^k \mathrm{div}(au)\,dx.
\end{aligned} \tag{3.28}$$

Then we use the same estimates as carried out in [13] for the Navier–Stokes to obtain

$$-\int\limits_{\mathbb{R}^3} \nabla^k u_t \nabla\nabla^k a\,dx \le -\frac{d}{dt}\int\limits_{\mathbb{R}^3} \nabla^k u\nabla\nabla^k a\,dx + C\|\nabla^{k+1}u\|_{L^2}^2 + C\sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}a\|_{L^2}^2.$$

We now estimate those terms in J_u and J_w.

Term J_{u1}:

$$J_{u1} \le C\sum_{0\le l\le k} \|\nabla^l \mathfrak{h}(a)\nabla^{k-l}(\bar{B}\cdot\nabla)B\|_{L^2} := C\sum_{0\le l\le k} T(J_{u1},k,l),$$

where

$$\begin{aligned}
T(J_{u1},0,0) &\lesssim \|\mathfrak{h}(a)\|_{L^3}\|(\bar{B}\cdot\nabla)B\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla^2 B\|_{L^2}. \\
T(J_{u1},1,0)+T(J_{u1},1,1) &\le \|\nabla\mathfrak{h}(a)(\bar{B}\cdot\nabla)B\|_{L^2} + \|\mathfrak{h}(a)\nabla(\bar{B}\cdot\nabla)B\|_{L^2} \\
&\lesssim \|\nabla\mathfrak{h}(a)\|_{L^6}\|\nabla B\|_{L^3} + \|\mathfrak{h}(a)\|_{L^3}\|\nabla^2 B\|_{L^6} \\
&\lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^2 a\|_{L^2} + \|\nabla^3 B\|_{L^2}\right). \\
T(J_{u1},2,0) &\lesssim \|\mathfrak{h}(a)\|_{L^\infty}\|\nabla^3 B\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 B\|_{L^2}, \\
T(J_{u1},2,1) &\lesssim \|\nabla a\|_{L^2}\|\nabla^3 B\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 B\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
T(J_{u1},2,2) &\lesssim \|\nabla^2 a\|_{L^6}\|\nabla B\|_{L^3} + \|\nabla a\|_{L^\infty}\|\nabla a\|_{L^4}\|\nabla B\|_{L^4} \\
&\lesssim \|\nabla B\|_{L^3}\|\nabla^3 a\|_{L^2} + \|\nabla a\|_{L^\infty}\|a\|_{L^3}^{\frac12}\|\nabla^3 a\|_{L^2}^{\frac12}\|B\|_{L^3}^{\frac12}\|\nabla^3 B\|_{L^2}^{\frac12} \\
&\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 a\|_{L^2} + \|\nabla^3 B\|_{L^2}).
\end{aligned}$$

Therefore, for $k=0,1,2$, we have arrived at

$$J_{u1} \lesssim \sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2} + \|\nabla^{k+1}B\|_{L^2} + \|\nabla^{k+2}B\|_{L^2}\right).$$

Term J_{u2}:

$$J_{u2} \lesssim \sum_{0\le l\le k} \|\nabla^l \mathfrak{f}(a)\nabla^{k-l+1}a\|_{L^2} := \sum_{0\le l\le k} T(J_{u2},k,l),$$

where

$$\begin{aligned}
T(J_{u2},0,0) &\lesssim \|\mathfrak{f}(a)\|_{L^\infty}\|\nabla a\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla a\|_{L^2},\\
T(J_{u2},1,0)+T(J_{u2},1,1) &\lesssim \|\nabla\mathfrak{f}(a)\nabla a\|_{L^2} + \|\mathfrak{f}(a)\nabla^2 a\|_{L^2}\\
&\lesssim \|\nabla a\|_{L^3}\|\nabla a\|_{L^6} + \|\mathfrak{f}(a)\|_{L^\infty}\|\nabla^2 a\|_{L^2}\\
&\lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla^2 a\|_{L^2},\\
T(J_{u2},2,0) &\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2},\\
T(J_{u2},2,1) &\lesssim \|\nabla a\|_{L^3}\|\nabla^2 a\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
T(J_{u2},2,2) &\lesssim \|\nabla a\|_{L^\infty}\|\nabla a\|_{L^4}^2 + \|\nabla^2 a\|_{L^6}\|\nabla a\|_{L^3}\\
&\lesssim \|\nabla a\|_{L^\infty}\|a\|_{L^3}\|\nabla^3 a\|_{L^2} + \|\nabla a\|_{L^3}\|\nabla^3 a\|_{L^2}\\
&\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2},
\end{aligned}$$

therefore, we have for $k=0,1,2$, the following estimate

$$J_{u2} \lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla^{k+1}a\|_{L^2}.$$

Term J_{u3}:

$$J_{u3} \lesssim \sum_{0\le l\le k} \|\nabla^l u\nabla^{k-l+1}u\|_{L^2} := \sum_{0\le l\le k} T(J_{u3},k,l),$$

where

$$\begin{aligned}
T(J_{u3},0,0) &\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla u\|_{L^2},\\
T(J_{u3},1,0)+T(J_{u3},1,1) &\lesssim \|u\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla u\|_{L^6}\\
&\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^2 u\|_{L^2},\\
T(J_{u3},2,0) &\lesssim \|u\|_{L^\infty}\|\nabla^3 u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 u\|_{L^2},\\
T(J_{u3},2,1) &\lesssim \|\nabla u\|_{L^3}\|\nabla^2 u\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 u\|_{L^2},
\end{aligned}$$

and

$$T(J_{u3},2,2)\lesssim\|\nabla^2u\|_{L^6}\|\nabla u\|_{L^3}\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^3u\|_{L^2}.$$

Hence, we finally obtain, for $k=0,1,2$, that

$$J_{u3}\lesssim\sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}u\|_{L^2}+\|\nabla^{k+2}u\|_{L^2}\right).$$

Term J_{u4}:

$$J_{u4}\lesssim\sum_{0\le l\le k}\|\nabla^l\mathfrak{h}(a)\nabla^{k-l+2}u\|_{L^2}:=\sum_{0\le l\le k}T(J_{u4},k,l),$$

where

$$\begin{aligned}T(J_{u4},0,0)&\lesssim\|\mathfrak{h}(a)\|_{L^\infty}\|\nabla^2u\|_{L^2}\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^2u\|_{L^2},\\T(J_{u4},1,0)+T(J_{u4},1,1)&\lesssim\|\mathfrak{h}(a)\|_{L^\infty}\|\nabla^3u\|_{L^2}+\|\nabla\mathfrak{h}(a)\|_{L^3}\|\nabla^2u\|_{L^6}\\&\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^3u\|_{L^2}.\\T(J_{u4},2,0)+T(J_{u4},2,1)&\lesssim\|\mathfrak{h}(a)\|_{L^\infty}\|\nabla^4u\|_{L^2}+\|\nabla\mathfrak{h}(a)\|_{L^3}\|\nabla^3u\|_{L^6}\\&\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^4u\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}T(J_{u4},2,2)&\lesssim\|\nabla^2a\|_{L^3}\|\nabla^2u\|_{L^6}+\|\nabla a\|_{L^\infty}\|\nabla a\|_{L^3}\|\nabla^2u\|_{L^6}\\&\lesssim\|\nabla a\|_{L^6}^{\frac12}\|\nabla^3a\|_{L^2}^{\frac12}\|\nabla^3u\|_{L^2}+\|\nabla a\|_{L^\infty}\|\nabla a\|_{L^3}\|\nabla^3u\|_{L^2}\\&\lesssim\sqrt{\mathcal{E}_0^3}\,(\|\nabla^3u\|_{L^2}).\end{aligned}$$

Thus, it holds, for $k=0,1,2$, that

$$J_{u4}\lesssim\sqrt{\mathcal{E}_0^3}\,(\|\nabla^{k+1}u\|_{L^2}+|\nabla^{k+2}u\|_{L^2}).$$

Term J_{u5}:

$$J_{u5}\lesssim\sum_{0\le l\le k}\|\nabla^l\mathfrak{h}_1(a)\nabla^{k-l+1}(|B|^2)\|_{L^2}:=\sum_{0\le l\le k}T(J_{u5},k,l),$$

where $\mathfrak{h}_1(a)=\mathfrak{h}(a)-1$. Here, we proceed as follows,

$$\begin{aligned}T(J_{u5},0,0)&\lesssim\|\mathfrak{h}_1(a)\|_{L^\infty}\|B\|_{L^3}\|\nabla B\|_{L^6}\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^2B\|_{L^2}.\\T(J_{u5},1,0)&\lesssim\|\mathfrak{h}_1(a)\|_{L^\infty}\|\nabla B\|_{L^4}^2+\|\mathfrak{h}_1(a)\|_{L^\infty}\|B\|_{L^3}\|\nabla^2B\|_{L^6}\\&\lesssim\|B\|_{L^3}\|\nabla^3B\|_{L^2}+\|B\|_{L^3}\|\nabla^3B\|_{L^2}\\&\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^3B\|_{L^2},\end{aligned}$$

$$
\begin{aligned}
T(J_{u5},1,1) &\lesssim \|\nabla a\|_{L^3}\|B\|_{L^\infty}\|\nabla B\|_{L^6}\\
&\lesssim \sqrt{\mathcal{E}_0^3}\,(\|\nabla^2 B\|_{L^2}).
\end{aligned}
$$

$$
\begin{aligned}
T(J_{u5},2,0) &\lesssim \|\mathfrak{h}_1(a)\|_{L^\infty}\|\nabla^2(B\cdot\nabla B)\|_{L^2}\\
&\lesssim \|\nabla B\cdot\nabla^2 B\|_{L^2}+\|B\cdot\nabla^3 B\|_{L^2}\\
&\lesssim \|\nabla B\|_{L^3}\|\nabla^2 B\|_{L^6}+\|B\|_{L^3}\|\nabla^3 B\|_{L^6}\\
&\lesssim \sqrt{\mathcal{E}_0^3}\,(\|\nabla^3 B\|_{L^2}+\|\nabla^4 B\|_{L^2}),
\end{aligned}
$$

$$
\begin{aligned}
T(J_{u5},2,1) &\lesssim \|\nabla\mathfrak{h}_1(a)\nabla(B\cdot\nabla B)\|_{L^2}\\
&\lesssim \|\nabla a\|_{L^\infty}\|\nabla B\|_{L^4}^2+\|\nabla a\|_{L^\infty}\|B\|_{L^3}\|\nabla^2 B\|_{L^6}\\
&\lesssim \|\nabla a\|_{L^\infty}\|B\|_{L^3}\|\nabla^3 B\|_{L^2}\\
&\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^3 B\|_{L^2}),
\end{aligned}
$$

and

$$
\begin{aligned}
T(J_{u5},2,2) &= \|\nabla^2\mathfrak{h}_1(a)(B\cdot\nabla B)\|_{L^2}\\
&\lesssim \|\nabla a\|_{L^4}^2\|B\|_{L^\infty}\|\nabla B\|_{L^\infty}+\|\nabla^2 a\|_{L^6}\|B\|_{L^\infty}\|\nabla B\|_{L^3}\\
&\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2}.
\end{aligned}
$$

Summing up estimates for $k=0,1,2$, we know that

$$
J_{u5}\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}a\|_{L^2}+\|\nabla^{k+1}B\|_{L^2}+\|\nabla^{k+2}B\|_{L^2}).
$$

Combining the estimates about J_{u1} to J_{u5}, we find that

$$
J_u\lesssim \sqrt{\mathcal{E}_0^3}\,\Big(\|(\nabla^{k+1}a,\nabla^{k+1}u,\nabla^{k+1}B)\|_{L^2}+\|(\nabla^{k+2}u,\nabla^{k+2}B)\|_{L^2}\Big).
$$

Term J_{w1}: Using Lemma 2.7, if we can show the following estimate

$$
\|\nabla^k(a\nabla a)\|_{L^2}\lesssim \sqrt{\mathcal{E}_0^3}\,\|\nabla^{k+1}a\|_{L^2}, \tag{3.29}
$$

then we have

$$
\begin{aligned}
J_{w1}=\|\nabla^k(a\nabla a|w|^4)\|_{L^2} &\lesssim \|\nabla^k(a\nabla a)\|_{L^2}\|w\|_{L^\infty}^4+\|a\nabla a\|_{L^\infty}\|\nabla^k(|w|^4)\|_{L^2}\\
&\lesssim (1+t)^{-9/2}\sqrt{\mathcal{E}_0^3}\,\Big(\|\nabla^{k+1}a\|_{L^2}+\|\nabla^k w\|_{L^2}\Big),
\end{aligned}
$$

where for the term we used the following fact

$$
\|\nabla^k|w|^4\|_{L^2}\lesssim \|w\|_{L^\infty}^2\|\nabla^k|w|^2\|_{L^2}\lesssim \|w\|_{L^\infty}^3\|\nabla^k w\|_{L^2}.
$$

We now prove (3.29). In fact,

$$\|\nabla^k(a\nabla a)\|_{L^2} \lesssim \sum_{0\le l\le k} \|\nabla^l a \nabla^{k+1-l} a\|_{L^2} := \sum_{0\le l\le k} T(a,k,l),$$

where,

$$T(a,0,0) = \|a\nabla a\|_{L^2} \le \|a\|_{L^\infty}\|\nabla a\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla a\|_{L^2},$$

$$T(a,1,0) = \|a\nabla^2 a\|_{L^2} \le \|a\|_{L^\infty}\|\nabla^2 a\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^2 a\|_{L^2},$$

$$T(a,1,1) = \|\nabla a\nabla a\|_{L^2} \le \|\nabla a\|_{L^3}\|\nabla a\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^2 a\|_{L^2},$$

$$T(a,2,0) = \|a\nabla^3 a\|_{L^2} \le \|a\|_{L^\infty}\|\nabla^3 a\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2},$$

$$T(a,2,1) = \|\nabla a\nabla^2 a\|_{L^2} \le \|\nabla a\|_{L^3}\|\nabla^2 a\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2},$$

and

$$T(a,2,2) = \|\nabla a\nabla^2 a\|_{L^2} \le \|\nabla a\|_{L^3}\|\nabla^2 a\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2}.$$

Therefore, we completed the proof of (3.29), and hence the term J_{w1}.
Term J_{w2}: For the term J_{w2}, we simply have

$$\begin{aligned} J_{w2} &= \|\nabla^k(\nabla a|w|^4)\|_{L^2} \\ &\lesssim \|\nabla^{k+1}a\|_{L^2}\|w\|_{L^\infty}^4 + \|\nabla a\|_{L^\infty}\|\nabla^k(|w|^4)\|_{L^2} \\ &\lesssim (1+t)^{-9/2}\sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2} + \|\nabla^k w\|_{L^2}\right). \end{aligned}$$

Term J_{w3}: For the term J_{w3}, we have

$$J_{w3} \lesssim \sum_{0\le l\le k} \|\nabla^l a\nabla^{k-l+1}(|w|^4)\|_{L^2} := \sum_{0\le l\le k} T(J_{w3},k,l).$$

We proceed it order by order.

$$T(J_{w3},0,0) \lesssim \|a\|_{L^\infty}\|w\|_{L^\infty}^3\|\nabla w\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(1+t)^{-\frac{9}{2}}\|\nabla w\|_{L^2}, \tag{3.30}$$

$$\begin{aligned} T(J_{w3},1,0) &\lesssim \|a\|_{L^\infty}\|w\|_{L^\infty}^2(\|\nabla w\|_{L^3}\|\nabla w\|_{L^6} + \|w\|_{L^\infty}\|\nabla^2 w\|_{L^2}) \\ &\lesssim \sqrt{\mathcal{E}_0^3}(1+t)^{-3}\|\nabla^2 w\|_{L^2}, \end{aligned} \tag{3.31}$$

$$T(J_{w3},1,1) \lesssim \|w\|_{L^\infty}^3\|\nabla a\|_{L^3}\|\nabla w\|_{L^6} \lesssim \sqrt{\mathcal{E}_0^3}(1+t)^{-\frac{9}{2}}\|\nabla^2 w\|_{L^2}. \tag{3.32}$$

$$\begin{aligned} T(J_{w3},2,0) &= \|a\nabla^3(|w|^4)\|_{L^2} \\ &\lesssim \|a\|_{L^\infty}\|w\|_{L^\infty}^3\|\nabla^3 w\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(1+t)^{-\frac{9}{2}}\|\nabla^3 w\|_{L^2}, \end{aligned} \tag{3.33}$$

$$\begin{aligned} T(J_{w3},2,1) &\lesssim \|\nabla a\|_{L^\infty}\|w\|_{L^\infty}^2\|\nabla w\|_{L^4}^2 + \|\nabla a\|_{L^3}\|w\|_{L^\infty}^3\|\nabla^2 w\|_{L^6} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(1+t)^{-3}\|\nabla^3 w\|_{L^2}, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} T(J_{w3},2,2) = \|\nabla^2 a\nabla(|w|^4)\| &\le \|\nabla^2 a\|_{L^6}\|w\|_{L^\infty}^3\|\nabla w\|_{L^3} \\ &\lesssim (1+t)^{-\frac{9}{2}}\sqrt{\mathcal{E}_0^3}\|\nabla^3 a\|_{L^2}. \end{aligned} \tag{3.35}$$

Combining estimates from (3.30) to (3.35), we have

$$J_{w3} \lesssim (1+t)^{-3}\sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2} + \|\nabla^{k+1}w\|_{L^2}\right).$$

Term J_{w4}: For the term J_{w4}, we find that

$$\begin{aligned} J_{w4} &= \|\nabla^{k+1}(|w|^4)\|_{L^2} \\ &\lesssim \|w\|_{L^\infty}^3\|\nabla^{k+1}w\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}\,(1+t)^{-3}\|\nabla^{k+1}w\|_{L^2}. \end{aligned}$$

Combining the estimates from J_{w1} to J_{w4}, we obtain

$$J_w \lesssim (1+t)^{-3}\sqrt{\mathcal{E}_0^3}\left(\|\nabla^{k+1}a\|_{L^2} + \|(\nabla^k w, \nabla^{k+1}w)\|_{L^2}\right).$$

With the estimates for J_u, J_w and inequalities (3.27), (3.28), and the smallness of $\sqrt{\mathcal{E}_0^3}$, we can easily complete the proof of this lemma. □

4. Proof of the Theorem 1.1: negative Besov estimates

In this section, we will derive the evolution of the solution to (1.34) in the negative Besov space. We will establish the following lemma.

Lemma 4.1. *Suppose* $\|w\circ Y\|_{L^\infty} \simeq \|w\|_{L^\infty}$ *and* $\|\nabla(w\circ Y)\|_{L^2} \simeq \|\nabla w\|_{L^2}$. *For* $s \in (0, \frac{1}{2}]$, *we have*

$$\begin{aligned} &\frac{d}{dt}\|(a,u,B)\|_{B_{2,\infty}^{-s}}^2 + \frac{1}{2}\|(\nabla u, \nabla B)\|_{B_{2,\infty}^{-s}}^2 \\ &\quad\lesssim \left(\|(\nabla a, \nabla u, \nabla B)\|_{H^1}^2 + \|w\|_{L^\infty}^2\right)\|(a,u,B)\|_{B_{2,\infty}^{-s}}. \end{aligned}$$

For $s \in (\frac{1}{2}, \frac{3}{2}]$, *we have*

$$\begin{aligned}
&\frac{d}{dt}\|(a,u,B)\|^2_{B^{-s}_{2,\infty}} + \frac{1}{2}\|(\nabla u, \nabla B)\|^2_{B^{-s}_{2,\infty}} \\
&\quad \lesssim \Big(\|(\nabla a, \nabla u, \nabla B)\|_{L^2}^{\frac{5}{2}-s} \|(a,u,B)\|_{L^2}^{s-\frac{1}{2}} \\
&\qquad + \|w\|^2_{L^\infty}\|(a,w)\|_{L^2}^{s-\frac{1}{2}}\|(\nabla a, \nabla w)\|_{L^2}^{\frac{3}{2}-s} \\
&\qquad\qquad + \|\nabla^2 u\|_{L^2}\|a\|_{L^2}^{s-\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{3}{2}-s}\Big)\|(a,u,B)\|_{B^{-s}_{2,\infty}}.
\end{aligned}$$

Proof. We consider (1.34) in the vector form (3.4). We then apply the operator $\Lambda^{-s}\Delta_q$ to (3.4) and we make the scalar product of the resulting equation by $\Lambda^{-s}\Delta_q V$ to obtain

$$\frac{d}{dt}\|\Lambda^{-s}\Delta_q V\|^2_{L^2} + \big(\mathbf{A}_0(\partial_x)\Lambda^{-s}\Delta_q V \mid \Lambda^{-s}\Delta_q V\big) = \big(\Lambda^{-s}\Delta_q \mathbf{H} \mid \Lambda^{-s}\Delta_q V\big), \tag{4.1}$$

where we use $(\mathbf{A}(\partial_x)W \mid W) = (\mathbf{A}_0(\partial_x)W \mid W)$. We then obtain

$$\frac{d}{dt}\|\Lambda^{-s}\Delta_q V\|^2_{L^2} + \frac{1}{2}\|\Lambda^{-s}\Delta_q(\nabla u, \nabla B)\|^2_{L^2} \lesssim W_1 + \cdots + W_{13}, \tag{4.2}$$

where

$$\begin{gathered}
W_1 = -(\Lambda^{-s}\Delta_q(a\mathrm{div}u)|\Lambda^{-s}\Delta_q a), \quad W_2 = -(\Lambda^{-s}\Delta_q(u\cdot\nabla a)|\Lambda^{-s}\Delta_q a), \\
W_3 = -\frac{1}{2}(\Lambda^{-s}\Delta_q((\mathfrak{h}(a)-1)\nabla(|B|^2))|\Lambda^{-s}\Delta_q u), \\
W_4 = (\Lambda^{-s}\Delta_q(\mathfrak{h}(a)\nabla(\bar{B}\cdot B))|\Lambda^{-s}\Delta_q u), \quad W_5 = -(\Lambda^{-s}\Delta_q(\mathfrak{h}(a)(\bar{B}\cdot\nabla)B)|\Lambda^{-s}\Delta_q u), \\
W_6 = (\Lambda^{-s}\Delta_q((\mathfrak{h}(a)-1)B\cdot\nabla B)|\Lambda^{-s}\Delta_q u), \quad W_7 = -(\Lambda^{-s}\Delta_q(\mathfrak{f}(a)\nabla a)|\Lambda^{-s}\Delta_q u), \\
W_8 = -(\Lambda^{-s}\Delta_q(\mathfrak{h}(a)(\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u))|\Lambda^{-s}\Delta_q u), \\
W_{10} = \alpha(\Lambda^{-s}\Delta_q(\frac{1}{1+a}\nabla(g'(\frac{1}{1+a})h(|w\circ Y|^2)))|\Lambda^{-s}\Delta_q u), \\
W_9 = -(\Lambda^{-s}\Delta_q(u\cdot\nabla u)|\Lambda^{-s}\Delta_q u), \quad W_{11} = -(\Lambda^{-s}\Delta_q(u\cdot\nabla B)|\Lambda^{-s}\Delta_q B), \\
W_{12} = -(\Lambda^{-s}\Delta_q(\mathrm{div}u B)|\Lambda^{-s}\Delta_q B), \quad W_{13} = (\Lambda^{-s}\Delta_q(B\cdot\nabla u)|\Lambda^{-s}\Delta_q B).
\end{gathered}$$

Firstly, we restrict $s \in (0, \frac{1}{2}]$. Using Lemma 6.4, for $1/p = \frac{1}{2} + s/3$, we get

$$\begin{aligned}
|W_1| &\le \|a\mathrm{div}u\|_{B^{-s}_{2,\infty}}\|a\|_{B^{-s}_{2,\infty}} \\
&\lesssim \|a\|_{L^p}\|a\|_{B^{-s}_{2,\infty}} \\
&\lesssim \|a\|_{L^{3/s}}\|\nabla u\|_{L^2}\|a\|_{B^{-s}_{2,\infty}} \\
&\lesssim \|\nabla a\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2 a\|_{L^2}^{\frac{1}{2}-s}\|\nabla u\|_{L^2}\|a\|_{B^{-s}_{2,\infty}} \\
&\lesssim \left(\|\nabla a\|^2_{H^1} + \|\nabla u\|^2_{L^2}\right)\|a\|_{B^{-s}_{2,\infty}}.
\end{aligned}$$

Except for W_{10}, all other terms can be estimated in the similar way as the one for W_1 above as follows,

$$|W_2| \lesssim \left(\|\nabla u\|_{H^1}^2 + \|\nabla a\|_{L^2}^2\right)\|a\|_{B^{-s}_{2,\infty}}, \quad |W_3|, |W_6| \lesssim \left(\|\nabla a\|_{H^1}^2 + \|\nabla B\|_{H^1}^2\right)\|a\|_{B^{-s}_{2,\infty}},$$
$$|W_4|, |W_5| \lesssim \left(\|\nabla a\|_{H^1}^2 + \|\nabla B\|_{L^2}^2\right)\|u\|_{B^{-s}_{2,\infty}}, \quad |W_7| \lesssim \|\nabla a\|_{H^1}^2\|u\|_{B^{-s}_{2,\infty}},$$
$$|W_8| \lesssim \left(\|\nabla a\|_{H^1}^2 + \|\nabla^2 u\|_{L^2}^2\right)\|u\|_{B^{-s}_{2,\infty}}, \quad |W_9| \lesssim \|\nabla u\|_{H^1}^2\|u\|_{B^{-s}_{2,\infty}}$$
$$|W_{11}|, |W_{13}| \lesssim \left(\|\nabla u\|_{H^1}^2 + \|\nabla B\|_{L^2}^2\right)\|B\|_{B^{-s}_{2,\infty}}, \quad |W_{12}| \lesssim \left(\|\nabla B\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\right)\|B\|_{B^{-s}_{2,\infty}}.$$

For W_{10}, like J_{13} in the proof of Lemma 3.1, we need to estimate

$$\tilde I_1 = |(\Lambda^{-s}\Delta_q(a\nabla a|w|^4)|\Lambda^{-s}\Delta_q u)|, \quad \tilde I_2 = |(\Lambda^{-s}\Delta_q(\nabla a|w|^4)|\Lambda^{-s}\Delta_q u)|,$$
$$\tilde I_3 = |(\Lambda^{-s}\Delta_q(a|w|^2\nabla(|w|^2))|\Lambda^{-s}\Delta_q u)|, \quad \tilde I_4 = |(\Lambda^{-s}\Delta_q(|w|^2\nabla(|w|^2))|\Lambda^{-s}\Delta_q u)|.$$

For $\tilde I_1$, with $1/p = \frac{1}{2} + s/3$ in Lemma 6.4, we have

$$\begin{aligned}\tilde I_1 &\lesssim \|a\nabla a|w|^4\|_{B^{-s}_{2,\infty}}\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \|a\nabla a|w|^4\|_{L^p}\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \|a\|_{L^{3/s}}\|\nabla a\|_{L^2}\|w\|_{L^\infty}^4\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \|w\|_{L^\infty}^4\|\nabla a\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2 a\|_{L^2}^{\frac{1}{2}-s}\|\nabla a\|_{L^2}\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \|\nabla a\|_{H^1}^2\|u\|_{B^{-s}_{2,\infty}}.\end{aligned}$$

Similar arguments imply

$$\tilde I_2 \lesssim \|w\|_{L^\infty}^2\left(\|\nabla w\|_{H^1}^2 + \|\nabla a\|_{L^2}^2\right)\|u\|_{B^{-s}_{2,\infty}}.$$

For the term $\tilde I_3$, we have

$$\begin{aligned}\tilde I_3 &\lesssim \|a|w|^2\nabla(|w|^2)\|_{B^{-s}_{2,\infty}}\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \|a\|_{L^{3/s}}\||w|^2\nabla(|w|^2)\|_{L^2}\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \||w|^2\nabla(|w|^2)\|_{L^2}\|\nabla a\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2 a\|_{L^2}^{\frac{1}{2}-s}\|u\|_{B^{-s}_{2,\infty}}\\ &\lesssim \|\nabla w\|_{L^2}\|w\|_{L^\infty}^3\|\nabla a\|_{H^1}\|u\|_{B^{-s}_{2,\infty}},\end{aligned}$$

where we used Lemma 6.4. For the term $\tilde I_4$, we have

$$
\begin{aligned}
\tilde{I}_4 &\lesssim \||w|^2\nabla(|w|^2)\|_{B^{-s}_{2,\infty}}\|u\|_{B^{-s}_{2,\infty}}\\
&\lesssim \|w\|_{L^{3/s}}\||w|\nabla(|w|^2)\|_{L^2}\|u\|_{B^{-s}_{2,\infty}}\\
&\lesssim \|\nabla w\|^2_{H^1}\|w\|^2_{L^\infty}\|u\|_{B^{-s}_{2,\infty}},
\end{aligned}
$$

where we also used Lemma 6.4. Therefore, we thus have for $s \in (0, \frac{1}{2}]$ that

$$
|W_{10}| \lesssim \left(\|\nabla a\|^2_{H^1} + \|w\|^2_{L^\infty}\right)\|u\|_{B^{-s}_{2,\infty}}.
$$

Combining all the estimates about W_j for $j = 1, \cdots, 13$, (4.2) gives the proof of this lemma for $s \in (0, \frac{1}{2}]$.

Next, we need to deal with the case $s \in (\frac{1}{2}, \frac{3}{2}]$. Using Lemma 6.1 and Lemma 6.4, we have

$$
\begin{aligned}
|W_1| &\lesssim \|a\|_{L^{3/s}}\|\nabla u\|_{L^2}\|a\|_{B^{-s}_{2,\infty}}\\
&\lesssim \|a\|^{2-\frac{1}{2}}_{L^2}\|\nabla a\|^{\frac{3}{2}-s}_{L^2}\|\nabla u\|_{L^2}\|a\|_{B^{-s}_{2,\infty}}.
\end{aligned}
$$

Similarly we can get the estimates for other terms and give the results as follows,

$$
\begin{aligned}
|W_2| &\lesssim \|u\|^{s-\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{3}{2}-s}_{L^2}\|\nabla a\|_{L^2}\|a\|_{B^{-s}_{2,\infty}},\\
|W_3|, |W_6| &\lesssim \|a\|^{s-\frac{1}{2}}_{L^2}\|\nabla a\|^{\frac{3}{2}-s}_{L^2}\|\nabla B\|_{L^2}\|u\|_{B^{-s}_{2,\infty}}\\
&\quad + \|B\|^{s-\frac{1}{2}}_{L^2}\|\nabla B\|^{\frac{3}{2}-s}_{L^2}\|\nabla B\|_{L^2}\|u\|_{B^{-s}_{2,\infty}},\\
|W_4|, |W_5| &\lesssim \|a\|^{s-\frac{1}{2}}_{L^2}\|\nabla a\|^{\frac{3}{2}-s}_{L^2}\|\nabla B\|_{L^2}\|u\|_{B^{-s}_{2,\infty}},\\
|W_7| &\lesssim \|a\|^{s-\frac{1}{2}}_{L^2}\|\nabla a\|^{\frac{3}{2}-s}_{L^2}\|\nabla a\|_{L^2}\|u\|_{B^{-s}_{2,\infty}},\\
|W_8| &\lesssim \|a\|^{s-\frac{1}{2}}_{L^2}\|\nabla a\|^{\frac{3}{2}-s}_{L^2}\|\nabla^2 u\|_{L^2}\|u\|_{B^{-s}_{2,\infty}},\\
|W_9| &\lesssim \|u\|^{s-\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{3}{2}-s}_{L^2}\|\nabla u\|_{L^2}\|u\|_{B^{-s}_{2,\infty}},\\
|W_{11}| &\lesssim \|u\|^{s-\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{3}{2}-s}_{L^2}\|\nabla B\|_{L^2}\|B\|_{B^{-s}_{2,\infty}},\\
|W_{12}|, |W_{13}| &\lesssim \|B\|^{s-\frac{1}{2}}_{L^2}\|\nabla B\|^{\frac{3}{2}-s}_{L^2}\|\nabla u\|_{L^2}\|B\|_{B^{-s}_{2,\infty}}.
\end{aligned}
$$

It remains to give the estimates on W_{10}. For the term $\tilde{I}_1$, using Lemma 6.1 and Lemma 6.4, we have

$$
\begin{aligned}
\tilde{I}_1 &\lesssim \|a\|_{L^{3/s}}\|\nabla a\|_{L^2}\|w\|^4_{L^\infty}\|u\|_{B^{-s}_{2,\infty}}\\
&\lesssim \|w\|^4_{L^\infty}\|a\|^{s-\frac{1}{2}}_{L^2}\|\nabla a\|^{\frac{3}{2}-s}_{L^2}\|u\|_{B^{-s}_{2,\infty}}.
\end{aligned}
$$

Similarly, we could obtain that

$$\tilde{I}_2 \lesssim \|w\|_{L^2}^{s-\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{3}{2}-s} \|\nabla a\|_{L^2} \|w\|_{L^\infty}^3 \|u\|_{B_{2,\infty}^{-s}},$$

$$\tilde{I}_3 \lesssim \|a\|_{L^2}^{s-\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{3}{2}-s} \|\nabla w\|_{L^2} \|w\|_{L^\infty}^3 \|u\|_{B_{2,\infty}^{-s}},$$

and

$$\tilde{I}_4 \lesssim \|w\|_{L^2}^{s-\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{3}{2}-s} \|\nabla w\|_{L^2} \|w\|_{L^\infty}^2 \|u\|_{B_{2,\infty}^{-s}}.$$

Form the estimates about I_1, I_2, I_3 and I_4, we could get

$$|W_{10}| \lesssim \|w\|_{L^\infty}^2 \|(a, w)\|_{L^2}^{s-\frac{1}{2}} \|(\nabla a, \nabla w)\|_{L^2}^{\frac{3}{2}-s} \|u\|_{B_{2,\infty}^{-s}}.$$

So at this point, we can easily complete the proof for the case $s \in (\frac{1}{2}, \frac{3}{2}]$. $\square$

5. Proof of the main theorems: existence and time-decay

In this section, we shall combine all the energy estimates that we have derived in the previous two sections and the Besov interpolation inequalities to complete the proof.

We first close the energy estimates at each l-th level. Let $0 \le l \le m-1$ with $1 \le m \le 3$. Summing up the estimates in Lemma 3.1 from $k = l+1$ to m, since $\sqrt{\mathcal{E}_0^3} \le \delta$ is small, we obtain

$$\begin{aligned} &\frac{d}{dt} \sum_{k=l+1}^{m} \|(\nabla^k a, \nabla^k u, \nabla^k B)\|_{L^2}^2 + \frac{1}{4} \sum_{k=l+2}^{m+1} \|(\nabla^k u, \nabla^k B)\|_{L^2}^2 \\ &\lesssim \delta \left(\sum_{k=l+1}^{m} \|\nabla^k a\|_{L^2}^2 + \sum_{k=l+1}^{m+1} \|(\nabla^k u, \nabla^k B)\|_{L^2}^2 + (1+t)^{-6} \|w\|_{H^3}^2 \right). \end{aligned} \tag{5.1}$$

Summing up the estimates in Lemma 3.2 from $k = l$ to $m-1$, since $\sqrt{\mathcal{E}_0^3} \le \delta$ is small, we obtain

$$\begin{aligned} &\frac{d}{dt} \sum_{k=l}^{m-1} \|(\nabla^k a, \nabla^k u, \nabla^k B)\|_{L^2}^2 + \frac{1}{4} \sum_{k=l+1}^{m} \|(\nabla^k u, \nabla^k B)\|_{L^2}^2 \\ &\lesssim \delta \left(\sum_{k=l+1}^{m} \|(\nabla^k a, \nabla^k u, \nabla^k B)\|_{L^2}^2 + (1+t)^{-6} \|w\|_{H^3}^2 \right). \end{aligned} \tag{5.2}$$

Summing up (5.1) and (5.2), it holds for a positive constant C_1 that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{k=l}^{m}\|(\nabla^k a,\nabla^k u,\nabla^k B)\|_{L^2}^2+\frac{1}{8}\sum_{k=l+1}^{m+1}\|(\nabla^k u,\nabla^k B)\|_{L^2}^2\\
&\quad\le C_1\delta\left(\sum_{k=l+1}^{m}\|\nabla^k a\|_{L^2}^2+(1+t)^{-6}\|w\|_{H^2}^2\right).
\end{aligned}
\tag{5.3}
$$

Summing up the estimates in Lemma 3.3 from $k=l$ to $m-1$, we obtain for a positive constant C_2 that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{k=l}^{m-1}\int_{\mathbb{R}^3}\nabla^k u\nabla\nabla^k a\,dx+\frac{1}{4}\sum_{k=l+1}^{m}\|\nabla^k a\|_{L^2}^2\\
&\quad\le C_2\left(\sum_{k=l+1}^{m+1}\|(\nabla^k u,\nabla^k B)\|_{L^2}^2+\delta(1+t)^{-6}\|w\|_{H^3}^2\right).
\end{aligned}
\tag{5.4}
$$

If we compute $16(1+C_2)$(5.3) + (5.4), with the help of Cauchy–Schwartz inequality, it is clear that, for δ small, there is a positive constant C_3 such that

$$
\frac{d}{dt}\mathcal{E}_l^m+\|\nabla^{l+1}a\|_{H^{m-l-1}}^2+\|(\nabla^{l+1}u,\nabla^{l+1}B)\|_{H^{m-l}}^2\le C_3\delta\delta_0(1+t)^{-6},
\tag{5.5}
$$

where

$$
\mathcal{E}_l^m\simeq\|(\nabla^l a,\nabla^l u,\nabla^l B)\|_{H^{m-l}}^2.
$$

Therefore, setting $l=0$ and $m=3$, choosing $\delta\le\delta_0^{1/2}$, and integrating in t, we arrive at

$$
\|(a,u,B)(\cdot,t)\|_{H^3}^2+\int_0^t(\|\nabla a(\cdot,\tau)\|_{H^2}^2+\|(\nabla u,\nabla B)(\cdot,\tau)\|_{H^3}^2)\,d\tau\le C_4\delta_0^{3/2},
\tag{5.6}
$$

for some positive constant C_4.

In addition, from Lemma 2.7, we obtain for some positive constant C_5 that

$$
\|(a(x,t),u(x,t),B(x,t),w(y,t))\|_{H^3}\le C_5\delta_0.
\tag{5.7}
$$

By a standard continuation argument, this closes the a priori estimates (3.1) if at the initial time we assume that $\delta\le\delta_0^{1/2}$ is sufficiently small.

To conclude the proof of Theorem 1.1, it remains to prove the time-decay estimates, which imply the global existence and establish, in particular, (2.19).

Define $\mathcal{E}_{-s}(t)=\|(a(x,t),u(x,t),B(x,t))\|_{B_{2,\infty}^{-s}}^2$. We will proceed in three steps for the range of s. The idea is to start from smaller $s\in(0,\frac{1}{2}]$ to gain some decay estimates and then push forward to the larger $s\in(1,\frac{3}{2}]$ for better decay rates. Here, for the reader's convenience, we remark that our condition in Theorem 1.1 ensure smallness of initial data in Besov space $B^{-l}_{2,\infty}$ with $l \in (0, 1]$. Actually, let $l \in (0, 1]$, $s \in (1, \frac{3}{2}]$, we have

$$\begin{aligned}
\|a_0, u_0, B_0\|_{B^{-\ell}_{2,\infty}} &= \sup_j 2^{-\ell j} \|\Delta_j a_0, \Delta_j u_0, \Delta_j B_0\|_{L^2} \\
&\le \sup_{j<0} 2^{-\ell j} \|\Delta_j a_0, \Delta_j u_0, \Delta_j B_0\|_{L^2} + \sup_{j\ge 0} 2^{-\ell j} \|\Delta_j a_0, \Delta_j u_0, \Delta_j B_0\|_{L^2} \\
&\le \sup_{j<0} 2^{-sj} \|\Delta_j a_0, \Delta_j u_0, \Delta_j B_0\|_{L^2} + \sup_{j\ge 0} \|\Delta_j a_0, \Delta_j u_0, \Delta_j B_0\|_{L^2} \\
&\le \|a_0, u_0, B_0\|_{B^{-s}_{2,\infty}} + \|a_0, u_0, B_0\|_{L^2} \le \delta_0.
\end{aligned}$$

Case 1: $s \in (0, \frac{1}{2}]$. For $s \in (0, \frac{1}{2}]$, we read from Lemma 4.1 that

$$\frac{d}{dt}\|(a, u, B)\|_{B^{-s}_{2,\infty}} \lesssim \|(\nabla a, \nabla u, \nabla B)\|^2_{H^1} + \|w\|^2_{L^\infty},$$

integrating in time and using (5.6), for some positive constant C_6, we have

$$\sup_{0\le\tau\le t} \mathcal{E}_{-s}(t) \le C_6\delta_0^2, \quad \text{for } s \in (0, \frac{1}{2}]. \tag{5.8}$$

For $l = 0, 1, 2$, Lemma 6.5 implies that there is a positive constant C_7 such that

$$\|(\nabla^l a, \nabla^l u, \nabla^l B)\|_{L^2} \le C_7 \delta_0^{\frac{1}{l+s+1}} \|(\nabla^{l+1} a, \nabla^{l+1} u, \nabla^{l+1} B)\|_{L^2}^{\frac{l+s}{l+s+1}},$$

which implies that

$$\|(\nabla^{l+1} a, \nabla^{l+1} u, \nabla^{l+1} B)\|^2_{L^2} \ge C_8 \delta_0^{-\frac{2}{l+s}} \left(\|(\nabla^l a, \nabla^l u, \nabla^l B)\|^2_{L^2}\right)^{\frac{(l+s+1)}{l+s}}, \tag{5.9}$$

where $C_8 = C_7^{-\frac{2(l+s+1)}{l+s}}$. Therefore, (5.5) implies that for $l = 0, 1, 2$,

$$\frac{d}{dt}\mathcal{E}^m_l + C_8\delta_0^{-\frac{2}{l+s}} \left(\mathcal{E}^m_l\right)^{\frac{l+s+1}{l+s}} \le C_3\delta_0^2(1+t)^{-6}. \tag{5.10}$$

Setting $m = 3$ and define $A(t) := (1+t)^{l+s}\mathcal{E}^3_l(t)$, we thus deduce

$$\frac{d}{dt}A(t) - (1+t)^{-1}\left[(l+s) - C_8\delta_0^{-\frac{2}{l+s}} A(t)^{\frac{1}{l+s}}\right] A(t) \le C_3\delta_0^2(1+t)^{(l+s-6)}. \tag{5.11}$$

If for some time interval $t \in [t_1, t_2]$ for $0 \le t_1 < t_2$, $A(t) \le \left(\frac{l+s}{C_8}\right)^{l+s} \delta_0^2$, then we have on such interval,

$$A(t) \le C_9\delta_0^2, \quad C_9 = \left(\frac{l+s}{C_8}\right)^{l+s}, \quad t \in [t_1, t_2]. \tag{5.12}$$

On the other hand, if for some time interval $t \in [t_3, t_4]$ for $0 \le t_3 < t_4$, $A(t) \ge \left(\frac{l+s}{C_8}\right)^{l+s} \delta_0^2$, then on such interval, we find from (5.11) that

$$\frac{d}{dt} A(t) \le C_3 \delta_0^2 (1+t)^{(l+s-6)},$$

which implies that

$$A(t) \le C_3 \delta_0^2 \int_0^{t_4} (1+t)^{(l+s-6)}\, dt \le \frac{C_3}{5-(l+s)} \delta_0^2 \le 2C_3 \delta_0^2, \tag{5.13}$$

where we have used the fact that $l+s \le 2 + \frac{3}{2} < 5$. We thus conclude that for all $t > 0$, it holds that

$$A(t) \le C_{10} \delta_0^2, \qquad \text{for } C_{10} = \max\{C_9, 2\}. \tag{5.14}$$

This is equivalent to

$$\mathcal{E}_l^3(t) \le C_{11} \delta_0^2 (1+t)^{-(l+s)}, \quad s \in (0, \frac{1}{2}],\ l = 0, 1, 2. \tag{5.15}$$

We remark that, we do need the smallness of $\mathcal{E}_{-s}(t)$ for $s \in (0, \frac{1}{2}]$ in this proof, because of the structure in (5.10) due to the interaction terms, which is different from the models with only fluid parts.

Case 2: $s \in (\frac{1}{2}, 1]$. For $s \in (\frac{1}{2}, \frac{3}{2}]$, we have from Lemma 4.1 that

$$\begin{aligned} \frac{d}{dt} \|(a,u,B)\|_{B^{-s}_{2,\infty}} \lesssim \Big(& \|(\nabla a, \nabla u, \nabla B)\|_{L^2}^{\frac{5}{2}-s} \|(a,u,B)\|_{L^2}^{s-\frac{1}{2}} + \|w\|_{L^\infty}^2 \\ & + \|\nabla^2 u\|_{L^2} \|a\|_{L^2}^{s-\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{3}{2}-s} \Big). \end{aligned} \tag{5.16}$$

From the result in Case 1 above, we set $s = \frac{1}{2}$ and find for $l = 0, 1, 2$ that

$$\|\nabla^l a(t)\|_{H^{3-l}}^2 + \|\nabla^l u(t)\|_{H^{3-l}}^2 + \|\nabla^l B(t)\|_{H^{3-l}}^2 \lesssim \delta_0^2 (1+t)^{-(l+\frac{1}{2})}. \tag{5.17}$$

Therefore, we integrate (5.16) in time to find for some positive constant C_{11} that

$$\begin{aligned} \|(a,u,B)(\cdot,t)\|_{B^{-s}_{2,\infty}} \le & \|(a,u,B)(\cdot,0)\|_{B^{-s}_{2,\infty}} \\ & + C_{11} \delta_0^2 \int_0^t \left((1+\tau)^{\frac{7}{4}-\frac{s}{2}} + (1+\tau)^{-3} + (1+\tau)^{\frac{9}{4}-\frac{s}{2}} \right) d\tau, \end{aligned} \tag{5.18}$$

which implies that there is a positive constant C_{12} such that

$$\mathcal{E}_{-s}(t) \le C_{12} \delta_0^2, \quad \text{for } s \in (\frac{1}{2}, 1]. \tag{5.19}$$

Now we use this estimate, and repeat the steps in Case 1 above to obtain

$$\mathcal{E}_l^3(t) \le C_{11}\delta_0^2(1+t)^{-(l+s)}, \quad s \in (\frac{1}{2}, 1],\ l = 0, 1, 2. \tag{5.20}$$

Case 3: $s \in (1, \frac{3}{2}]$. From the result in Case 2 above, we set $s = 1$ to find for $l = 0, 1, 2$, that

$$\|\nabla^l a(t)\|_{H^{3-l}}^2 + \|\nabla^l u(t)\|_{H^{3-l}}^2 + \|\nabla^l B(t)\|_{H^{3-l}}^2 \lesssim \delta_0^2(1+t)^{-(l+1)}, \tag{5.21}$$

with $l = 0, 1, 2$. Replacing (5.18) with this better decay estimates, the same argument used in Case 2 gives

$$\mathcal{E}_l^3(t) \le C_{11}\delta_0^2(1+t)^{-(l+s)}, \quad s \in (1, \frac{3}{2}],\ l = 0, 1, 2. \tag{5.22}$$

We summarize all cases above to achieve that, for $s \in (0, \frac{3}{2}]$ and $l = 0, 1, 2$,

$$\|\nabla^l a(t)\|_{H^{N-l}}^2 + \|(\nabla^l u(t), \nabla^l B(t))\|_{H^{N-l}}^2 \lesssim \delta_0^2(1+t)^{-(l+s)}.$$

In particular, when $s \in (1, \frac{3}{2}]$, one finds that

$$\|\nabla u\|_{H^2} \lesssim \delta_0(1+t)^{-\frac{1+s}{2}},$$

with $\frac{1+s}{2} > 1$. Therefore, by choosing δ_0 sufficiently small, we see the validity of main a priori hypothesis (2.19). We thus complete the proof of Theorem 1.1.

Proof of Theorem 1.2. We outline the proof of Theorem 1.2. When $w \equiv 0$, we see all the estimates up to (5.6) are valid without using the a priori hypothesis (2.19) since it is not necessary to change norms between Eulerian and Lagrangian coordinates. In particular, (5.5) reads as

$$\frac{d}{dt}\mathcal{E}_l^m + \|\nabla^{l+1} a\|_{H^{m-l-1}}^2 + \|(\nabla^{l+1} u, \nabla^{l+1} B)\|_{H^{m-l}}^2 \le 0. \tag{5.23}$$

Therefore, the global existence of classical solution $(a, u, B)(x, t)$ is established. For the decay estimates, we start with the case $s \in (0, \frac{1}{2}]$. The same argument with $w = 0$ in derivation of (5.8) gives

$$\sup_{0\le\tau\le t} \mathcal{E}_{-s}(t) \le \tilde{C}_6, \quad \text{for } s \in (0, \frac{1}{2}], \tag{5.24}$$

for some positive constant $\tilde{C}_6$, without using the smallness of Besov norms. Then (5.10) becomes

$$\frac{d}{dt}\mathcal{E}_l^m + \tilde{C}_8\delta_0^{-\frac{2}{l+s}}\left(\mathcal{E}_l^m\right)^{\frac{l+s+1}{l+s}} \le 0, \tag{5.25}$$

for some positive constant $\tilde{C}_8$. This differential inequality implies directly that

$$\mathcal{E}_l^m \le \tilde{C}_9(1+t)^{-(l+s)}, \quad \text{for } s \in (0, \frac{1}{2}].$$

The other two cases when $s \in (\frac{1}{2}, 1]$ and $s \in (1, \frac{3}{2}]$ can be carried out similarly. We omit the details. We thus completes the proof of Theorem 1.2. □

6. Appendix

For the reader's convenience, we give some useful lemmas, which used frequently in this paper.

Lemma 6.1. *[13] Let $0 \le m, \alpha \le l$, then one has in $\mathbb{R}^3$ that*

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|_{L^q}^{1-\theta} \|\nabla^l f\|_{L^r}^{\theta},$$

where $0 \le \theta \le 1$ and α satisfies

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta.$$

Here when $p = \infty$ we require that $0 < \theta < 1$.

Lemma 6.2. *[13] Assume that $\|a\|_{L^\infty} \le 1$. Let $g(a)$ be a smooth function of a with bounded derivatives of any order then for any integer $m \ge 1$ we have*

$$\|\nabla^m (g(a))\|_{L^\infty} \lesssim \|\nabla^m a\|_{L^\infty}.$$

Lemma 6.3. *[13] For all $m \in \mathbb{N}$, if α is a multi-index with $|\alpha| \le m$, there exists $C > 0$ such that, for $u, v \in H^m \cap L^\infty$,*

$$\|\nabla^\alpha (uv)\|_{L^2} \lesssim \left(\|u\|_{L^\infty} \|D^{|\alpha|} v\|_{L^2} + \|D^{|\alpha|} u\|_{L^2} \|v\|_{L^\infty}\right).$$

Lemma 6.4. *[15,11] Suppose that $s > 0$ and $1 \le p < 2$. One has*

$$\|f\|_{B^{-s}_{r,\infty}} \lesssim \|f\|_{L^p},$$

with $1/p - 1/r = s/n$. In particular, this holds with $s = n/2$, $r = 2$ and $p = 1$.

Lemma 6.5. *[15] Suppose $k \ge 0$ and $m, \beta > 0$. Then the following inequality holds*

$$\|\Lambda^k f\|_{L^2} \lesssim \|\Lambda^{k+m} f\|_{L^2}^{\theta} \|f\|_{B^{-\beta}_{2,\infty}}^{1-\theta}, \tag{6.1}$$

with $\theta = \frac{\beta+k}{\beta+k+m}$. (6.1) is also true for ∂^α with $|\alpha| = k$ (k is a nonnegative integer).

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