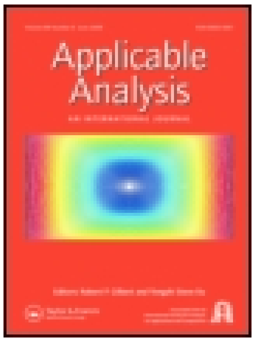

Taylor & Francis
Taylor & Francis Group

Applicable Analysis
An International Journal

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: http://www.tandfonline.com/loi/gapa20

Optimal time decay rate for compressible viscoelastic equations in critical spaces

Junxiong Jia & Jigen Peng

To cite this article: Junxiong Jia & Jigen Peng (2016): Optimal time decay rate for compressible viscoelastic equations in critical spaces, Applicable Analysis

To link to this article: http://dx.doi.org/10.1080/00036811.2016.1200721

APPLICABLE ANALYSIS, 2016
http://dx.doi.org/10.1080/00036811.2016.1200721

Taylor & Francis
Taylor & Francis Group

Optimal time decay rate for compressible viscoelastic equations in critical spaces

Junxiong Jia[a,b] and Jigen Peng[a,b]

[a]Department of Mathematics, Xi'an Jiaotong University, Xi'an, China; [b]Beijing Center for Mathematics and Information Interdisciplinary Sciences (BCMIIS), Beijing, China

ABSTRACT
In this paper, we are concerned with the optimal time convergence rate of the global strong solution to some constant equilibrium states for the compressible viscoelastic fluids in the whole space. Green's matrix method and energy estimate method are used to obtain the optimal time decay rate under the critical Besov space framework. Our result implies the optimal L^2-time decay rate and only need the initial datum to be small in some critical Besov space which have very low regularity compared with the classical Sobolev space.

ARTICLE HISTORY
Received 13 March 2016
Accepted 8 June 2016

COMMUNICATED BY
B. Zhang

KEYWORDS
Compressible viscoelastic fluids; critical space framework; optimal time decay

AMS SUBJECT CLASSIFICATIONS
76N10; 35Q35; 35D35

1. Introduction and main results

Many fluids do not satisfy the Newtonian law. A viscoelastic fluid of the Oldroyd type is a classical non-Newtonian fluid that exhibits elastic behavior, such as memory effects. The elastic properties of the fluid are described by associating the fluid motions with an energy functional of the deformation tensor U. Let us assume that the elastic energy is $W(U)$, then the compressible viscoelastic system can be written as follows

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla P(\rho) = \mathrm{div}(2\mu \mathcal{D}(v) + \nabla(\lambda \mathrm{div}(v))) \\ \qquad + \mathrm{div}\left(\frac{W_U(U)U^T}{\det(U)}\right), \\ \partial_t U + v \cdot \nabla U = \nabla v U. \end{cases} \tag{1.1}$$

Here, ρ is the density and $v(x,t)$ is the velocity of the fluid. The pressure $P(\rho)$ is a given state equation with $P'(\rho) > 0$ for any ρ and $\mathcal{D}(v) = \frac{1}{2}(\nabla v + \nabla v^T)$ is the strain tensor. The Lamé coefficients μ and λ are assumed to satisfy

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0. \tag{1.2}$$

Such a condition ensures the ellipticity of the operator $-\nabla(2\mu\mathcal{D}\,\cdot\,) - \nabla(\lambda\nabla\,\cdot\,)$ and is satisfied in the physical case, where $\lambda + 2\mu/N \approx 0$. Moreover, $W_U(U)$ is the Piola–Kirchhoff tensor and $\frac{W_U(U)U^T}{\det(U)}$ is the Cauchy–Green tensor. For the special case of the Hookean linear elasticity, $W(U) = |U|^2$.

CONTACT Junxiong Jia jjx323@mail.xjtu.edu.cn

Many important investigations on incompressible viscoelastic fluids have been conducted recently. In [1], authors obtain the well posedness of incompressible viscoelastic fluids and found the relation

$$\nabla_k F^{ij} - \nabla_j F^{ik} = F^{lj}\nabla_l F^{ik} - F^{lk}\nabla_l F^{ij},$$

with $F = U - I$. This relation indicates that the linear term $\nabla \times F$ is actually a higher order term. Lin et al. [2,3] provide the local well posedness in Hilbert space H^s, and obtain the global well posedness for small initial data. To prove the global part, they capture the damping mechanism on F through very subtle energy estimates. Tang and Fang [4] prove the global well posedness of the incompressible version of the system (1.1) in the critical L^p framework, thereby allowing the construction of a unique global solution with highly oscillating initial velocity.

For compressible viscoelastic fluids, the author of [5,6] prove the local and global well-posedness in the L^2 based critical Besov type space. They deeply use the properties of the viscoelastic fluids and their results indicate that the role of the deformation tensor U is similar to that of the density ρ. It should be mentioned that the global existence of a smooth solution is still an open problem, even for incompressible viscoelastic fluids. Lions and Masmoudi [7] obtain the global existence of a weak solution with general initial data in the case the contribution of the strain rate in the constitutive equation is neglected. Recently, the global well-posedness in L^p based critical Besov spaces have been archived in [8].

In addition to well-posedness theory, the problem of the optimal time decay rate is another important subject. There are many papers concerned with the optimal time decay rate for the compressible Navier–Stokes system [9–16]. However, because of the complexity of compressible viscoelastic equations, there are few results on viscoelastic equations. Recently, Hu and Wu [17] provide a detailed analysis on the time decay rate in Sobolev space framework. They divide the whole system into two small systems, which in turn facilitated the analysis. In [8], the authors use estimates in homogeneous space to provide a slow decay rate when the initial data are only small in Besov spaces with low regularity. The main goal of this paper is to obtain the optimal time decay rate when the initial data are only small in the framework of critical Besov spaces. Hence, we can link the results from [17] and [8] to provide a more elaborate characterization of the optimal time decay rate for compressible viscoelastic equations.

In [5,6], the authors provide the following proposition, which reveals some intrinsic properties of compressible viscoelastic equations.

Proposition 1.1: *The density ρ and deformation tensor U in (1.1) satisfy the following relations:*

$$\begin{aligned} &\operatorname{div}\left(\frac{U^T}{\det U}\right) = 0, \quad \operatorname{div}(\rho U^T) = 0, \quad \rho \det U = 1, \\ &\text{and} \quad U^{lk}\nabla_l U^{ij} - U^{lj}\nabla_l U^{ik} = 0, \end{aligned} \tag{1.3}$$

if the initial data $(\rho, U)|_{t=0} = (\rho_0, U_0)$ satisfies

$$\begin{aligned} &\operatorname{div}\left(\frac{U_0^T}{\det U_0}\right) = 0, \quad \operatorname{div}(\rho_0 U_0^T) = 0, \quad \rho_0 \det U_0 = 1, \\ &\text{and} \quad U_0^{lk}\nabla_l U_0^{ij} - U_0^{lj}\nabla_l U_0^{ik} = 0, \end{aligned} \tag{1.4}$$

respectively.

Using Proposition 1.1, the last term in the second equation of (1.1) can be rewritten as

$$\nabla_j\left(\frac{\frac{\partial W(U)}{\partial U^{ik}} U^{jk}}{\det U}\right) = \frac{1}{\det U} U^{jk}\nabla_j\left(\frac{\partial W(U)}{\partial U^{ik}}\right) = \rho U^{jk}\nabla_j\left(\frac{\partial W(U)}{\partial U^{ik}}\right). \tag{1.5}$$

As in [6], without loss of generality, the Hookean linear elasticity, $W(U) = |U|^2$ is considered in the following parts of this paper. This usage does not reduce the essential difficulty. In view of (1.5), we consider the following system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho \partial_t v^i + \rho v \cdot \nabla v^i - \mathrm{div}(2\mu \mathcal{D}(v)) - \nabla(\lambda \mathrm{div} v) + \nabla P(\rho) = \rho U^{jk} \nabla_j U^{ik}, \\ \partial_t U + v \cdot \nabla U = \nabla v U, \\ (\rho, v, U)|_{t=0} = (\rho_0, v_0, U_0) \end{cases} \tag{1.6}$$

where the initial data satisfy (1.4).

We now state the main result of this paper which provides the optimal L^2-time decay rate for the strong solutions of (1.6) in critical Besov spaces.

Theorem 1.2: *Assume that dimension $n = 3$, $\bar{\rho}$ be a constant and I stands for the identity vector $(1, 1, 1)^t$. There exists $\delta > 0$ such that if $v_0 \in B_{2,1}^{n/2-1} \cap \dot{B}_{1,\infty}^0$, $\rho_0 - \bar{\rho} \in B_{2,1}^{n/2} \cap \dot{B}_{1,\infty}^0$, $U - I \in B_{2,1}^{n/2} \cap \dot{B}_{1,\infty}^0$ and*

$$\|(\rho_0 - \bar{\rho}, U - I)\|_{B_{2,1}^{n/2} \cap \dot{B}_{1,\infty}^0} + \|v_0\|_{B_{2,1}^{n/2-1} \cap \dot{B}_{1,\infty}^0} \leq \delta,$$

then problem (1.6) has a unique global solution $(\rho - \bar{\rho}, v, U - I) \in C(\mathbb{R}^+; B_{2,1}^{n/2}) \times \left(C(\mathbb{R}^+; B_{2,1}^{n/2-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{n/2+1})\right)^n \times \left(C(\mathbb{R}^+; B_{2,1}^{n/2})\right)^{n\times n}$. Furthermore, there exists constant $C_0 > 0$, and we have

$$\|(\rho - \bar{\rho}, v, U - I)(t)\|_{B_{2,1}^{n/2-1}} \leq C_0(1 + t)^{-n/4}, \tag{1.7}$$

for $t \geq 0$.

Remark 1.3: From [17], we know that the optimal L^2-time decay rate for the compressible viscoelastic equations is

$$\|(\rho - \bar{\rho}, v, U - I)(t)\|_{L^2} \leq C(1 + t)^{-n/4}. \tag{1.8}$$

The convergence rate (1.7) is optimal because $B_{2,1}^{n/2-1} \subset L^2$.

To prove Theorem 1.2, we split the system (1.6) by Littlewood–Paley operator to low frequency and high frequency parts. For the low frequency part, we further decompose the system into three small systems and analyze the Green's matrix of each small system, as in [8,17]. We can then combine the estimates for each small system to finally obtain an estimate of the whole system because of the fine properties of homogeneous space and singular operators. For the high frequency part, we reformulate the system as presented in [5] and use energy estimates in critical Besov spaces to obtain an appropriate a priori estimate.

The paper is organized as follows. In Section 2, we introduce the notations, some properties of Besov space and some important lemmas. In Section 3, we split the system into three small systems and provide the estimates for the low frequency part. In Section 4, we convert the system into an equivalent form and prove an a priori estimate for the high frequency part. In Section 5, we provide the proof for Theorem 1.2.

2. Preliminaries

In this section, we first introduce some notations to be used throughout the paper. Secondly, we provide some basic knowledge about Besov space. Lastly, we present some useful lemmas and theorems.

2.1. Notation

Let n stands for the dimension, $L^p(1 \le p \le \infty)$ denote the usual L^p-Lebesgue space on $\mathbb{R}^n$. $[z]$ stands for the integer part of a number $z \in \mathbb{R}$. The inner product of L^2 is denoted by $(\cdot,\cdot)$. If S is any nonempty set, then the sequence space $\ell^p(S)$ denotes the usual ℓ^p sequence space on S. For any integer $\ell \ge 0$, $\nabla^\ell f$ denotes all of the ℓth derivatives of f.

For a function f, its Fourier transform denoted by $\mathcal{F}[f] = \hat{f}$:

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\mathrm{d}x.$$

The inverse of $\mathcal{F}$ is denoted by $\mathcal{F}^{-1}[f] = \check{f}$:

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}\mathrm{d}\xi.$$

In the following, C stands for a 'harmless' constant, and we sometimes use the notation $A \lesssim B$ as an equivalent of $A \le CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2.2. Besov spaces

This section provides some basic knowledge on Besov spaces, which can be found in [19]. We first introduce the dyadic partition of unity. We can use for instance any $(\phi,\chi) \in C^\infty$, such that ϕ is supported in $\{\xi \in \mathbb{R}^n : 3/4 \le |\xi| \le 8/3\}$, χ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \le 4/3\}$ such that

$$\chi(\xi) + \sum_{q\ge 0}\phi(2^{-q}\xi) = 1 \quad \xi \in \mathbb{R}^n,$$
$$\sum_{q\in\mathbb{Z}}\phi(2^{-q}\xi) = 1 \quad \text{if} \quad \xi \ne 0.$$

Denoting $h = \mathcal{F}^{-1}[\phi]$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, we define the dyadic blocks as follows

$$\Delta_{-1}u = \chi(D)u = \tilde{h} * u,$$
$$\Delta_q u = \phi(2^{-q}D)u = 2^{qn}\int_{\mathbb{R}^n} h(2^q y)u(x-y)\mathrm{d}y \quad \text{if} \quad q \ge 0,$$
$$\dot{\Delta}_q u = \phi(2^{-q}D)u = 2^{qn}\int_{\mathbb{R}^n} h(2^q y)u(x-y)\mathrm{d}y \quad \text{if} \quad q \in \mathbb{Z}.$$

The low frequency cut-off operator is defined by

$$S_q u = \sum_{-1\le k\le q-1}\Delta_q u, \quad \dot{S}_q u = \sum_{k\le q-1}\dot{\Delta}_k u.$$

The following two formal decompositions

$$u = \sum_{q\ge -1}\Delta_q u, \quad u = \sum_{q\in\mathbb{Z}}\dot{\Delta}_q u$$

are called inhomogeneous and homogeneous Littlewood–Paley decomposition, respectively.

Let us give the definition of inhomogeneous Besov space as follows.

Definition 2.1: For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, and $u \in \mathcal{S}'$. The inhomogeneous Besov space $B^s_{p,r}$ consists of distributions u in $\mathcal{S}'$ such that

$$\|u\|_{B^s_{p,r}} := \left(\sum_{q \ge -1} 2^{rjs} \|\Delta_q u\|^r_{L^p} \right)^{1/r} < +\infty.$$

Let us now introduce the homogeneous Besov space.

Definition 2.2: We denote by $\mathcal{S}'_h$ the space of tempered distributions u such that

$$\lim_{q \to -\infty} S_q u = 0 \quad \text{in} \quad \mathcal{S}'.$$

Definition 2.3: Let s be a real number and (p, r) be in $[1, \infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of distributions u in $\mathcal{S}'_h$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_{q \in \mathbb{Z}} 2^{rjs} \|\dot{\Delta}_q u\|^r_{L^p} \right)^{1/r} < +\infty.$$

From now on, the notation $\dot{B}^s_p$, B^s_p means $\dot{B}^s_{p,1}$ and $B^s_{p,1}$, respectively. The notation $\dot{B}^s$, B^s means $\dot{B}^s_{2,1}$ and $B^s_{2,1}$, respectively.

The study of non-stationary PDE's usually requires spaces of type $L^r_T(X) := L^r(0, T; X)$ for appropriate Banach spaces X. In our case, X is expected to be a Besov space, so that it is natural to localize equations through Littlewood–Paley decomposition. We then obtain estimates for each dyadic block and perform integration in time. However, in doing so, we obtain bounds in spaces that are not of type $L^r(0, T; B^s_p)$ or $L^r(0, T; \dot{B}^s_p)$. This approach was initiated in [18] naturally leading to the following definitions for the inhomogeneous Besov space.

Definition 2.4: Let $(r, p) \in [1, +\infty]^2$, $T \in (0, +\infty]$ and $s \in \mathbb{R}$. We set

$$\|u\|_{\tilde{L}^r_T(B^s_p)} := \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\Delta_q u(t)\|^r_{L^p} \, \mathrm{d}t \right)^{1/r}$$

and

$$\tilde{L}^r_T(B^s_p) := \left\{ u \in L^r_T(B^s_p), \|u\|_{\tilde{L}^r_T(B^s_p)} < +\infty \right\}.$$

Owing to Minkowski inequality, we have $\tilde{L}^r_T(B^s_p) \hookrightarrow L^r_T(B^s_p)$. That embedding is strict in general if $r > 1$. We will denote by $\tilde{C}_T(B^s_p)$ the set of functions u belonging to $\tilde{L}^\infty_T(B^s_p) \cap C([0, T]; B^s_p)$. For the homogeneous Besov space, we can define similarly.

Let X stands for B or $\dot{B}$, we have the following interpolation inequality

$$\|u\|_{\tilde{L}^r_T(X^s_p)} \le \|u\|^{\theta}_{\tilde{L}^{r_1}_T(X^{s_1}_p)} \|u\|^{1-\theta}_{\tilde{L}^{r_2}_T(X^{s_2}_p)},$$

with

$$\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} \quad \text{and} \quad s = \theta s_1 + (1-\theta) s_2,$$

and the following embeddings

$$\tilde{L}^r_T(X^{n/p}_p) \hookrightarrow L^r_T(\mathcal{C}_0) \quad \text{and} \quad \tilde{C}_T(X^{n/p}_p) \hookrightarrow C([0,T]\times\mathbb{R}^n).$$

Next, let us give the definition and some properties about hybrid Besov spaces.

Definition 2.5: let $s,t\in\mathbb{R}$. We set

$$\|u\|_{B^{s,t}_{q,p}} := \sum_{k\le R_0} 2^{qs}\|\dot{\Delta}_k u\|_{L^q} + \sum_{k>R_0} 2^{qt}\|\dot{\Delta}_k u\|_{L^p}.$$

and

$$B^{s,t}_{q,p}(\mathbb{R}^N) := \left\{u\in \mathcal{S}'_h(\mathbb{R}^N) : \|u\|_{B^{s,t}_{q,p}} < +\infty\right\},$$

where R_0 is a fixed large enough number.

Lemma 2.6:

(1) *We have $B^{s,s}_{2,2} = \dot{B}^s$.*
(2) *If $s\le t$ then $B^{s,t}_{p,p} = \dot{B}^s_p\cap\dot{B}^t_p$. Otherwise, $B^{s,t}_{p,p} = \dot{B}^s_p\oplus\dot{B}^t_p$.*
(3) *The space $B^{0,s}_{p,p}$ coincide with the usual inhomogeneous Besov space.*
(4) *If $s_1\le s_2$ and $t_1\ge t_2$ then $B^{s_1,t_1}_{p,p}\hookrightarrow B^{s_2,t_2}_{p,p}$.*
(5) *Interpolation: For $s_1,s_2,\sigma_1,\sigma_2\in\mathbb{R}$ and $\theta\in[0,1]$, we have*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2,\theta\sigma_1+(1-\theta)\sigma_2}_{2,p}} \le \|f\|^{\theta}_{B^{s_1,\sigma_1}_{2,p}}\|f\|^{1-\theta}_{B^{s_2,\sigma_2}_{2,p}}.$$

From now on, the notations $B^{s,t}_p$, $B^{s,t}$ means $B^{s,t}_{p,p}$ and $B^{s,t}_{2,2}$, respectively. For more information on Besov spaces and hybrid Besov spaces, we give references [19–22].

For the reader's convenience, we list here an important lemma [5,21] which will be used in the following.

Lemma 2.7: *Let F be a homogeneous smooth function of degree m. Suppose $1-n/2<\rho\le 1+n/2$ and $-1/n<\rho'\le n/2+1$. Then the following inequialities hold:*

$$\begin{aligned}
&|(F(D)\Delta_q(v\cdot\nabla c)|F(D)\Delta_q c)| \\
&\quad\le C\alpha_q 2^{-q(\rho'-m)}\|v\|_{\dot{B}^{n/2+1}}\|c\|_{\dot{B}^{\rho'}}\|F(D)\Delta_q c\|_{L^2},\\
&|(F(D)\Delta_q v\cdot\nabla c|F(D)\Delta_q c)| \\
&\quad\le C\alpha_q 2^{-q(\rho-m)}\min(2^q,1)\|v\|_{\dot{B}^{n/2+1}}\|c\|_{B^{\rho-1,\rho}}\|F(D)\Delta_q c\|_{L^2},\\
&|(F(D)\Delta_q(v\cdot\nabla c)|\Delta_q d)| + |(\Delta_q(v\cdot d)|F(D)\Delta_q c)| \\
&\quad\le C\alpha_q 2^{-q(\rho-m)}\min(2^q,1)\|v\|_{\dot{B}^{n/2+1}}(\|c\|_{B^{\rho-1,\rho}}\|\Delta_q d\|_{L^2} + \|d\|_{B^{\rho-1,\rho}}\|\Delta_q c\|_{L^2}),\\
&|(F(D)\Delta_q(v\cdot c)|\Delta_q d)| + |(\Delta_q(v\cdot\nabla d)|F(D)\Delta_q c)| \\
&\quad\le C\alpha_q\|v\|_{\dot{B}^{n/2+1}}(2^{-q\rho'}\|F(D)\Delta_q c\|_{L^2}\|d\|_{\dot{B}^{\rho'}} \\
&\qquad + 2^{-q(\rho-m)}\min(2^q,1)\|d\|_{B^{\rho-1,\rho}}\|\Delta_q d\|_{L^2}).
\end{aligned}$$

2.3. Useful theorems

In this part, we will list two theorems for (1.6) which are essential for our proof of Theorem 1.2. We denote

$$E_p(T) := \left\{v\in C([0,T];B^{n/p}_p),\ \partial_t v,\nabla^2 v\in L^1\left(0,T;B^{n/p}_p\right)\right\}.$$

For $v \in E_p(T)$ will be endowed with the norm

$$\|v\|_{E_p(T)} := \|v\|_{L^\infty_T(B_p^{n/p-1})} + \|\partial_t v, \nabla^2 v\|_{L^1_T(B_p^{n/p-1})}.$$

Employing similar methods used in [8] or by simple changing the low frequency estimates in [5], we obtain Theorem 2.8. We omit the proof because it has no new features.

Theorem 2.8: *Let $1 < p < 2n$ and $n \geq 2$. Let v_0 be vector field in $B_p^{n/p-1}$. Assume ρ_0 satisfies $a_0 := \rho_0 - 1 \in B_p^{n/p}$, U_0 satisfies $F_0 := U_0 - I \in B_p^{n/p}$ and*

$$\inf_x \rho_0(x) > 0. \tag{2.1}$$

Then system (1.6) has a unique local solution (ρ, v, U) with $v \in E_p(T)$, $U - I \in C([0,T]; B_p^{n/p})$, ρ bounded away from 0 and $\rho - 1 \in C([0,T]; B_p^{n/p})$.

Denote:

$$\begin{aligned}\mathcal{E}^s := \Big\{(a,u,F) \in\ & (L^1(0,\infty; B_{2,p}^{s_p+1,s}) \cap \tilde{L}^\infty(0,\infty; B_{2,p}^{s_p-1,s}))\\ & \times (L^1(0,\infty; B_{2,p}^{s_p+1,s+1}) \cap \tilde{L}^\infty(0,\infty; B_{2,p}^{s_p-1,s-1}))^n\\ & \times (L^1(0,\infty; B_{2,p}^{s_p+1,s}) \cap \tilde{L}^\infty(0,\infty; B_{2,p}^{s_p-1,s}))^{n\times n}\Big\},\end{aligned}$$

where $s_p = s - \frac{n}{p} + \frac{n}{2}$.

The following Theorem 2.9 shows that system (1.6) has a unique global solution in critical Besov spaces.

Theorem 2.9: [8] *Let $\bar{\rho} > 0$ be a constant such that $P'(\bar{\rho}) > 0$. Suppose that $n = 3$. There exist two positive constants α_0 and C such that for all (ρ_0, v_0, U_0) with $\rho_0 - \bar{\rho} \in B_{2,p}^{n/2-1,n/p}$, $U_0 - I \in B_{2,p}^{n/2-1,n/p}$, $v_0 \in B_{2,p}^{n/2-1,n/p-1}$, and*

$$\|\rho_0 - \bar{\rho}\|_{B_{2,p}^{n/2-1,n/p}} + \|v_0\|_{B_{2,p}^{n/2-1,n/p-1}} + \|U_0 - I\|_{B_{2,p}^{n/2-1,n/p}} \leq \alpha_0, \tag{2.2}$$

then if $2 \leq p < n$, system (1.6) has a unique global solution $(\rho - \bar{\rho}, v, U - I) \in \mathcal{E}^{n/p}$ with

$$\begin{aligned}\|(\rho - \bar{\rho}, v, U - I)\|_{\mathcal{E}^{n/p}} \leq C\Big(&\|\rho_0 - \bar{\rho}\|_{B_{2,p}^{n/2-1,n/p}} + \|v_0\|_{B_{2,p}^{n/2-1,n/p-1}}\\ &+ \|U_0 - \bar{U}\|_{B_{2,p}^{n/2-1,n/p}}\Big).\end{aligned}$$

Remark 2.10: Taking $p = 2$ in Theorem 2.9, we gain global well-posedness in critical homogeneous Besov spaces. Assume

$$\|(\rho_0 - \bar{\rho}, U - I)\|_{B^{n/2} \cap \dot{B}^0_{1,\infty}} + \|v_0\|_{B^{n/2-1} \cap \dot{B}^0_{1,\infty}} \leq \delta,$$

as in our main Theorem 1.2. If $\delta > 0$ is taken to be small enough, then the above-mentioned assumption implies (2.2) for $p = 2$. Hence, we obtain the results in Theorem 2.9. In particular, we know that

$$\int_0^\infty \|v\|_{\dot{B}^{n/2+1}} \mathrm{d}t \leq C\delta. \tag{2.3}$$

In the following sections, this estimate plays an essential role.

3. Analysis of low frequency part

In this section, we first decompose the system into three small systems and then carefully analyze the semigroup of the low frequency part. Without losing generality, we assume $P'(1) = 1$, $\bar{\rho} = 1$ and set $\nu = \lambda + 2\mu$, $\mathcal{A} = \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$. Define

$$\begin{gathered} K(a) = \frac{P'(1+a)}{1+a} - 1, \quad d = \Lambda^{-1}\mathrm{div}v, \\ \Omega = \Lambda^{-1}\mathrm{curl}v \quad \text{with } (\mathrm{curl}v)_{ij} = \partial_{x_j}v^i - \partial_{x_i}v^j, \\ \mathcal{E}_{ij} = \Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}(F^{ij} + F^{ji}), \\ \mathcal{W} = \Lambda^{-1}\partial_{x_k}(F^{lk}\nabla_l F^{ij} - F^{lj}\nabla_l F^{ik}) - \Lambda^{-1}\partial_{x_k}(F^{lk}\nabla_l F^{ji} - F^{li}\nabla_l F^{jk}). \end{gathered}$$

where

$$\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s\hat{f}) \quad \text{for } s \in \mathbb{R}.$$

We adopt decompositions used in [6] to obtain the following equations

$$\begin{cases} \partial_t a + \Lambda d = L - v\cdot\nabla a, \\ \partial_t d - \mu\Delta d - 2\Lambda a = G - v\cdot\nabla d, \\ \partial_t \mathcal{E} + 2\Lambda d = J - v\cdot\nabla\mathcal{E}, \\ \partial_t(F^T - F) + \Lambda\Omega = I - v\cdot\nabla(F^T - F), \\ \partial_t\Omega - \mu\Delta\Omega - \Lambda(F^T - F) = H - v\cdot\nabla\Omega. \end{cases} \tag{3.1}$$

In addition, equation of d has the following equivalent form

$$\partial_t d - \nu\Delta d - \Lambda\mathcal{E} = K - v\cdot\nabla d, \tag{3.2}$$

where

$$\begin{aligned} L &= -a\mathrm{div}v, \\ G &= v\cdot\nabla d + \Lambda^{-1}\mathrm{div}\left(-v\cdot\nabla v + F\nabla F - K(a)\nabla a - \frac{a}{1+a}\mathcal{A}v - \mathrm{div}(aF)\right), \\ H &= v\cdot\nabla\Omega + \Lambda^{-1}\mathrm{curl}\left(-v\cdot\nabla v + F\nabla F - K(a)\nabla a - \frac{a}{1+a}\mathcal{A}v\right) + \mathcal{W}, \\ I &= (\nabla vF)^T - \nabla vF, \\ J &= -[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}, v^k]\partial_{x_k}(F^{ij} + F^{ji}) \\ &\quad + \Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}((\nabla vF)^{ij} + (\nabla vF)^{ji}), \\ K &= v\cdot\nabla d + \Lambda^{-1}\mathrm{div}(-v\cdot\nabla v + F\nabla F - K(a)\nabla a - \frac{a}{1+a}\mathcal{A}v + \mathrm{div}(aF)). \end{aligned}$$

Here, we denote

$$\begin{aligned} M_1(t) &:= \sup_{0\le\tau\le t}(1+\tau)^{n/4}\left(\|a(\tau)\|_{B^{n/2-1,n/2}} + \|d(\tau)\|_{\dot{B}^{n/2-1}}\right), \\ M_2(t) &:= \sup_{0\le\tau\le t}(1+\tau)^{n/4}\left(\|\mathcal{E}(\tau)\|_{B^{n/2-1,n/2}} + \|d(\tau)\|_{\dot{B}^{n/2-1}}\right), \\ M_3(t) &:= \sup_{0\le\tau\le t}(1+\tau)^{n/4}\left(\|(F^T - F)(\tau)\|_{B^{n/2-1,n/2}} + \|\Omega(\tau)\|_{\dot{B}^{n/2-1}}\right), \end{aligned}$$

$$M_4(t) := \sup_{0\le\tau\le t} (1+\tau)^{n/4}\left(\|a\|_{L^2} + \|F\|_{L^2} + \|v\|_{L^2}\right),$$

and

$$M(t) := \sup_{0\le\tau\le t} (1+\tau)^{n/4}\left(\|a\|_{B^{n/2}} + \|F\|_{B^{n/2}} + \|v\|_{B^{n/2-1}}\right).$$

From basic properties of Besov spaces, we easily know

$$M(t) \approx M_1(t) + M_2(t) + M_3(t) + M_4(t),$$

under smallness conditions of the initial data, where we used (5.13) to (5.15) in [8]. Here, we also need to denote

$$\bar{M}_1 = \begin{pmatrix} L - v\cdot\nabla a \\ G - v\cdot\nabla s \end{pmatrix}, \quad \bar{M}_2 = \begin{pmatrix} J - v\cdot\nabla\mathcal{E} \\ K - v\cdot\nabla d \end{pmatrix}, \tag{3.3}$$

and

$$\bar{M}_3 = \begin{pmatrix} I - v\cdot\nabla(F^T - F) \\ H - v\cdot\nabla\Omega \end{pmatrix}. \tag{3.4}$$

Now, we need to introduce the following linearized system with convection terms.

$$\begin{cases} \partial_t a + \Lambda d = L - v\cdot\nabla a, \\ \partial_t d - \mu\Delta d - 2\Lambda a = G - v\cdot\nabla d, \\ \partial_t\mathcal{E} + 2\Lambda d = J - v\cdot\nabla\mathcal{E}, \\ \partial_t(F^T - F) + \Lambda\Omega = I - v\cdot\nabla(F^T - F), \\ \partial_t\Omega - \mu\Delta\Omega - \Lambda(F^T - F) = H - v\cdot\nabla\Omega. \end{cases} \tag{3.5}$$

We can decompose the above system into three subsystems.

$$\begin{cases} \partial_t a + \Lambda d = L - v\cdot\nabla a, \\ \partial_t d - \mu\Delta d - 2\Lambda a = G - v\cdot\nabla d. \end{cases} \tag{3.6}$$

$$\begin{cases} \partial_t\mathcal{E} + 2\Lambda d = J - v\cdot\nabla\mathcal{E}, \\ \partial_t d - \nu\Delta d - \Lambda\mathcal{E} = K - v\cdot\nabla d. \end{cases} \tag{3.7}$$

$$\begin{cases} \partial_t(F^T - F) + \Lambda\Omega = I - v\cdot\nabla(F^T - F), \\ \partial_t\Omega - \mu\Delta\Omega - \Lambda(F^T - F) = H - v\cdot\nabla\Omega. \end{cases} \tag{3.8}$$

The above three systems have similar mathematical structures, so that we only need to study the following linear system, which captures the main structures of the systems (3.6)–(3.8).

$$\begin{cases} \partial_t c + \alpha\Lambda u = 0, \\ \partial_t u - \kappa\Delta u - \beta\Lambda c = 0, \end{cases} \tag{3.9}$$

where c, u are scalar functions and α, β, κ are positive constants. We first provide some important properties of the Green's matrix of the above system (3.9).

Lemma 3.1: *Let $\mathcal{G}$ be the Green matrix of system (3.9). Then we have the following explicit expression of $\hat{\mathcal{G}}$:*

$$\hat{\mathcal{G}}(\xi,t) = \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -\alpha\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)|\xi| \\ -\beta\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)|\xi| & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix}$$

where

$$\lambda_\pm = -\frac{1}{2}\kappa|\xi|^2 \pm \frac{1}{2}\sqrt{\kappa^2|\xi|^4 - 4\alpha\beta|\xi|^2}.$$

Lemma 3.2: *Given $R > 0$, there is a positive number $\subseteq$ depends on R such that, for any multi-indices γ and $|\xi| \le R$,*

$$|D_\xi^\gamma \hat{\mathcal{G}}(\xi,t)| \le Ce^{-\subseteq|\xi|^2 t}(1+|\xi|)^{|\gamma|}(1+t)^{|\gamma|}$$

where $C = C(R, |\gamma|)$.

The proof of the above two lemmas follows those of Lemma 3.1 and Theorem 3.2 in [23], so that we omit the proof for simplicity. Next, we prove an important lemma that has a key role in the low frequency analysis.

Lemma 3.3: *Let $\mathcal{G}$ be the Green matrix of system (3.9), dimension $n = 3$. Denote $U_0 = (c_0, u_0)$, then $\mathcal{G}(t)$ satisfies the estimate*

$$\sum_{q\le R} \|\mathcal{G}(t)\dot{\Delta}_q U_0\|_{L^2} \le C(1+t)^{-n/4}\|U_0\|_{\dot{B}^0_{1,\infty}}$$

for $t \ge 0$ and $R > 0$ is a large positive constant.

Proof: Using Placherel's theorem and Lemma 3.2, we have

$$\begin{aligned} \|\mathcal{G}(t)\dot{\Delta}_q U_0\|_{L^2} &\lesssim \left(\int_{\frac{3}{4}2^q<|\xi|<\frac{8}{3}2^q} \left|e^{\hat{\mathcal{G}}(\xi)t}\phi_q(\xi)\hat{U}_0\right|^2 \mathrm{d}\xi\right)^{1/2} \\ &\lesssim \left(\int_{\frac{3}{4}2^q<|\xi|<\frac{8}{3}2^q} e^{-\theta|\xi|^2 t}\left|\phi_q(\xi)\hat{U}_0\right|^2 \mathrm{d}\xi\right)^{1/2} \\ &\lesssim \|\dot{\Delta}_q U_0\|_{L^1}\left(\int_{\frac{3}{4}2^q<|\xi|<\frac{8}{3}2^q} e^{-\theta|\xi|^2 t}\mathrm{d}\xi\right)^{1/2} \\ &\lesssim t^{-n/4}\|\dot{\Delta}_q U_0\|_{L^1}\left(\int_{\sqrt{t}\frac{3}{4}2^q<|\xi|<\sqrt{t}\frac{8}{3}2^q} r^{\frac{n-1}{2}}e^{-\theta r^2}\mathrm{d}r\right)^{1/2} \\ &\lesssim t^{-n/4}\|U_0\|_{\dot{B}^0_{1,\infty}} e^{-\frac{2}{9}4^q t\theta}\left(1 - e^{-\frac{20}{3}4^q t\theta}\right)^{1/2} \end{aligned} \tag{3.10}$$

Now, we perform some calculations about $\sum_{q\le R} e^{-\frac{2}{9}4^q t\theta}\left(1-e^{-\frac{20}{3}4^q t\theta}\right)^{1/2}$. Let $k=-q$, then we have

$$\begin{aligned}\sum_{q\le R} e^{-\frac{2}{9}4^q t\theta}\left(1-e^{-\frac{20}{3}4^q t\theta}\right)^{1/2} &= \sum_{k=-R}^{\infty} e^{-\frac{2}{9}\left(\frac{1}{4}\right)^k t\theta}\left(1-e^{-\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta}\right)^{1/2}\\ &= I+II+III,\end{aligned}$$

where

$$\begin{aligned} I &= \sum_{k=-R}^{0} e^{-\frac{2}{9}\left(\frac{1}{4}\right)^k t\theta}\left(1-e^{-\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta}\right)^{1/2},\\ II &= \sum_{k=1}^{\left[\log_4 \frac{20}{3}t\theta\right]} e^{-\frac{2}{9}\left(\frac{1}{4}\right)^k t\theta}\left(1-e^{-\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta}\right)^{1/2}\end{aligned}$$

and

$$III = \sum_{k=\left[\log_4 \frac{20}{3}t\theta\right]+1}^{\infty} e^{-\frac{2}{9}\left(\frac{1}{4}\right)^k t\theta}\left(1-e^{-\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta}\right)^{1/2}.$$

For I, we have $I\le R\cdot 1\le C<\infty$. For an arbitrary $t>0$, there exists a positive integer $N>0$ such that $t\theta\le \frac{3}{20}4^N$. Hence, without loss of generality, we can choose $t\theta=\frac{3}{20}4^N$. For II, we have

$$\begin{aligned} II &= \sum_{k=1}^{N} e^{-\frac{2}{9}\left(\frac{1}{4}\right)^k \frac{3}{20}4^N}\left(1-e^{-\left(\frac{1}{4}\right)^k 4^N}\right)^{1/2}\\ &\le C\sum_{k=1}^{N} e^{-\frac{1}{30}\left(\frac{1}{4}\right)^k 4^N} \le C\sum_{k=1}^{N} e^{-\frac{1}{30}4^{-(k-N)}}\\ &\le C\sum_{m=0}^{N-1} e^{-\frac{1}{30}4^m}\le C<\infty.\end{aligned}$$

Using Taylor's formula, we have

$$1-e^{-\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta} = \frac{20}{3}\left(\frac{1}{4}\right)^k t\theta+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n!}\left(\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta\right)^n.$$

When $k>\log_4 \frac{20}{3}t\theta$, we have $\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta<1$. Through properties of alternating series, we know

$$1-e^{-\frac{20}{3}\left(\frac{1}{4}\right)^k t\theta}\le \frac{40}{3}\left(\frac{1}{4}\right)^k t\theta. \tag{3.11}$$

Substituting (3.11) into III, we obtain

$$III\le C\sum_{k=\left[\log_4 \frac{20}{3}t\theta\right]+1}^{\infty} e^{-\frac{2}{9}\left(\frac{1}{4}\right)^k t\theta}\left(\frac{1}{4}\right)^{k/2}\sqrt{t\theta}$$

$$\leq C\sqrt{t\theta}e^{-\frac{2}{9}t\theta} \sum_{k=\left[\log_4 \frac{20}{3}t\theta\right]+1}^{\infty} \left(\frac{1}{2}\right)^k$$
$$\leq C < \infty.$$

Summing up the estimates for I, II, III, we finally get

$$\sum_{q\leq R} e^{-\frac{2}{9}4^q t\theta} \left(1 - e^{-\frac{20}{3}4^q t\theta}\right)^{1/2} \leq C < \infty, \tag{3.12}$$

where C does not depend on t.

Through estimates (3.10) and (3.12), we obtain

$$\begin{aligned} \sum_{q\leq R} \|\mathcal{G}(t)\dot{\Delta}_q U_0\|_{L^2} &\leq Ct^{-n/4}\|U_0\|_{\dot{B}^0_{1,\infty}} \sum_{q\leq R} e^{-\frac{2}{9}4^q t\theta} \left(1 - e^{-\frac{20}{3}4^q t\theta}\right)^{1/2} \\ &\leq Ct^{-n/4}\|U_0\|_{\dot{B}^0_{1,\infty}}. \end{aligned} \tag{3.13}$$

Similarly, we also find that

$$\begin{aligned} \sum_{q\leq R} \|\mathcal{G}(t)\dot{\Delta}_q U_0\|_{L^2} &\leq C\sum_{q\leq R} \|\dot{\Delta}_q U_0\|_{L^1} \left(\int_{\frac{3}{4}2^q\leq r\leq \frac{8}{3}2^q} r^{n-1}e^{-\theta r^2 t}dr\right)^{1/2} \\ &\leq C\|U_0\|_{\dot{B}^0_{1,\infty}} \sum_{q\leq R} (\sqrt{8})^q \leq C < \infty. \end{aligned} \tag{3.14}$$

Through estimates (3.13) and (3.14), we finally obtain our desired results. □

Remark 3.4: Let $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ stand for the Green matrix of system (3.6)–(3.8), respectively. Denote $V_0^1 = (a_0, d_0)$, $V_0^2 = (\mathcal{E}_0, d_0)$ and $V_0^3 = (F_0^T - F_0, \Omega_0)$. By similar methods used to prove Lemma 3.3, we will have

$$\sum_{q\leq R} \|\mathcal{G}_i(t)\dot{\Delta}_q V_0^i\|_{L^2} \leq C(1+t)^{-n/4}\|V_0^i\|_{\dot{B}^0_{1,\infty}}$$

for $i = 1, 2, 3$.

Next, we need to consider estimates for $\bar{M}_1, \bar{M}_2, \bar{M}_3$ defined in (3.3) and (3.4).

Lemma 3.5: *There exists a constant $\delta > 0$ such that if*

$$\|a_0\|_{B^{n/2}} + \|v_0\|_{B^{n/2-1}} + \|F_0\|_{B^{n/2}} \leq \delta,$$

then there exists a constant $C > 0$ independent of time T such that

$$\|\bar{M}_1, \bar{M}_2, \bar{M}_3\|_{\dot{B}^0_{1,\infty}} \leq C(1+t)^{-n/4}M(t)f(t) + C(1+t)^{-n/2}M^2(t)$$

for $t \in [0, T]$, where $f(t) = \|v(t)\|_{\dot{B}^{n/2+1}} \in L^1(0, \infty)$.

Proof: Now, we start with $\bar{M}_1$. For the term $v\cdot\nabla a$, we have

$$\begin{aligned}\|v\cdot\nabla a\|_{\dot{B}^0_{1,\infty}} &\le C\,\|v\cdot\nabla a\|_{L^1} \le\; C\,\|v\|_{L^2}\|\nabla a\|_{L^2}\\ &\le C\,(1+t)^{-n/4}M_4(t)\|a\|_{B^{n/2-1,n/2}}\\ &\le C\,(1+t)^{-n/2}M^2(t).\end{aligned} \tag{3.15}$$

For the term $a\mathrm{div}v$, we have

$$\begin{aligned}\|a\mathrm{div}v\|_{\dot{B}^0_{1,\infty}} &\le C\,\|a\mathrm{div}v\|_{L^1} \le\; C\,\|a\|_{L^2}\|\nabla v\|_{L^2}\\ &\le C\,(1+t)^{-n/4}M_4(t)\left(\sum_{q\le R}\|\dot{\Delta}_q\nabla v\|_{L^2}+\sum_{q>R}\|\dot{\Delta}_q\nabla v\|_{L^2}\right)\\ &\le C\,(1+t)^{-n/4}M_4(t)\left(\|v\|_{L^2}+\|v\|_{\dot{B}^{n/2+1}}\right)\\ &\le C\,(1+t)^{-n/2}M_4(t)\left(M_4(t)+f(t)\right)\\ &\le C\,(1+t)^{-n/2}M^2(t)+C\,(1+t)^{-n/4}M(t)f(t).\end{aligned} \tag{3.16}$$

For the term $v\cdot\nabla d$, we have

$$\begin{aligned}\|v\cdot\nabla d\|_{\dot{B}^0_{1,\infty}} &\le C\,\|v\cdot\nabla d\|_{L^1} \le\; C\,\|v\|_{L^2}\|\nabla d\|_{L^2}\\ &\le C\,\|v\|_{L^2}\left(\|v\|_{L^2}+\|v\|_{\dot{B}^{n/2+1}}\right)\\ &\le C\,(1+t)^{-n/2}M^2(t)+C\,(1+t)^{-n/4}M(t)f(t).\end{aligned} \tag{3.17}$$

For the term $\Lambda^{-1}\mathrm{div}(v\cdot\nabla v)$, we have

$$\begin{aligned}\|\Lambda^{-1}\mathrm{div}(v\cdot\nabla v)\|_{\dot{B}^0_{1,\infty}} &\le C\,\|v\cdot\nabla v\|_{\dot{B}^0_{1,\infty}} \le\; C\,\|v\cdot\nabla v\|_{L^1}\\ &\le C\,\|v\|_{L^2}\left(\|v\|_{L^2}+\|v\|_{\dot{B}^{n/2+1}}\right)\\ &\le C\,(1+t)^{-n/2}M^2(t)+C\,(1+t)^{-n/4}M(t)f(t).\end{aligned} \tag{3.18}$$

For the term $\Lambda^{-1}\mathrm{div}(F\nabla F)$, similar to (3.18) and (3.15), we can obtain

$$\|\Lambda^{-1}\mathrm{div}(F\nabla F)\|_{\dot{B}^0_{1,\infty}} \le\; C\,(1+t)^{-n/2}M^2(t). \tag{3.19}$$

Using composition rules (for example: Theorem 2.61 in [19]), we could obtain

$$\begin{aligned}\left\|\Lambda^{-1}\mathrm{div}\left(\frac{a}{1+a}\mathcal{A}v\right)\right\|_{\dot{B}^0_{1,\infty}} &\le C\,\left\|\frac{a}{1+a}\mathcal{A}v\right\|_{L^1}\\ &\le C\,\|a\|_{L^2}\|\nabla^2 v\|_{L^2}\\ &\le C\,\|a\|_{L^2}\left(\|v\|_{L^2}+\|v\|_{\dot{B}^{n/2+1}}\right)\\ &\le C\,(1+t)^{-n/2}M^2(t)\\ &\quad +C\,(1+t)^{-n/4}M(t)f(t),\end{aligned} \tag{3.20}$$

where we used $n/2+1>2$. Summing up estimates from (3.15) to (3.20), we could get

$$\|\bar{M}_1\|_{\dot{B}^0_{1,\infty}} \le\; C\,(1+t)^{-n/2}M^2(t)+C\,(1+t)^{-n/4}M(t)f(t). \tag{3.21}$$

Next, let us estimate $\bar{M}_2$. The terms $v\cdot\nabla\mathcal{E}$, $v\cdot\nabla d$ and K can all be estimated similar to the terms appeared in $\bar{M}_1$, so we just need to give the following estimates about J. Since

$$\begin{aligned}\left\|\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\nabla_{x_j}\left[\left(\nabla vF\right)^{ij}+\left(\nabla vF\right)^{ji}\right]\right\|_{\dot{B}^0_{1,\infty}} &\le C\|\nabla v\|_{L^2}\|F\|_{L^2}\\ &\le C(1+t)^{-n/2}M^2(t)+C(1+t)^{-n/4}M(t)f(t),\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}&\|\left[\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j},v^k\right]\partial_{x_k}(F^{ij}+F^{ji})\|_{\dot{B}^0_{1,\infty}}\\ &\quad\le\|\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}(v^k\partial_{x_k}(F^{ij}+F^{ji}))\|_{\dot{B}^0_{1,\infty}}\\ &\qquad+\|v^k\cdot\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}(\partial_{x_k}(F^{ij}+F^{ji}))\|_{\dot{B}^0_{1,\infty}}\\ &\quad\le C(1+t)^{-n/2}M^2(t)+C\|v\|_{L^2}\|\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}(\partial_{x_k}(F^{ij}+F^{ji}))\|_{\dot{B}^0_{1,\infty}}\\ &\quad\le C(1+t)^{-n/2}M^2(t)+C\|v\|_{L^2}\left(\|F\|_{L^2}+\|F\|_{B^{n/2-1,n/2}}\right)\\ &\quad\le C(1+t)^{-n/2}M^2(t),\end{aligned}\tag{3.23}$$

we have

$$\|\bar{M}_2\|_{\dot{B}^0_{1,\infty}}\le C(1+t)^{-n/2}M^2(t)+C(1+t)^{-n/4}M(t)f(t).\tag{3.24}$$

Since all terms appeared in $\bar{M}_3$ can be estimated similar to the terms appeared in $\bar{M}_1$ and $\bar{M}_2$, here, we just give the estimates as follows

$$\|\bar{M}_3\|_{\dot{B}^0_{1,\infty}}\le C(1+t)^{-n/2}M^2(t)+C(1+t)^{-n/4}M(t)f(t).\tag{3.25}$$

At this stage, we easily finished the proof by just summing up (3.21), (3.24) and (3.25). □

Denote $V=(a,v,F)$, $V_1=(a,d)$, $V_2=(\mathcal{E},d)$, $V_3=(F^T-F,\Omega)$ and define $\Delta_R f:=\Delta_{-1}f+\sum_{0\le q\le R}\Delta_q f$ for a tempered distribution f. Now, we can prove the following proposition which is the main result for the low frequency part.

Proposition 3.6: *Let $n=3$, there exists a constant $\delta>0$ such that if*

$$\|a_0\|_{B^{n/2}}+\|v_0\|_{B^{n/2-1}}+\|F_0\|_{B^{n/2}}\le\delta,$$

then there exists a constant $C>0$ independent of time T such that

$$\sup_{0\le\tau\le t}(1+\tau)^{n/4}\|\Delta_R V(\tau)\|_{L^2}\le C\|V_0\|_{\dot{B}^0_{1,\infty}}+C\delta M(t)+CM^2(t)$$

for $t\in[0,T]$.

Proof: Through the properties of the Littlewood–Paley operator and (5.13), (5.14), (5.15) in [8], we have

$$\begin{aligned}\|\Delta_R V(\tau)\|_{L^2}&\le\sum_{q\le R}\|\dot{\Delta}_q\Delta_R V(\tau)\|_{L^2}\lesssim\sum_{q\le R}\|\dot{\Delta}_q V(\tau)\|_{L^2}\\ &\lesssim\sum_{q\le R}\|\dot{\Delta}_q V_1(\tau)\|_{L^2}+\sum_{q\le R}\|\dot{\Delta}_q V_2(\tau)\|_{L^2}\\ &\quad+\sum_{q\le R}\|\dot{\Delta}_q V_3(\tau)\|_{L^2}+\sum_{q\le R}\|\dot{\Delta}_q\Lambda^{-1}(F\nabla F)(\tau)\|_{L^2}\end{aligned}\tag{3.26}$$

For the last term appearing in the above inequality (3.26), we have

$$\sum_{q\le R}\|\dot\Delta_q\Lambda^{-1}(F\nabla F)(\tau)\|_{L^2}\le C\,\|F\|_{\dot B^0_{2,2}}\|F\|_{\dot B^{n/2}}\\ \le C\,(1+t)^{-n/2}M^2(t), \tag{3.27}$$

where we used Lemma A.4 in [8](take $\tilde t=s=0, \tilde s=t=\frac{1}{2}, p=2$ and $\gamma=0$). From (3.1) and (3.2), we easily obtain

$$\begin{aligned}\dot\Delta_q V_1(t)&=\mathcal{G}_1(t)\dot\Delta_q V_{10}+\int_0^t\mathcal{G}_1(t-s)\dot\Delta_q\bar M_1(s)ds,\\ \dot\Delta_q V_2(t)&=\mathcal{G}_2(t)\dot\Delta_q V_{20}+\int_0^t\mathcal{G}_2(t-s)\dot\Delta_q\bar M_2(s)ds,\\ \dot\Delta_q V_3(t)&=\mathcal{G}_3(t)\dot\Delta_q V_{30}+\int_0^t\mathcal{G}_3(t-s)\dot\Delta_q\bar M_3(s)ds.\end{aligned} \tag{3.28}$$

By using Remark 3.4 and Lemma 3.5, we will obtain

$$\begin{aligned}\|\Delta_R V(\tau)\|_{L^2}&\lesssim\sum_{q\le R}\|\dot\Delta_q V_1(\tau)\|_{L^2}+\sum_{q\le R}\|\dot\Delta_q V_2(\tau)\|_{L^2}\\ &\quad+\sum_{q\le R}\|\dot\Delta_q V_3(\tau)\|_{L^2}+(1+\tau)^{-n/2}M^2(\tau)\\ &\lesssim I+II+(1+\tau)^{-n/2}M^2(\tau),\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}I&=\sum_{q\le R}\left\{\|\mathcal{G}_1(\tau)\dot\Delta_q V_{10}\|_{L^2}+\|\mathcal{G}_2(\tau)\dot\Delta_q V_{20}\|_{L^2}+\|\mathcal{G}_3(\tau)\dot\Delta_q V_{30}\|_{L^2}\right\}\\ &\le C\,(1+\tau)^{-n/4}\left\{\|V_{10}\|_{\dot B^0_{1,\infty}}+\|V_{20}\|_{\dot B^0_{1,\infty}}+\|V_{30}\|_{\dot B^0_{1,\infty}}\right\}\\ &\le C\,(1+\tau)^{-n/4}\|V_0\|_{\dot B^0_{1,\infty}},\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}II&=\sum_{q\le R}\int_0^\tau\Big\{\|\mathcal{G}_1(\tau-s)\dot\Delta_q\bar M_1(s)\|_{L^2}+\|\mathcal{G}_2(\tau-s)\dot\Delta_q\bar M_2(s)\|_{L^2}\\ &\quad+\|\mathcal{G}_3(\tau-s)\dot\Delta_q\bar M_3(s)\|_{L^2}\Big\}\mathrm{d}s\\ &\le C\int_0^\tau(1+\tau-s)^{-n/4}\left\{\|\bar M_1\|_{\dot B^0_{1,\infty}}+\|\bar M_2\|_{\dot B^0_{1,\infty}}+\|\bar M_3\|_{\dot B^0_{1,\infty}}\right\}\mathrm{d}s\\ &\le C\,(1+\tau)^{-n/4}M(t)\int_0^\tau f(s)\mathrm{d}s+C\,(1+\tau)^{-n/2}M^2(\tau)\\ &\le C\,(1+\tau)^{-n/4}M(t)\delta+C\,(1+\tau)^{-n/2}M^2(\tau),\end{aligned} \tag{3.31}$$

where we used Lemma 3.5 in the above estimates. Summing up (3.29)–(3.31), we finally obtain

$$\sup_{0\le\tau\le t}(1+\tau)^{n/4}\|\Delta_R V(\tau)\|_{L^2}\le\ C\,\|V_0\|_{\dot B^0_{1,\infty}}+C\,\delta M(t)+C\,M^2(t). \tag{3.32}$$

□

4. Analysis of high frequency part

In this part, we need to convert the system into another form and obtain the estimates in the high frequency domain which is completely different from the low frequency domain.

Without loss of generality, assume $\bar{\rho} = 1$ and $\gamma = \sqrt{P'(\bar{\rho})} - 1$. Denote $a = \rho - 1$, $F = U - I$, $\Lambda = (-\Delta)^{1/2}$, $d = \Lambda^{-1}\mathrm{div}v$, $e^{ij} = \Lambda^{-1}\nabla_j v^i$. From $U^{\ell k}\nabla_\ell U^{ij} - U^{\ell j}\nabla_\ell U^{ik} = 0$, we easily know

$$\Lambda^{-1}(\nabla_j\nabla_k F^{ik}) = -\Lambda F^{ij} - \Lambda^{-1}\nabla_k\left(F^{\ell j}\nabla_\ell F^{ik} - F^{\ell k}\nabla_\ell F^{ij}\right).$$

Hence, we can convert the system (1.6) into the following new form.

$$\begin{aligned} &\partial_t a + v\cdot\nabla a + \Lambda d = G_1,\\ &\partial_t e^{ij} + v\cdot\nabla e^{ij} - \mu\Delta e^{ij} - (\lambda+\mu)\nabla_i\nabla_j d\\ &\qquad + \Lambda^{-1}\nabla_i\nabla_j a + \Lambda F^{ij} = G_2^{ij}\\ &\partial_t F^{ij} + v\cdot\nabla F^{ij} - \Lambda e^{ij} = G_3^{ij}, \end{aligned} \tag{4.1}$$

where

$$G_1 = a\,\mathrm{div}v, \quad G_3^{ij} = \nabla_k v^i F^{kj},$$

and

$$\begin{aligned} G_2^{ij} = {}& v\cdot\nabla e^{ij} - \Lambda^{-1}\nabla_j\Big[v\cdot\nabla v^i + C(a)\mathcal{A}v + F^{jk}\nabla_j F^{ik}\Big]\\ &+ \Lambda^{-1}\nabla_k(F^{\ell j}\nabla F^{ik} - F^{\ell k}\nabla_\ell F^{ij}) \end{aligned}$$

with $C(a) = \frac{a}{1+a}$, $K(a) = \frac{P'(1+a)}{1+a} - 1$. Moreover, we have

$$\nabla_i F^{ij} = -\nabla_j a + G_0^j, \quad G_0^j = -\nabla_i(aF^{ij}). \tag{4.2}$$

Now, we provide the main estimate for the high frequency part in the following proposition.

Proposition 4.1: *There exists a constant $\delta > 0$ such that if*

$$\|a_0\|_{B^{n/2}} + \|v_0\|_{B^{n/2-1}} + \|F_0\|_{B^{n/2}} \le \delta,$$

then there holds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_q(t) + c_0 E_q(t) \le C\Big\{&\alpha_q(1+t)^{-n/4}M(t)f(t)\\ &+ \alpha_q\|G_1, G_3\|_{B^{n/2-1,n/2}} + \alpha_q\|G_0, G_2\|_{\dot{B}^{n/2-1}}\Big\} \end{aligned}$$

for $t \in [0,T]$ and $q \ge R$, where $\sum_{q\ge 1}\alpha_q \le 1$,

$$\int_0^\infty f(t)\mathrm{d}t = \int_0^\infty \|v(t)\|_{\dot{B}^{n/2+1}}\mathrm{d}t \le C\delta$$

and c_0 does not depend on q. Here, $E_q(t)$ is equivalent to $2^{\frac{n}{2}q}\|\dot\Delta_q a\|_{L^2}+2^{\frac{n}{2}q}\|\dot\Delta_q F\|_{L^2}+2^{(\frac{n}{2}-1)q}\|\dot\Delta_q e\|_{L^2}$ defined as

$$E_q(t)=2^{(\frac{n}{2}-1)q}\Big(\|\dot\Delta_q e\|_{L^2}^2+\xi(\lambda+2\mu)\|\Lambda\dot\Delta_q a\|_{L^2}^2+\xi\mu\|\Lambda\dot\Delta_q F\|_{L^2}^2 \\ +\xi(\lambda+\mu)\|\Lambda^{-1}\nabla_i\nabla_j\dot\Delta_q F^{ij}\|_{L^2}^2-2\xi(\Lambda\dot\Delta_q a|\dot\Delta_q d)+2\xi(\Lambda\dot\Delta_q F|\dot\Delta_q e)\Big)$$

with $\xi>0$ is a small enough positive constant. That is, there exists a constant D_1 such that

$$\frac{1}{D_1}\tilde E_q\le E_q\le D_1\tilde E_q$$

where

$$\tilde E_q=2^{\frac{n}{2}q}\|\dot\Delta_q a,\dot\Delta_q F\|_{L^2}+2^{(\frac{n}{2}-1)q}\|\dot\Delta_q e\|_{L^2}$$

Proof: Applying the operator $\dot\Delta_q$ to the system (4.1), we find that (a,e,F) satisfies

$$\begin{aligned}&\dot\Delta_q\partial_t a+\Lambda\dot\Delta_q d=\dot\Delta_q G_1-\dot\Delta_q(v\cdot\nabla a),\\ &\dot\Delta_q\partial_t e^{ij}-\mu\Delta\dot\Delta_q e^{ij}-(\lambda+\mu)\nabla_i\nabla_j\dot\Delta_q d+\Lambda^{-1}\nabla_i\nabla_j\dot\Delta_q a\\ &\qquad+\Lambda\dot\Delta_q F^{ij}=\dot\Delta_q G_2^{ij}-\dot\Delta_q(v\cdot\nabla e^{ij}),\\ &\dot\Delta_q\partial_t F^{ij}-\Lambda\dot\Delta_q e^{ij}=\dot\Delta_q G_3^{ij}-\dot\Delta_q(v\cdot\nabla F^{ij}),\end{aligned}\tag{4.3}$$

where $i,j=1,2,3$. Taking the L^2-product of the second equation of (4.3) with $\dot\Delta_q e^{ij}$, then summing up the resulting equation with respect to indexes i,j, we can get

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\dot\Delta_q e\|_{L^2}^2+\mu\|\Lambda\dot\Delta_q e\|_{L^2}^2+(\lambda+\mu)\|\Lambda\dot\Delta_q d\|_{L^2}^2-(\dot\Delta_q a|\Lambda\dot\Delta_q d) \\ +(\Lambda\dot\Delta_q F|\dot\Delta_q e)=(\dot\Delta_q G_2|\dot\Delta_q e)-(\dot\Delta_q(v\cdot\nabla e)|\dot\Delta_q e),\tag{4.4}$$

where we used the fact $d=-\Lambda^{-2}\nabla_i\nabla_j e^{ij}$. We apply the operator Λ to the first equation of (4.3) and take the L^2-product of the resulting equation with $-\dot\Delta_q d$ and take the L^2-product of the second equation of (4.3) with $\Lambda^{-1}\nabla_i\nabla_j\dot\Delta_q a$. Then, summing up the resulting equations yields that

$$-\frac{\mathrm d}{\mathrm dt}(\Lambda\dot\Delta_q a|\dot\Delta_q d)-\|\Lambda\dot\Delta_q d\|_{L^2}^2+\|\Lambda\dot\Delta_q a\|_{L^2}^2-(\lambda+2\mu)(\Lambda^2\dot\Delta_q d|\Lambda\Delta_q a) \\ +(\dot\Delta_q F^{ij}|\nabla_i\nabla_j\dot\Delta_q a)=-(\Lambda\dot\Delta_q G_1|\dot\Delta_q d)+(\dot\Delta_q G_2^{ij}|\Lambda^{-1}\nabla_i\nabla_j\dot\Delta_q a) \\ +(\Lambda\dot\Delta_q(v\cdot\nabla a)|\dot\Delta_q d)-(\dot\Delta_q(v\cdot\nabla e^{ij})|\Lambda^{-1}\nabla_i\nabla_j\dot\Delta_q a).\tag{4.5}$$

We apply the operator Λ to the third equation of (4.3) and take the L^2-product of the resulting equation with $\dot\Delta_q e^{ij}$ and take the L^2-product of the second equation of (4.3) with $\Lambda\dot\Delta_q F^{ij}$. Then, summing up the resulting equations yields

$$\frac{\mathrm d}{\mathrm dt}(\Lambda\dot\Delta_q F|\dot\Delta_q e)-\|\Lambda\dot\Delta_q e\|_{L^2}^2+\|\Lambda\dot\Delta_q F\|_{L^2}^2+\mu(\Lambda^2\dot\Delta_q e|\Lambda\dot\Delta_q F) \\ +(\lambda+\mu)(\nabla_i\nabla_j\dot\Delta_q d|\Lambda\dot\Delta_q F^{ij})+(\nabla_i\nabla_j\dot\Delta_q a|\dot\Delta_q F^{ij}) \\ =(\dot\Delta_q G_2|\Lambda\dot\Delta_q F)+(\Lambda\dot\Delta_q G_3|\dot\Delta_q e)-(\Lambda\dot\Delta_q(v\cdot\nabla e)|\dot\Delta_q F) \\ -(\Lambda\dot\Delta_q(v\cdot\nabla F)|\dot\Delta_q e).\tag{4.6}$$

Now, applying the operator Λ to the first and the third equations of (4.3), then taking the L^2 product of the resulting equations with $\Lambda\dot{\Delta}_q a$ and $\Lambda\dot{\Delta}_q F^{ij}$, we will obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\dot{\Delta}_q a\|_{L^2}^2 + (\Lambda^2\dot{\Delta}_q d|\Lambda\dot{\Delta}_q a)\\ = (\Lambda\dot{\Delta}_q G_1|\Lambda\dot{\Delta}_q a) - (\Lambda\dot{\Delta}_q(v\cdot\nabla a)|\Lambda\dot{\Delta}_q a),\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\dot{\Delta}_q F\|_{L^2}^2 - (\Lambda^2\dot{\Delta}_q e|\Lambda\dot{\Delta}_q F)\\ = (\Lambda\dot{\Delta}_q G_3|\Lambda\dot{\Delta}_q F) - (\dot{\Delta}_q(v\cdot\nabla F)|\Lambda\dot{\Delta}_q F).\end{aligned}\tag{4.8}$$

We apply the operator $\Lambda^{-1}\nabla_i\nabla_j$ to the third equation of (4.3) and take the summation with respect to i, j, then we take the L^2 times the resulting equation with $\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q F^{ij}$ to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q F^{ij}\|_{L^2}^2 + (\Lambda\dot{\Delta}_q d|\nabla_i\nabla_j\dot{\Delta}_q F^{ij})\\ = (\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q G_3^{ij}|\Lambda^{-1}\nabla_k\nabla_\ell\dot{\Delta}_q F^{k,\ell})\\ - (\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q(v\cdot\nabla F^{ij})|\Lambda^{-1}\nabla_k\nabla_\ell\dot{\Delta}_q F^{k,\ell}).\end{aligned}\tag{4.9}$$

For small $\xi > 0$, performing the following calculation

$$(4.4) + \xi(4.5) + \xi(4.6) + \xi(\lambda+2\mu)(4.7) + \xi\mu(4.8) + \xi(\lambda+\mu)(4.9)$$

yields that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}f_q^2 + \tilde{f}_q^2 + 2(\dot{\Delta}_q F^{ij}|\nabla_i\nabla_j\dot{\Delta}_q a) &= (\dot{\Delta}_q G_2|\dot{\Delta}_q e) - \xi(\Lambda\dot{\Delta}_q G_1|\dot{\Delta}_q d)\\ &- \xi(\dot{\Delta}_q G_2^{ij}|\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q a) + \xi(\dot{\Delta}_q G_2|\Lambda\dot{\Delta}_q F) + \xi(\Lambda\dot{\Delta}_q G_3|\dot{\Delta}_q e)\\ &+ \xi(\lambda+2\mu)(\Lambda\dot{\Delta}_q G_1|\Lambda\dot{\Delta}_q a) + \xi\mu(\Lambda\dot{\Delta}_q G_3|\Lambda\dot{\Delta}_q F)\\ &+ \xi(\lambda+\mu)(\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q G_3|\Lambda^{-1}\nabla_k\nabla_\ell\dot{\Delta}_q F^{k\ell}) + F_q,\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}f_q^2 &= \|\dot{\Delta}_q e\|_{L^2}^2 + \xi(\lambda+2\mu)\|\Lambda\dot{\Delta}_q a\|_{L^2}^2 + \xi\mu\|\Lambda\dot{\Delta}_q F\|_{L^2}^2\\ &\quad + \xi(\lambda+\mu)\|\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q F^{ij}\|_{L^2}^2 - 2\xi(\Lambda\dot{\Delta}_q a|\dot{\Delta}_q d) + 2\xi(\Lambda\dot{\Delta}_q F|\dot{\Delta}_q e),\\ \tilde{f}_q^2 &= (\mu-\xi)\|\Lambda\dot{\Delta}_q e\|_{L^2}^2 + (\lambda+\mu-\xi)\|\Lambda\dot{\Delta}_q d\|_{L^2}^2 + \xi\|\Lambda\dot{\Delta}_q a\|_{L^2}^2 + \xi\|\Lambda\dot{\Delta}_q F\|_{L^2}^2\\ &\quad - (\dot{\Delta}_q a|\Lambda\dot{\Delta}_q d) + (\Lambda\dot{\Delta}_q F|\dot{\Delta}_q e),\\ F_q &= -(\dot{\Delta}_q(v\cdot\nabla e)|\dot{\Delta}_q e) + \xi\Big((\Lambda\dot{\Delta}_q(v\cdot\nabla a)|\dot{\Delta}_q d) + (\dot{\Delta}_q(v\cdot\nabla e^{ij})|\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q a)\Big)\\ &\quad + \xi\Big((\Lambda\dot{\Delta}_q(v\cdot\nabla e)|\dot{\Delta}_q F) - (\Lambda\dot{\Delta}_q(v\cdot\nabla F)|\dot{\Delta}_q e)\Big) - \xi\mu(\Lambda\dot{\Delta}_q(v\cdot\nabla F)|\Lambda\dot{\Delta}_q F)\\ &\quad - \xi(\lambda+\mu)(\Lambda^{-1}\nabla_i\nabla_j\dot{\Delta}_q(v\cdot\nabla F)|\Lambda^{-1}\nabla_k\nabla_\ell\dot{\Delta}_q F^{k\ell})\\ &\quad - \xi(\lambda+2\mu)^2(\Lambda\dot{\Delta}_q(v\cdot\nabla a)|\nabla\dot{\Delta}_q a)\end{aligned}$$

Here, we can choose ξ small enough and R to be a large enough fixed constant. For $q > R$, we can easily obtain

$$f_q^2 \approx 2^{2q}\|\dot{\Delta}_q a\|_{L^2}^2 + \|\dot{\Delta}_q e\|_{L^2}^2 + 2^{2q}\|\dot{\Delta}_q F\|_{L^2}^2, \tag{4.11}$$

and

$$2^{2q}\|\dot{\Delta}_q a\|_{L^2}^2 + \|\dot{\Delta}_q e\|_{L^2}^2 + 2^{2q}\|\dot{\Delta}_q F\|_{L^2}^2 \lesssim \tilde{f}_q^2. \tag{4.12}$$

Using the identity (4.2), we find that

$$\begin{aligned}(\dot{\Delta}_q F^{ij}|\nabla_i\nabla_j a) &= (\nabla_i\nabla_j\dot{\Delta}_q F^{ij}|\dot{\Delta}_q a)\\ &= \|\Lambda\dot{\Delta}_q a\|_{L^2}^2 + (\Lambda\dot{\Delta}_q a|\Lambda^{-1}\nabla_j G_0^j).\end{aligned} \tag{4.13}$$

Let $E_q(t) = 2^{(\frac{n}{2}-1)q} f_q$, we have

$$E_q(t) \approx 2^{\frac{n}{2}q}\|\dot{\Delta}_q a\|_{L^2} + 2^{(\frac{n}{2}-1)q}\|\dot{\Delta}_q e\|_{L^2} + 2^{\frac{n}{2}q}\|\dot{\Delta}_q F\|_{L^2} \tag{4.14}$$

By (4.10)–(4.14) and Lemma 2.7, we finally obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_q(t) + c_0 E_q(t) \le{}& C\alpha_q(1+t)^{-n/4}M(t)f(t) + C\alpha_q\|G_1, G_3\|_{B^{n/2-1,n/2}}\\ &+ C\alpha_q\|G_0, G_2\|_{\dot{B}^{n/2-1}}.\end{aligned}$$

□

5. Derivation of the optimal time decay rate

Given the analysis on the low and high frequency parts, we now provide the proof of Theorem 1.2. From Proposition 4.1, we know that

$$\begin{aligned}E_q(t) \le{}& e^{-c_0 t}E_q(0) + C\int_0^t e^{-c_0(t-\tau)}\Big(\alpha_q(1+\tau)^{-n/4}M(\tau)f(\tau)\\ &+ \alpha_q\|G_1, G_3\|_{B^{n/2-1,n/2}} + \alpha_q\|G_0, G_2\|_{\dot{B}^{n/2-1}}\Big)\mathrm{d}\tau.\end{aligned} \tag{5.1}$$

Through homogeneous para-differential calculus, we could get

$$\begin{aligned}\|G_1\|_{B^{n/2-1,/2}} &\le C\,\|a\|_{B^{n/2-1,n/2}}\|\mathrm{div}v\|_{\dot{B}^{n/2}}\\ &\le C\,(1+\tau)^{-n/4}M(\tau)f(\tau),\end{aligned} \tag{5.2}$$

$$\begin{aligned}\|G_3\|_{B^{n/2-1,n/2}} &\le C\,\|F\|_{B^{n/2-1,n/2}}\|\nabla v\|_{\dot{B}^{n/2}}\\ &\le C\,(1+\tau)^{-n/4}M(\tau)f(\tau),\end{aligned} \tag{5.3}$$

$$\begin{aligned}\|G_0\|_{\dot{B}^{n/2-1}} &\le C\,\|aF\|_{\dot{B}^{n/2}} \le C\,\|a\|_{\dot{B}^{n/2}}\|F\|_{\dot{B}^{n/2}}\\ &\le C\,(1+\tau)^{-n/2}M^2(\tau).\end{aligned} \tag{5.4}$$

For the term G_2, we need to estimate term by term carefully as follows

$$\|v\cdot\nabla e\|_{\dot{B}^{n/2-1}} + \|v\cdot\nabla v\|_{\dot{B}^{n/2-1}} \le C\,\|v\|_{\dot{B}^{n/2-1}}\|\nabla v\|_{\dot{B}^{n/2}} \tag{5.5}$$

$$\le C(1+\tau)^{-n/4}M(\tau)f(\tau). \tag{5.6}$$

Since $C(0) = K(0) = 0$, we get using Lemma 3 and Remark 6 in [5] that

$$\|C(a)\mathcal{A}v\|_{\dot{B}^{n/2-1}} \leq C\|\nabla^2 v\|_{\dot{B}^{n/2-1}}\|C(a)\|_{\dot{B}^{n/2}} \leq C(1+\tau)^{-n/4}M(\tau)f(\tau), \tag{5.7}$$

$$\|K(a)\nabla a\|_{\dot{B}^{n/2-1}} \leq C\|K(a)\|_{\dot{B}^{n/2}}\|\nabla a\|_{\dot{B}^{n/2-1}} \leq C(1+\tau)^{-n/2}M^2(\tau), \tag{5.8}$$

$$\|F\nabla F\|_{\dot{B}^{n/2-1}} \leq C\|F\|_{\dot{B}^{n/2}}\|\nabla F\|_{\dot{B}^{n/2-1}} \leq C(1+\tau)^{-n/2}M^2(\tau). \tag{5.9}$$

From the above estimates (5.5)–(5.9), we obtain

$$\|G_2\|_{\dot{B}^{n/2-1}} \leq C(1+\tau)^{-n/2}M^2(\tau) + C\,(1+\tau)^{-n/4}M(\tau)f(\tau). \tag{5.10}$$

Substitute (5.2)–(5.4) and (5.10) into (5.1), we will have

$$\begin{aligned}
\sum_{q\geq R} E_q(t) &\leq e^{-c_0 t}\sum_{q\geq R} E_q(0) + C\int_0^t e^{-c_0(t-\tau)}\Big((1+\tau)^{-n/2}M^2(\tau) \\
&\quad + (1+\tau)^{-n/4}M(\tau)f(\tau)\Big)\mathrm{d}\tau \\
&\leq e^{-c_0 t}\sum_{q\geq R} E_q(0) + M(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-n/4}f(\tau)\mathrm{d}\tau \\
&\quad + M^2(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-n/2}\mathrm{d}\tau \\
&\leq e^{-c_0 t}\sum_{q\geq R} E_q(0) + C(1+t)^{-n/4}\delta M(t) + C(1+t)^{-n/2}M^2(t).
\end{aligned}$$

So we obtain

$$(1+t)^{n/4}\sum_{q\geq R} E_q(\tau) \leq C\left(\|(a_0, F_0)\|_{B^{n/2}} + \|v_0\|_{B^{n/2-1}}\right) + C\delta M(t) + CM^2(t).$$

Combining the above inequality, Remark 3.4 and using properties of homogeneous Besov space, we could obtain

$$M(t) \leq C\left(\|(a_0, F_0)\|_{B^{n/2}} + \|v_0\|_{B^{n/2-1}}\right) + C\delta M(t) + CM^2(t). \tag{5.11}$$

By taking $\delta > 0$ suitably small, we finally have

$$M(t) \leq C\left(\|(a_0, F_0)\|_{B^{n/2}} + \|v_0\|_{B^{n/2-1}}\right) \tag{5.12}$$

for all $0 \leq t \leq T$. It follows from local well-posedness Theorem 2.8 and the above estimate (5.12) that

$$M(t) \leq C < \infty$$

for all $t > 0$. Hence, we obtain the desired decay estimates in Theorem 1.2.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

J. Jia was supported by the National Natural Science Foundation of China [grant number 11501439]; the Postdoctoral Science Foundation Project of China [grant number 2015M580826]; the Natural Science Foundation Project of Shannxi [grant number 2016JQ1020]. J. Peng was supported partially by National Natural Science Foundation of China [grant number 11131006], [grant number 41390454].

References

[1] Lei Z, Liu C, Zhou Y. Global solutions of incompressible viscoelastic fluids. Arch. Rational Mech. Anal. 2008;188:371–398.
[2] Lin F, Liu C, Zhang P. On hydrodynamics of viscoelastic fluids. Commun. Pure Appl. Math. 2005;58:1437–1471.
[3] Lin F, Zhang P. On the initial-boundary value problem of the incompressible viscolastic fluid system. Commun. Pure Appl. Math. 2008;61:539–558.
[4] Zhang T, Fang D. Global existence of strong solution for equations related to the incompressible viscoelastic fluids in the critical L^p framework. SIAM J. Math. Anal. 2013;44:2266–2288.
[5] Qian J, Zhang Z. Global well-posedness for compressible viscoelastic fluids near equilibrium. Arch. Rational Mech. Anal. 2010;198:835–868.
[6] Hu X, Wang D. Global existence for the multi-dimensional compressible viscoelastic flows. J. Differ. Equ. 2011;250:1200–1231.
[7] Lions P, Masmoudi N. Global solutions for some Oldroyd models of non-Newtonian flows. Chin. Ann. Math. Ser. B. 2000;21:131–146.
[8] Jia J, Peng J, Mei Z. Well-posedness and time-decay for compressible viscoelastic fluids in critical Besov space. J. Math. Anal. Appl. 2014;418:638–675.
[9] Ponce G. Global existence of small solution to a class of nonlinear evolution equations. Nonlinear Anal. 1985;9:339–418.
[10] Guo Y, Wang Y. Decay of dissipative equations and negative sobolev spaces. Commun. Partial Differ. Equ. 2012;37:2165–2208.
[11] Li H, Zhang T. Large time bahavior of isentropic compressible Navier–Stokes system in $\mathbb{R}^3$. Math. Meth. Appl. Sci. 2011;34:670–682.
[12] Hoff D, Zumbrun K. Multi-dimensional diffusion waves for the Navier–Stokes equations of compressible flow. Indiana Univ. Math. J. 1995;44:604–676.
[13] Hoff D, Zumbrun K. Pointwise decay estimates for multi-dimensional Navier–Stokes diffusion waves. Z. Angew. Math. Phys. 1997;48:597–614.
[14] Liu T, Wang W. The pointwise estimates of diffusion waves for the Navier–Stokes equaions in odd multi-dimensions. Commun. Math. Phys. 1998;196:145–173.
[15] Li DL. The green's function of the Navier–Stokes equations for gas dynamics in $\mathbb{R}^3$. Commun. Math. Phys. 2005;257:579–619.
[16] Okita M. Optimal decay rate for strong solutions in critical spaces to the compressible Navier–Stokes equations. J. Differ. Equ. 2014;257:3850–3867.
[17] Hu X, Wu G. Global existence and optimal decay rates for three-dimensional compressible viscoelastic flows. (English summary). SIAM J. Math. Anal. 2013;45:2815–2833.
[18] Chemin J-Y, Lerner N. Flot de champs de vecteurs non lipschitziens etéquations de Navier–Stokes [Flow fields of non-Lipschitz vectors and Navier-Stokes equations]. J. Differ. Equ. 1992;121:314–328.
[19] Bahouri H, Chemin J, Danchin R. Fourier analysis and nonlinear partial differential equations. Grundlehren der mathematischen Wissenschaften [Basics of mathematical Sciences]. Berlin: Springer-Verlag; 2011.
[20] Danchin R. Fourier analysis methods for PDE's. Lecture notes. 2005. November 14. Available from: http://perso-math.univ-mlv.fr/users/danchin.raphael/cours/courschine.pdf.
[21] Chen Q, Miao C, Zhang Z. Global well-posedness for compressible Navier–Stokes equations with highly oscillating initial velocity. Commun. Pure. Appl. Math. 2010;LXIII:1173–1224.
[22] Charve F, Danchin R. A gloal existence result for the compressible Navier–Stokes equations in the critical L^p framework. Arch. Ration. Mech. Anal. 2010;198:233–271.
[23] Hoff D, Zumbrun K. Mutli-dimensional diffusion waves for the Navier–Stokes equations of compressible flow. Indiana Univ. Math. J. 1995;44:603–676.