J Sci Comput (2015) 64:745–760
DOI 10.1007/s10915-014-9843-z

On Isentropic Approximations for Compressible Euler Equations

Junxiong Jia · Ronghua Pan

Received: 30 December 2013 / Revised: 11 March 2014 / Accepted: 14 March 2014 /
Published online: 27 March 2014

Abstract In this paper, we first generalize the classical results on Cauchy problem for positive symmetric quasilinear systems to more general Besov space. Through this generalization, we obtain the local well-posedness with initial data in the space $B_{2,1}^{\frac{d}{2}+1}(\mathbb{R}^d)$ which has critical regularity index. We then apply these results to give an explicit characterization on the isentropic approximation for full compressible Euler equations in $\mathbb{R}^3$. This characterization tells us that isentropic compressible Euler equations is a reasonable approximation to Non-isentropic compressible Euler equations in the regime of classical solutions. The failure of such characterization was illustrated when singularities occur in the solutions.

Keywords Compressible Euler equation · Isentropic approximation · Critical regularity · Besov spaces

Mathematics Subject Classification 35L65

1 Introduction

This note is devoted to the explicit characterization and mathematical justification on the isentropic approximation for the compressible inviscid fluid flow. For this purpose, we consider the following Cauchy problem of compressible Euler equations

J. Jia
Department of Mathematics, Xi'an Jiaotong University, Xi'an 710049, China
e-mail: jiajunxiong@163.com

R. Pan (✉)
Department of Mathematics, Georgia Institute of Technology,
686 Cherry Street, Skiles Building, Atlanta, GA 30332-0160, USA
e-mail: panrh@math.gatech.edu

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho\, \mathrm{div} u = 0, \quad x \in \mathbb{R}^d, \\ \rho\, (\partial_t u + u \cdot \nabla u) + \nabla p = 0, \\ \partial_t s + u \cdot \nabla s = 0, \\ \rho(x, 0) = \rho_0(x) \geq 0, \quad u(x, 0) = u_0(x), \qquad s(x, 0) = s_0(x), \end{cases} \tag{1.1}$$

where ρ is the density, u is the velocity field, s stands for the specific entropy and $p(\rho, s) = \rho^\gamma e^s$ is the pressure law for polytropic gas, with the adiabatic exponent $\gamma > 1$. In many applications, if in the thermodynamical process the specific entropy has only very small changes near a constant equilibrium $\bar{s}$, an isentropic approximation is applied by assuming $s(x, t) = \bar{s}$ which reduces (1.1) to the isentropic Euler equations

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho\, \mathrm{div} u = 0, \quad x \in \mathbb{R}^d \\ \rho\, (\partial_t u + u \cdot \nabla u) + \nabla \tilde{p} = 0, \\ \rho(x, 0) = \rho_0(x) \geq 0, \quad u(x, 0) = u_0(x), \end{cases} \tag{1.2}$$

where $\tilde{p}(\rho) = \rho^\gamma e^{\bar{s}}$. Now, if one assumes

$$s_0(x) = \bar{s}$$

in (1.1), the solutions of (1.1) are expected to equal to the corresponding one of (1.2) formally. The main goal of this paper is to address this issue. More precisely, we will study the limiting process from solutions of (1.1) to corresponding solutions of (1.2) when

$$(s_0(x) - \bar{s}) \to 0.$$

The main results of the current paper show that, when the solutions of (1.1) and (1.2) are classical, then such a picture can be justified with sharp error estimates. However, when the solutions of Euler equations blow up and singularities are developed, the expectation above is not true at least by the measurement of Sobolev norms.

The main idea of this paper is the following observation. When both (1.1) and (1.2) admit smooth solutions in $\mathbb{R}^n \times [0, T]$ for some positive T, the justification of isentropic limit as $(s_0(x) - \bar{s}) \to 0$ can be obtained by means of the continuity dependence of initial data for solutions of (1.1) near the initial data $(\rho_0, u_0, \bar{s})$. This comes along with the local well-posdeness theory for smooth initial data. Therefore, the justification of isentropic limit will be achieved by a careful energy method for the symmetric hyperbolic systems. One of the new ingredients of this paper is that the results will be established for solutions in the critical Besov space $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$. On the other hand, when singularities, say shock waves, occur in the solution, such picture breaks down. We will show this by an explicit example. Therefore, our results are somehow optimal for initial data with lowest possible regularity, which is $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$.

In Sect. 2, we will list some basic information on Besov spaces. In Sect. 3, we will discuss the local well-posedness theory for symmetric hyperbolic systems, and we will establish the corresponding theory in the critical Besov space $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$ which is applicable to compressible Euler Eq. (1.1). We will then justify the isentropic limit in Sect. 4 within the regime of classical solutions. Finally, we will discuss the failure of isentropic approximation when singularity occurs in the solutions in the last section.

For compressible Euler equations in one space dimension, such a problem was investigated by Saint–Raymond [12] for BV solutions, where the difference between solutions of isentropic and full Euler equations measured by BV-norm was shown to grow at most linearly in time. For the steady Euler flows, similar results were obtained in [2] and [8]. It remains an interesting open problem on how to offer a (physically and mathematically) sound explanation on the isentropic approximation for physically admissible weak solutions for compressible Euler equations.

2 Preliminaries

In this section, we collect some basic facts on Besov spaces that will be used in this paper. The following notations will be used throughout this paper.

- For any tempered distribution u, both $\widehat{u}$ and $\mathcal{F}(u)$ denote the Fourier transform of u.
- For every $p \in [1, \infty]$, $\|\cdot\|_{L^p}$ denotes the norm of the Lebesgue space L^p.
- The norm in the mixed space–time Lebesgue space $L^p([0,T]; L^r(\mathbb{R}^d))$ is denoted by $\|\cdot\|_{L^p_T L^r}$ (with the obvious generalization to $\|\cdot\|_{L^p_T X}$ for any normed space X).
- For any pair of operators P and Q on some Banach space X, the commutator $[P, Q]$ is given by $PQ - QP$.
- For any function u, $\partial_i u$ stands for $\partial_{x_i} u$ with $i = 1, 2, \ldots, d$.

Then, we give a short introduction to the Besov type spaces. Details about Besov type space can be found in [1] or [11]. There exist two radial positive functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that

- $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \forall q \geq 1, \mathrm{supp}\chi \cap \mathrm{supp}\varphi(2^{-q}\cdot) = \phi$,
- $\mathrm{supp}\varphi(2^{-j}\cdot) \cap \mathrm{supp}\varphi(2^{-k}\cdot) = \phi$, if $|j-k| \geq 2$,

For every $v \in S'(\mathbb{R}^d)$ we set

$$\Delta_{-1} v = \chi(D)v, \quad \forall q \in \mathbb{N}, \quad \Delta_j v = \varphi(2^{-j}D)v \quad \text{and} \quad S_j = \sum_{-1 \leq m \leq j-1} \Delta_m. \tag{2.1}$$

With our choice of φ, one can easily verify that

$$\Delta_j \Delta_k f = 0 \quad \text{if} \quad |j-k| \geq 2. \tag{2.2}$$

$$\Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if} \quad |j-k| \geq 5. \tag{2.3}$$

Like in Bony's decomposition, we split the product uv into three parts

$$uv = T_u v + T_v u + R(u, v), \tag{2.4}$$

with

$$T_u v = \sum_j S_{j-1} u \Delta_j v,$$

$$R(u, v) = \sum_j \Delta_j u \widetilde{\Delta}_j v,$$

where $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$.

We now define inhomogeneous Besov spaces. For $p, r \in [1, +\infty]$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,q}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,q}} := (2^{js} \|\Delta_j u\|_{L^p})_{\ell^q} < +\infty.$$

Notice that the usual Sobolev spaces H^s coincide with $B^s_{2,2}$ for every $s \in \mathbb{R}$. For simplicity, we will use B^s to stand for the Besov space $B^s_{2,1}$.

The following two Lemmas will be used frequently in this paper.

Lemma 2.1 [4] *Let $s > 0$ and $1 \le p, q \le \infty$. Then $B^s_{p,q} \cap L^\infty$ is an algebra and we have*

$$\|uv\|_{B^s_{p,q}} \le C(\|u\|_{L^\infty}\|v\|_{B^s_{p,q}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,q}}),$$

where C is a positive constant.

Lemma 2.2 [4] *Let I be an open interval of $\mathbb{R}$. Let $s > 0$ and σ be the smallest integer such that $\sigma \ge s$. Let $F : I \to \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in W^{\sigma,\infty}(I; \mathbb{R})$. Assume that $v \in B^s_{p,q}$ has values in $J \subset\subset I$. Then $F(v) \in B^s_{p,q}$ and there exists a constant C depending only on s, I, J and N, and such that*

$$\|F(v)\|_{B^s_{p,q}} \le C\,(1 + \|v\|_{L^\infty})^\sigma\,\|F'\|_{W^{\sigma,\infty}(I)}\|v\|_{B^s_{p,q}}.$$

3 Positive Symmetric Hyperbolic Systems

In this section, we concentrate on the following quasilinear system

$$\begin{cases} A_0(U, x, t)\partial_t U + \displaystyle\sum_{k=1}^{d} A_k(U, x, t)\partial_k U = B(U, x, t), \\ U(x, 0) = U_0(x), \end{cases} \tag{3.1}$$

where the $n \times n$ matrices $A_0(U, x, t)$, $A_k(U, x, t)$ and the source term $B(U, x, t)$ depend on $U \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, and $t \in [0, \infty)$ smoothly. For $0 \le \tau_1 < \tau_2$, and for subsets $\mathcal{O}$ of $\mathbb{R}^n$, and Ω of $\mathbb{R}^d$, (3.1) is called *positive symmetric hyperbolic* on $\mathcal{O} \times \Omega \times [\tau_1, \tau_2]$, if, for any $U \in \mathcal{O}$, $x \in \Omega$, and $t \in [\tau_1, \tau_2]$, A_k $(k = 0, \dots, d)$ are symmetric and A_0 is positive definite.

The following Theorem is due to [3,5,7,9], and [1].

Theorem 3.1 *Assume that (3.1) is positive symmetric hyperbolic for its arguments, and U_0 belongs to $H^s(\mathbb{R}^d)$ for some $s > \frac{d}{2} + 1$. There exits a positive time T such that (3.1) has a unique solution $U(x, t) \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$. Moreover, T can be bounded from below by $c_1\|U_0(x)\|^{-1}_{H^s}$, where c_1 depends only on A_k $(k = 0, \dots, d)$. The maximal time of existence T^* of such a solution is independent of s and satisfies*

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\nabla U(\cdot, t)\|_{L^\infty}\, dt = \infty.$$

The positive symmetric hyperbolic systems cover an important class of quasi-linear hyperbolic systems. Although many physical systems are not exactly in positive symmetric form, like (1.1) and (1.2), they are *symmetrizable*, in the sense that there exists a transformation of unknown functions rendering the system into postive symmertic hyperbolic systems in the new variables. In particular, when a quasi-linear hyperbolic system endowed with a convex entropy, it is symmetrizable by Herssian matrix of this entropy. Therefore, Theorem 3.1 finds applications for many physical systems including isentropic Euler (1.2) and full Euler systems (1.1).

One of our goals is to establish the analog theory of Theorem 3.1 with intial data of crticital regularity, i.e., in the Besov space $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$ for (3.1). In [1], this was achieved when $A_0 = I_n$, and the result there finds application to the isentropic Euler Eq. (1.2). In fact, if one introduces the quantity

$$\sigma = \frac{(4\gamma e^{\bar{s}})^{\frac{1}{2}}}{\gamma - 1} \rho^{\frac{\gamma-1}{2}}, \tag{3.2}$$

then the isentropic Euler system turns into

$$\begin{cases} \sigma_t + u \cdot \nabla \sigma + \frac{\gamma-1}{2} \sigma \operatorname{div} u = 0, \\ u_t + u \cdot \nabla u + \frac{\gamma-1}{2} \sigma \nabla \sigma = 0. \end{cases} \tag{3.3}$$

This, however, is not the case for the full Euler Eq. (1.1). In order to generalize this theory to cover full Euler equations, we will extend the results of [1] for $A_0 = I_n$ to (3.1) for positive definite $A_0 = A_0(U)$.

3.1 Linear Positive Symmetric System

We first consider the following positive symmetric linear system

$$A_0(x,t)\partial_t U + \sum_{k=1}^{d} A_k(x,t)\partial_k U = B(x,t)U + F(x,t). \tag{3.4}$$

where the $n \times n$ matrices A_k $(k = 0, \ldots, d)$ and B are smooth functions.

The following classical results (c.f. [6]) is often very useful.

Theorem 3.2 *Assume that* (3.4) *is positive symmetric for all* $x \in \mathbb{R}^d$ *and* $t \geq 0$, *such that* $A_0(x,t)$ *satisfying*

$$\beta I_n \leq A_0 \leq \beta^{-1} I_n,$$

for some positive constant β *in the sense of quadratic forms. Assume that* A_0, A_k, B *and their first order derivatives are bounded. Then, for all* $T > 0$ *and* $U \in C([0,T]; H^1) \cap C^1([0,T]; L^2)$, *it holds that*

$$\beta^2 \|U(t)\|_{L^2}^2 \leq e^{\gamma t} \|U(0)\|_{L^2}^2 + \int_0^t e^{\gamma(t-\tau)} \|F(\tau)\|_{L^2}^2 \, d\tau, \tag{3.5}$$

for all $t \in [0,T]$. γ *is chosen to be large enough, so that*

$$\beta(\gamma - 1) \geq \left\| \partial_t A_0 + \sum_{k=1}^{d} \partial_k (A_k) + B \right\|_{L^\infty}. \tag{3.6}$$

We now prove the following result.

Theorem 3.3 *Assume that* (3.4) *is positive symmetric for all* $x \in \mathbb{R}^d$ *and* $t \geq 0$, *such that* $A_0(x,t)$ *satisfying*

$$\beta I_n \leq A_0 \leq \beta^{-1} I_n,$$

for some positive constant β *in the sense of quadratic forms. Suppose that* $B \in L_T^\infty(B^{\frac{d}{2}+1}(\mathbb{R}^d))$, *and that* A_0, $A = \{A_k\}_{1 \leq k \leq d}$ *have the following form*

$$A_k(x,t) = \bar{A}_k + \tilde{A}_k(x,t), \; A_0(x,t) = \bar{A}_0 + \tilde{A}_0(x,t), \tag{3.7}$$

where $\bar{A}_k$, $\bar{A}_0$ *are constant matrices, and* $\tilde{A}_k \in L_T^\infty(B^{\frac{d}{2}+1})$, $\tilde{A}_0 \in L_T^\infty(B^{\frac{d}{2}+1})$, $\partial_t \tilde{A}_0 \in L_T^\infty(L^\infty)$. *Then, for all* $T > 0$ *and* $U \in C([0,T]; B^{\frac{d}{2}+1}) \cap C^1([0,T]; B^{\frac{d}{2}})$, *it holds that*

$$\beta\|U(t)\|_{B^{\frac{d}{2}+1}} \leq \|U(0)\|_{B^{\frac{d}{2}+1}} e^{C\beta^{-2}Mt} + CM_2 \int_0^t e^{C\beta^{-2}M(t-\tau)}\|A_0^{-1}F(\tau)\|_{B^{\frac{d}{2}+1}}\, d\tau, \tag{3.8}$$

where

$$\begin{aligned} M &= \|\partial_t A_0\|_{L_T^\infty(L^\infty)} + \|B\|_{L_T^\infty\left(B^{\frac{d}{2}+1}\right)} + \|\nabla A\|_{L_T^\infty\left(B^{\frac{d}{2}}\right)} \\ &\quad + \|A_0\|_{L_T^\infty(L^\infty)}\|A_0^{-1}B\|_{L_T^\infty\left(B^{\frac{d}{2}+1}\right)} + \|A_0\|_{L_T^\infty(L^\infty)}\|\nabla(A_0^{-1}A)\|_{L_T^\infty\left(B^{\frac{d}{2}}\right)}, \\ M_2 &= 1 + \|\tilde{A}_0\|_{L_T^\infty\left(B^{\frac{d}{2}+1}\right)}. \end{aligned}$$

Proof Taking the operator Δ_j on both sides of (3.4), we obtain

$$\Delta_j(A_0\partial_t U) + \sum_{k=1}^d \Delta_j(A_k\partial_k U) = \Delta_j(BU) + \Delta_j F. \tag{3.9}$$

Following [1], we define

$$\begin{aligned} &\bar{S}_j = S_j, \text{ if } j \geq 0; \text{ and } \bar{S}_j = \Delta_{-1}, \text{otherwise}, \\ &\bar{T}'_{\partial_k U}(A_k) = \sum_{j'\geq 0} S_{j'+2}\partial_k U \Delta_{j'} A_k, \end{aligned}$$

and therefore, (3.9) is reduced to

$$A_0\partial_t\Delta_j U + \sum_{k=1}^d \bar{S}_{j-1}(A_k)\partial_k\Delta_j U = \Delta_j(BU) + \Delta_j(F) + \sum_{l=1}^3 R_j^l - [\Delta_j, A_0]\partial_t U, \tag{3.10}$$

where

$$\begin{aligned} R_j^1 &= \sum_{|j'-j|\leq N_1} [\Delta_j, \bar{S}_{j'-1}(A_k)]\Delta_{j'}\partial_k U, \\ R_j^2 &= \sum_{|j'-j|\leq 1} \left(\bar{S}_{j'-1}(A_k) - \bar{S}_{j-1}(A_k)\right)\Delta_j\Delta_{j'}\partial_k U, \\ R_j^3 &= \Delta_j \sum_{k=1}^d \bar{T}'_{\partial_k U}(A_k), \end{aligned}$$

where N_1 is the fixed integer defined as in the proof Lemma 4.14 in [1] page 184 such that

$$\Delta_j \sum_j \bar{S}_{j'-1} A_k \Delta_{j'}\partial_k U = \Delta_j \sum_{|j'-j|\leq N_1} A_k\Delta_{j'}\partial_k U$$

Hence, we can easily give the estimates about R_j^1, R_j^2 and R_j^3 as in [1]

$$\begin{aligned} 2^{j\left(\frac{d}{2}+1\right)}\|R_j^1\|_{L^2} &\leq Cc_j\|\nabla A\|_{L^\infty}\|\nabla U\|_{B^{\frac{d}{2}}}, \\ 2^{j\left(\frac{d}{2}+1\right)}\|R_j^2\|_{L^2} &\leq Cc_j\|\nabla A\|_{L^\infty}\|\nabla U\|_{B^{\frac{d}{2}}}, \\ 2^{j\left(\frac{d}{2}+1\right)}\|R_j^3\|_{L^2} &\leq Cc_j\|\nabla A\|_{B^{\frac{d}{2}}}\|\nabla U\|_{B^{\frac{d}{2}}}. \end{aligned} \tag{3.11}$$

The first two terms on the right hand side of (3.10) can be estimated as follows

$$\begin{aligned} 2^{j\left(\frac{d}{2}+1\right)}\|\Delta_j(BU)\|_{L^2} &\le Cc_j\|B\|_{B^{\frac{d}{2}+1}}\|U\|_{B^{\frac{d}{2}+1}},\\ 2^{j\left(\frac{d}{2}+1\right)}\|\Delta_j(F)\|_{L^2} &= 2^{j\left(\frac{d}{2}+1\right)}\|\Delta_j(A_0A_0^{-1}F)\|_{L^2}\\ &\le Cc_j\left(1+\|\tilde{A}_0\|_{B^{\frac{d}{2}+1}}\right)\|A_0^{-1}F\|_{B^{\frac{d}{2}+1}}. \end{aligned} \tag{3.12}$$

Now, we consider the term $-[\Delta_j, A_0]\partial_t U$. Since

$$\begin{aligned} -[\Delta_j, A_0]\partial_t U &= -[\Delta_j, A_0]\left\{-\sum_{k=1}^d A_0^{-1}A_k\partial_k U + A_0^{-1}BU + A_0^{-1}F\right\}\\ &= \sum_{k=1}^d[\Delta_j, A_0]A_0^{-1}A_k\partial_k U - [\Delta_j, A_0]A_0^{-1}BU - [\Delta_j, A_0]A_0^{-1}F\\ &= I + II + III. \end{aligned}$$

For I, we have

$$\begin{aligned} I &= \sum_{k=1}^d\left(\Delta_j(A_k\partial_k U) - A_0\Delta_j\left(A_0^{-1}A_k\partial_k U\right)\right)\\ &= \sum_{k=1}^d[\Delta_j, A_k]\partial_k U - \sum_{k=1}^d A_0[\Delta_j, A_0^{-1}A_k]\partial_k U\\ &= I_1 + I_2. \end{aligned}$$

For I_1, we know that

$$\begin{aligned} 2^{j\left(\frac{d}{2}+1\right)}\|I_1\|_{L^2} &\le \sum_{k=1}^d 2^{j\left(\frac{d}{2}+1\right)}\|[\Delta_j, A_k]\partial_k U\|_{L^2}\\ &\le Cc_j\|\nabla A\|_{B^{\frac{d}{2}}}\|U\|_{B^{\frac{d}{2}+1}}, \end{aligned} \tag{3.13}$$

where we used Lemma 2.100 in [1]. For I_2, we have

$$\begin{aligned} 2^{j\left(\frac{d}{2}+1\right)}\|I_2\|_{L^2} &\le \sum_{k=1}^d 2^{j\left(\frac{d}{2}+1\right)}\|A_0[\Delta_j, A_0^{-1}A_k]\partial_k U\|_{L^2}\\ &\le Cc_j\|A_0\|_{L^\infty}\|\nabla\left(A_0^{-1}A\right)\|_{B^{\frac{d}{2}}}\|U\|_{B^{\frac{d}{2}+1}}, \end{aligned} \tag{3.14}$$

where we also used Lemma 2.100 in [1]. The term II will be treated as follows

$$\begin{aligned} 2^{j\left(\frac{d}{2}+1\right)}\|II\|_{L^2} &\le 2^{j\left(\frac{d}{2}+1\right)}\|[\Delta_j, A_0]A_0^{-1}BU\|_{L^2}\\ &\le 2^{j\left(\frac{d}{2}+1\right)}\left\{\|\Delta_j(BU)\|_{L^2} + \|A_0\Delta_j(A_0^{-1}BU)\|_{L^2}\right\}\\ &\le Cc_j\|B\|_{B^{d/2+1}}\|U\|_{B^{\frac{d}{2}+1}} + Cc_j\|A_0\|_{L^\infty}\|A_0^{-1}B\|_{B^{\frac{d}{2}+1}}\|U\|_{B^{\frac{d}{2}+1}}. \end{aligned} \tag{3.15}$$

For III, we have

$$\begin{aligned} 2^{j\left(\frac{d}{2}+1\right)}\|III\|_{L^2} &\le 2^{j\left(\frac{d}{2}+1\right)}\|[\Delta_j, A_0]A_0^{-1}F\|_{L^2}\\ &\le Cc_j\|\tilde{A}_0\|_{B^{\frac{d}{2}+1}}\|A_0^{-1}F\|_{B^{\frac{d}{2}+1}}. \end{aligned} \tag{3.16}$$

Due to the following two equalities

$$-\sum_{k=1}^{d}\left(\bar{S}_{j-1}(A_k)\partial_k\Delta_j U, \Delta_j U\right) = \frac{1}{2}\sum_{k=1}^{d}\left(\partial_k(\bar{S}_{j-1}A_k)\Delta_j U, \Delta_j U\right),$$

$$\frac{1}{2}\frac{d}{dt}\left(A_0\Delta_j U, \Delta_j U\right) = \left(A_0\partial_t\Delta_j U, \Delta_j U\right) + \frac{1}{2}\left(\partial_t A_0\Delta_j U, \Delta_j U\right),$$

we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(A_0\Delta_j U, \Delta_j U\right) = &\frac{1}{2}\sum_{k=1}^{d}\left(\partial_k(\bar{S}_{j-1}A_k)\Delta_j U, \Delta_j U\right) + \frac{1}{2}\left(\partial_t A_0\Delta_j U, \Delta_j U\right)\\ &+\left(\Delta_j(BU), \Delta_j U\right) + \left(\Delta_j(F), \Delta_j U\right)\\ &+\sum_{l=1}^{3}\left(R_j^l, \Delta_j U\right) + \left(I, \Delta_j U\right) - \left(II, \Delta_j U\right)\\ &-\left(III, \Delta_j U\right).\end{aligned}$$

Since A_0 is positive definite and symmetric, we know that there exist symmetric positive definite matrix $\sqrt{A_0}$ satisfies

$$\frac{d}{dt}\left(A_0\Delta_j U, \Delta_j U\right) = \frac{d}{dt}\left(\sqrt{A_0}\Delta_j U, \sqrt{A_0}\Delta_j U\right).$$

Combining this with Eq. (3.10) and the estimates (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), we have

$$\begin{aligned}\beta^{\frac{1}{2}}\|\Delta_j U(t)\|_{L^2} \le\ &\beta^{-\frac{1}{2}}\|\Delta_j U(0)\|_{L^2}\\ &+\beta^{-\frac{3}{2}}\int_0^t\left(\|\partial_t A_0\|_{L^\infty} + \|\nabla A\|_{L^\infty}\right)\|\Delta_j U(\tau)\|_{L^2}\,d\tau\\ &+C\beta^{-\frac{1}{2}}c_j 2^{-j\left(\frac{d}{2}+1\right)}M_1\int_0^t \|U(\tau)\|_{B^{\frac{d}{2}+1}}\,d\tau\\ &+C\beta^{-\frac{1}{2}}c_j 2^{-j\left(\frac{d}{2}+1\right)}M_2\int_0^t \|A_0^{-1}F(\tau)\|_{B^{\frac{d}{2}+1}}\,d\tau,\end{aligned}\tag{3.17}$$

where

$$\begin{aligned}M_1 =\ &\|B\|_{L_T^\infty(B^{\frac{d}{2}+1})} + \|\nabla A\|_{L_T^\infty\left(B^{\frac{d}{2}}\right)} + \|A_0\|_{L_T^\infty(L^\infty)}\|\nabla(A_0^{-1}A)\|_{L_T^\infty\left(B^{\frac{d}{2}}\right)}\\ &+\|A_0\|_{L_T^\infty(L^\infty)}\|A_0^{-1}B\|_{L_T^\infty\left(B^{\frac{d}{2}+1}\right)}.\\ M_2 =\ &1 + \|\tilde{A}_0\|_{L_T^\infty\left(B^{\frac{d}{2}+1}\right)}.\end{aligned}$$

Multiplying $2^{j\left(\frac{d}{2}+1\right)}$ on both sides of (3.17), and summing up for j, we obtain

$$\begin{aligned}\beta^{\frac{1}{2}}\|U(t)\|_{B^{d/2+1}} \le\ &\beta^{-\frac{1}{2}}\|U(0)\|_{B^{d/2+1}} + C\beta^{-2}M\int_0^t \beta^{\frac{1}{2}}\|U(\tau)\|_{B^{d/2+1}}\,d\tau\\ &+C\beta^{-\frac{1}{2}}M_2\int_0^t \|A_0^{-1}F(\tau)\|_{B^{d/2+1}}\,d\tau.\end{aligned}$$

Finally, by Gronwall's inequality, we complete the proof of this Lemma. □

Remark 3.1 Here, we actually proved the following stronger result

$$\beta \sum_{j\geq -1} 2^{j(d/2+1)} \|\Delta_j U\|_{L_t^\infty(L^2)} \leq \|U(0)\|_{B^{d/1+1}} e^{C\beta^{-2}Mt} + CM_2 \int_0^t e^{C\beta^{-2}M(t-\tau)} \|A_0^{-1}F(\tau)\|_{B^{d/2+1}}\, d\tau. \tag{3.18}$$

3.2 Quasilinear Positive Symmetric System

We now consider the following quasilinear system

$$\begin{cases} A_0(U)\partial_t U + \sum_{k=1}^d A_k(U)\partial_k U = B(U), \\ U(x,t) = U_0(x). \end{cases} \tag{3.19}$$

For this problem, we make the following assumptions.

Condition 3.1 *Assume the $n \times n$ symmetric matrices $A_k(U)$ and the source term $B(U)$ smoothly depend on $U \in \mathbb{R}^n$ and $B(0) = 0$. $A_k(U)$ has the following form*

$$A_k(U) = \bar{A}_k + \tilde{A}_k(U) \quad \textit{for } k = 0, 1, \ldots, d,$$

where $\bar{A}_k$ are constant matrices, $\tilde{A}_k(U)$ smoothly depend on $U \in \mathbb{R}^n$, $\tilde{A}_k(0) = 0$ and $A_0(U)$ is positive definite.

We now prove the following local theory.

Theorem 3.4 *Assume Condition 3.1. Let $\mathcal{O}$ be any open subset of $\mathbb{R}^n$. For the Cauchy problem (3.19) with initial data $U_0 \in B^{\frac{d}{2}+1}(\mathbb{R}^d)$ taking values in $\mathcal{O}$, there exists $T > 0$ such that the system has a unique solution $U \in C^1(\mathbb{R}^d \times [0,T])$. Furthermore, U belongs to $C([0,T]; B^{\frac{d}{2}+1}(\mathbb{R}^d)) \cap C^1([0,T]; B^{\frac{d}{2}}(\mathbb{R}^d))$.*

Proof The proof is based on the following iteration scheme

$$\begin{cases} A_0(U^n)\partial_t U^{n+1} + \sum_{k=1}^d A_k(U^n)\partial_k U^{n+1} = B(U^n), \\ U^{n+1}|_{t=0} = S_{N_2+n+1}U_0. \end{cases} \tag{3.20}$$

Since $S_{N_2}U_0$ tends to U_0 in $B^{\frac{d}{2}}(\mathbb{R}^d)$ when N_2 goes to $+\infty$, by the embedding

$$B^{\frac{d}{2}}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d),$$

there are N_2, $\delta > 0$ and a relatively compact open subset $\mathcal{V}$ of $\mathcal{O}$ such that any smooth function U satisfying the estimate

$$\|U - S_{N_2}U_0\|_{B^{\frac{d}{2}+1}} \leq \delta \tag{3.21}$$

takes values in $\mathcal{V}$.

The relatively compact subset $\mathcal{V}$ will serve as a reference for energy estimates. We take $\beta > 0$ so that

$$\beta I_n \leq A_0(U) \leq \beta^{-1} I_n, \tag{3.22}$$

for all $U \in \bar{\mathcal{V}}$.

We proceed by induction. Initially, we have $U^0(t) = S_{N_2}U_0$ and thus

$$\|U^0 - S_{N_2}U_0\|_{L^\infty_T(B^{\frac{d}{2}+1})} = 0 \le \delta,$$

for any T and δ. We assume that for all $l \le k$, U^l is defined inductively by (3.20) and satisfies the estimate

$$\|U^l - S_{N_2}U_0\|_{L^\infty_T(B^{\frac{d}{2}+1})} \le \delta. \tag{3.23}$$

We are going to show that the same estimate (3.23) holds for $l = k+1$, provided that T is suitably chosen.

We introduce the notations $V^{k+1} := U^{k+1} - S_{N_2}U_0$. By definition, V^{k+1} must solve the following Cauchy problem

$$\begin{cases} A_0(U^k)\partial_t V^{k+1} + \sum_{j=1}^d A_j(U^k)\partial_j V^{k+1} = B(U^k) - \sum_{j=1}^d A_j(U^k)\partial_j S_{N_2}U_0, \\ V^{k+1}(0) = S_{N_2+k+1}U_0 - S_{N_2}U_0. \end{cases} \tag{3.24}$$

Applying Theorem 3.3 to the above iterative system (3.24), we have

$$\beta\|V^{k+1}(t)\|_{B^{\frac{d}{2}+1}} \le \|V^{k+1}(0)\|_{B^{\frac{d}{2}+1}} e^{C\beta^{-2}Mt} + CM_2 e^{C\beta^{-2}Mt}\int_0^t G(\tau)\,d\tau. \tag{3.25}$$

where

$$\begin{aligned} G(t) &= \left\| A_0^{-1}(U^k)B(U^k) + \sum_{j=1}^d A_0^{-1}(U^k)A_j(U^k)\partial_j(S_{N_2}U_0)\right\|_{B^{\frac{d}{2}+1}}, \\ M &= \|\partial_t A_0(U^k)\|_{L^\infty_T(L^\infty)} + \|B(U^k)\|_{L^\infty_T\left(B^{\frac{d}{2}+1}\right)} + \|\nabla A(U^k)\|_{L^\infty_T\left(B^{\frac{d}{2}}\right)} \\ &\quad + \|A_0(U^k)\|_{L^\infty_T(L^\infty)}\|A_0^{-1}(U^k)B(U^k)\|_{L^\infty_T\left(B^{\frac{d}{2}+1}\right)} \\ &\quad + \|A_0(U^k)\|_{L^\infty_T(L^\infty)}\|\nabla(A_0^{-1}(U^k)A(U^k))\|_{L^\infty_T\left(B^{\frac{d}{2}}\right)}. \\ M_2 &= 1 + \|\tilde{A}_0(U^k)\|_{L^\infty_T\left(B^{\frac{d}{2}+1}\right)}, \end{aligned}$$

With he help of the inductive assumption, Lemma 2.1 and Lemma 2.2, we know that there exists a constant $M_0 > 0$ such that

$$\max\left\{M, M_2, \sup_{t\in[0,T]} G(t)\right\} \le M_0 < +\infty.$$

Hence, the inequality (3.25) implies

$$\beta\|V^{k+1}(t)\|_{B^{\frac{d}{2}+1}} \le \|V^{k+1}(0)\|_{B^{\frac{d}{2}+1}} e^{C\beta^{-2}M_0T} + CTM_0^2 e^{C\beta^{-2}M_0T}. \tag{3.26}$$

Taking N_2 large enough, T small enough, we finally obtain

$$\sup_{t\in[0,T]} \|V^{k+1}(t)\|_{B^{\frac{d}{2}+1}} \le \delta. \tag{3.27}$$

Using the similar method as in [6], we can prove U^n is a Cauchy sequence in $L_T^\infty(L^2)$ for small enough $T > 0$. By interpolation, U^n is also a Cauchy sequence in $L_T^\infty(B^{s'})$ for any $s' < \frac{d}{2} + 1$. The limit U of U^n is obvious a solution of (3.19). Using the Fatou property for the Besov space $B^{\frac{d}{2}+1}$, we conclude that U belongs to

$$L^\infty([0,T]; B^{\frac{d}{2}+1}) \cap C([0,T]; B^{s'}) \cap C^1([0,T]; B^{s'-1}),$$

for any $s' < \frac{d}{2} + 1$. In order to prove that U belongs to $C([0,T]; B^{\frac{d}{2}+1})$, we observe from (3.27) and Remark 3.1 that

$$\sum_{j\geq -1} 2^{j\left(\frac{d}{2}+1\right)} \|\Delta_j U\|_{L_T^\infty(L^2)} \leq \delta + \|S_{N_2} U_0\|_{B^{\frac{d}{2}+1}},$$

which implies that for any positive ϵ, there is some integer j_0 such that

$$\sum_{j\geq j_0} 2^{j\left(\frac{d}{2}+1\right)} \|\Delta_j U\|_{L_T^\infty(L^2)} \leq \frac{\epsilon}{4}.$$

Therefore, for $t, t' \in [0,T]$, we have

$$\begin{aligned}
\|U(\cdot,t) - U(\cdot,t')\|_{B^{\frac{d}{2}+1}} &\leq \sum_{j<j_0} 2^{j\left(\frac{d}{2}+1\right)} \|\Delta_j (U(\cdot,t) - U(\cdot,t'))\|_{L^2} \\
&\quad + 2\sum_{j\geq j_0} 2^{j\left(\frac{d}{2}+1\right)} \|\Delta_j U\|_{L_T^\infty(L^2)} \\
&\leq C 2^{j_0\left(\frac{d}{2}+1\right)} \|U(\cdot,t) - U(\cdot,t')\|_{L^2} + \frac{\epsilon}{2}.
\end{aligned}$$

Because U is in $C([0,T]; L^2)$, the first term on the right-hand side tends to 0 when $t' \to t$. This implies that U is continuous in time with values in $B^{\frac{d}{2}+1}$. $\square$

Now, one can follow the standard continuity argument to extend the existence time T of the solution to a maximal one; c.f. [1]. Therefore, the following theorem can be proved in the same manner as in [1].

Theorem 3.5 *Under the Condition 3.1, assume that (3.19) is positive symmetric hyperbolic for its arguments, with the initial data $U_0(x) \in B^{\frac{d}{2}+1}(\mathbb{R}^d)$. There exits a positive time T such that (3.19) has a unique solution $U(x,t) \in C([0,T]; B^{\frac{d}{2}+1}) \cap C^1([0,T]; B^{\frac{d}{2}})$. Moreover, T can be bounded from below by $c_1 \|U_0(x)\|^{-1}_{B^{\frac{d}{2}+1}}$, where c_1 depends only on A_k $(k = 0, \ldots, d)$. The maximal time of existence T^* of such a solution satisfies*

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\nabla U(\cdot,t)\|_{L^\infty}\, dt = \infty.$$

Remark 3.2 It is clear that if the initial data $U_0 - \bar{U} \in B^{\frac{d}{2}+1}$ for some constant $\bar{U}$. Replacing $B(0) = 0$ and $\tilde{A}_k(0) = 0$ with $B(\bar{U}) = 0$ and $\tilde{A}_k(\bar{U}) = 0$ in Condition 3.1, Theorem 3.5 holds for $U - \bar{U}$.

The following theorem is about the continuous dependence of the solutions, which will be useful in the next section for the description of isentropic approximations for compressible Euler equations.

Theorem 3.6 *Assume that A_0 is positive definite such that*

$$\beta I_n \le A_0 \le \beta^{-1} I_n$$

for $\beta > 0$ in the sense of quadratic forms. Let U^ϵ and U be solutions of (3.19) on $\mathbb{R}^d \times [0, T]$, with initial data $U_0^\epsilon, U_0 \in B^{\frac{d}{2}+1}(\mathbb{R}^d)$ respectively. Then

$$\sup_{t\in[0,T]} \|U^\epsilon(t) - U(t)\|_{L^2} \le \frac{e^{\frac{1}{2}(\lambda + C\beta^{-2})T}}{\beta} \|U_0^\epsilon - U_0\|_{L^2}.$$

where C depends on the L^∞-norms of ∇U and ∇U^ϵ, and λ is a large enough constant satisfing

$$\left\| \partial_t A_0(U) + \sum_{k=1}^{d} \partial_k A_k(U) \right\|_{L_T^\infty(L^\infty)} \le \beta(\lambda - 1).$$

Proof From Theorem 3.5, it is clear that $T > 0$ is smaller than the maximimal existence times of U^ϵ or U. Therefore, we have that

$$\sup_{0\le t\le T} (\|(U_t, \nabla U)(\cdot, t)\|_{L^\infty} + \|(U_t^\epsilon, \nabla U^\epsilon)(\cdot, t)\|_{L^\infty}) \le C,$$

for some positive constant C. Denote $\delta U = U^\epsilon - U$, we know that

$$A_0(U)\partial_t \delta U + \sum_{k=1}^{d} A_k(U)\partial_k \delta U = F, \tag{3.28}$$

where

$$\begin{aligned} F = {} & B(U^\epsilon) - B(U) - \left(A_0(U^\epsilon) - A_0(U)\right) [A_0(U^\epsilon)]^{-1} B(U^\epsilon) \\ & + \sum_{k=1}^{d} \left(A_0(U^\epsilon) - A_0(U)\right) [A_0(U^\epsilon)]^{-1} A_k(U^\epsilon)\partial_k U^\epsilon \\ & - \sum_{k=1}^{d} \left(A_k(U^\epsilon) - A_k(U)\right) \partial_k U^\epsilon. \end{aligned}$$

Using mean-value theorem, it holds that

$$\begin{aligned} \|F(\cdot, t)\|_{L^2} & \le C(\|(U, U^\epsilon)\|_{L_T^\infty(L^\infty)} + \|\nabla U^\epsilon\|_{L_T^\infty(L^\infty)})\|U^\epsilon(\cdot, t) - U(\cdot, t)\|_{L^2} \\ & \le C\|U^\epsilon(\cdot, t) - U(\cdot, t)\|_{L^2}. \end{aligned} \tag{3.29}$$

Multiplying both sides of (3.28) with $(\delta U)^T$, and then integrating over $\mathbb{R}^d$, integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dt}(A_0(U)\delta U, \delta U) & \le \beta(\lambda - 1)(\delta U, \delta U) + \beta(\delta U, \delta U) + \beta^{-1}(F, F) \\ & \le \lambda(A_0(U)\delta U, \delta U) + \beta^{-1}\|F(\cdot, t)\|_{L^2}^2 \\ & \le \lambda(A_0(U)\delta U, \delta U) + C\beta^{-1}(\delta U, \delta U) \\ & \le (\lambda + C\beta^{-2})(A_0(U)\delta U, \delta U). \end{aligned}$$

Therefore, we have

$$\sup_{t\in[0,T]} \|U^\epsilon(t) - U(t)\|_{L^2}^2 \le \frac{e^{(\lambda+C\beta^{-2})T}}{\beta^2}\|U_0^\epsilon - U_0\|_{L^2}^2.$$

Hence, the proof is completed. □

4 Explicit Characterization About Isentropic Approximation

In this section, we apply the general results obtained in Sect. 3 to the compressible Euler equations, and then give an explicit characterization about isentropic approximation. Firstly as in [10], we use

$$w = p^{\frac{(\gamma-1)}{2\gamma}} = \rho^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)s}{2\gamma}} \tag{4.1}$$

to transform Euler Eq. (1.1) into

$$\begin{cases} A_0(U)\partial_t U + \sum_{j=1}^{d} A_j(U)\partial_j U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{4.2}$$

for $U = (w, u_1, u_2, u_3, s)$, and the matrices

$$A_0(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(\gamma-1)^2}{4\gamma} e^{-\frac{s}{\gamma}} I_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_j(U) = \begin{pmatrix} u_j & \frac{\gamma-1}{2} w e_j & 0 \\ \frac{\gamma-1}{2} w e_j^T & \frac{(\gamma-1)^2}{4\gamma} e^{-\frac{s}{\gamma}} u_j I_3 & 0 \\ 0 & 0 & u_j \end{pmatrix} \quad (j = 1, 2, 3).$$

Here, e_j is the $j-$th row of I_3. Theorem 3.4 and Therem 3.5 give the following Theorem.

Theorem 4.1 *If for some constants $\bar\rho \ge 0$, $\bar s$ and $\bar w = \bar\rho^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)\bar s}{2\gamma}} \ge 0$, the initial data $U_0 = (w_0, u_0, s_0)$ satisfies that $U_0 - (\bar w, 0, \bar s) \in B^{\frac{5}{2}}(\mathbb{R}^3)$, there exists a constant $T > 0$ and a unique solution $U = (w, u, s)$ to the problem (4.2) such that*

$$U - (\bar w, 0, \bar s) \in C([0,T]; B^{\frac{5}{2}}(\mathbb{R}^3)) \cap C^1([0,T]; B^{\frac{3}{2}}(\mathbb{R}^3)).$$

If in addition $w_0(x) \ge 0$ for all $x \in \mathbb{R}^3$, then $w(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^3 \times [0,T]$. Equivalently, if $1 < \gamma \le 3$, $\rho_0 \in C^1(\mathbb{R}^3)$, $\rho_0 \ge 0$ and

$$U_0 - (\bar w, 0, \bar s) \in B^{\frac{5}{2}}(\mathbb{R}^3),$$

then there exists a positive number T and a unique solution $(\rho, u, s)(x,t) \in C^1([0,T]\times\mathbb{R}^3)$ to problem (1.1) such that $\rho(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^3 \times [0,T]$ and

$$U(x,t) = \left(\rho^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)s}{2\gamma}}, u, s\right)(x,t)$$

is the solution of (4.2) such that

$$U - (\bar w, 0, \bar s) \in C([0,T]; B^{\frac{5}{2}}(\mathbb{R}^3) \cap C^1([0,T]; B^{\frac{3}{2}}(\mathbb{R}^3)).$$

Remark 4.1 This Theorem includes the cases of initial data with or without vacuum. For the H^s theory with $s > \frac{5}{2}$, the case with initial data including vacuum was given in [10], and the case with initial data away from vacuum was given in [9].

In order to give a precise description on the isentropic approximation for compressible Euler equations, we now assume that

$$s_0(x) - \bar{s} = \varepsilon\phi(x) \tag{4.3}$$

for $\phi(x) \in B^{\frac{5}{2}}(\mathbb{R}^3)$, and $\varepsilon > 0$ is the controlling parameter. As explained in the introduction, given an initial data (ρ_0, u_0, s_0) or (w_0, u_0, s_0) as in Theorem 4.1, the Cauchy problem (1.1) has a smooth solution $(w, u, s)(x, t)$ defined on $\mathbb{R}^3 \times [0, T]$ for some positive T. We denote this solution as $U^\varepsilon(x, t) = (w, u, s)(x, t)$. If one assigns an initial data $(\rho_0, u_0, \bar{s})$ or $(\tilde{w}_0, u_0, \bar{s})$ with $\tilde{w} = w(\rho, \bar{s})$ to (1.1), the unique solution of (1.1) on $\mathbb{R}^3 \times [0, T]$ is the same one to (1.2) with initial data (ρ_0, u_0) or $(\tilde{w}_0, u_0)$, and we denote this solution by $U^I(x, t) = (w^I, u^I, \bar{s})$. We can now apply Theorem 3.6 to obtain the following theorem.

Theorem 4.2 *Suppose* $1 < \gamma \leq 3$, $\varepsilon \in (0, 1]$, $\rho_0 \in C^1(\mathbb{R}^3)$, $\rho_0 \geq 0$ *and for some* $\bar{\rho} \geq 0$,

$$\rho_0^{\frac{(\gamma-1)}{2}} - \bar{\rho}^{\frac{(\gamma-1)}{2}} \in B^{\frac{5}{2}}(\mathbb{R}^3), \quad u_0 \in B^{\frac{5}{2}}(\mathbb{R}^3), \quad \phi \in B^{\frac{5}{2}}(\mathbb{R}^3),$$

and

$$\rho_0^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)(\bar{s}+\varepsilon\phi)}{2\gamma}} - \bar{\rho}^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)\bar{s}}{2\gamma}} \in B^{\frac{5}{2}}(\mathbb{R}^3).$$

Then, (1.2) *has a unique solution*

$$U^I(x, t) = (w^I, u^I, \bar{s})(x, t),$$

and (1.1) *has a unique solution*

$$U^\varepsilon(x, t) = (w, u, s)(x, t),$$

both defined on $\mathbb{R}^3 \times [0, T]$. *Furthermore, the following estimate holds*

$$\sup_{t\in[0,T]} \|U^\varepsilon(\cdot, t) - U^I(\cdot, t)\|_{L^2} \leq C\varepsilon\|\phi\|_{L^2},$$

where C *is a positive constant depending on* C^1 *norms of* $U_0(x)$ *and* T, *but not on* ε.

Proof By Theorem 3.6, we only need to compute $\|U_0^\epsilon - U_0\|_{L^2}$ to complete the proof of this Theorem, since A_0 was bounded by $\|s\|_{L^\infty}$. From (4.3) and (4.1), we have

$$\begin{aligned}\|U_0^\varepsilon - U_0\|_{L^2} &\leq \varepsilon\|\phi\|_{L^2} + \left(\int_{\mathbb{R}^3} \left|\rho_0^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)\bar{s}}{2\gamma}} \left(e^{\frac{(\gamma-1)\varepsilon\phi}{2\gamma}} - 1\right)\right|^2 dx\right)^{1/2} \\ &\leq \varepsilon\|\phi\|_{L^2} + C(\|\rho_0\|_{L^\infty}, \gamma, \bar{s})\left(\int_{\mathbb{R}^3} \left(e^{\frac{(\gamma-1)\varepsilon\phi}{2\gamma}} - 1\right)^2 dx\right)^{1/2}. \end{aligned} \tag{4.4}$$

Since $\phi \in B^{\frac{5}{2}}(\mathbb{R}^3)$, it is clear that $e^{\frac{(\gamma-1)\varepsilon\phi}{2\gamma}} - 1$ tends to zero as $|x| \to \infty$. We now make the following calculations.

$$\begin{aligned}|e^{\frac{(\gamma-1)\varepsilon\phi}{2\gamma}} - 1| &\leq |e^{\frac{(\gamma-1)\varepsilon\phi}{2\gamma}} - 1 - \frac{(\gamma-1)\varepsilon\phi}{2\gamma}| + |\frac{(\gamma-1)\varepsilon\phi}{2\gamma}| \\ &\leq C(\|\phi\|_{L^\infty}, \gamma)(\varepsilon^2\phi^2 + \varepsilon|\phi|). \end{aligned} \tag{4.5}$$

Therefore, we conclude that

$$\begin{aligned}\|U_0^\varepsilon - U_0\|_{L^2} &\le \varepsilon\|\phi\|_{L^2} + C\varepsilon\|\phi\|_{L^2}\\ &\le C\varepsilon\|\phi\|_{L^2},\end{aligned}$$

which concludes the proof of this Theorem. □

Remark 4.2 This theorem gives a precise justification with an explicit error estimate on isentropic approximation for compressible Euler equations in the regime of smooth solutions.

5 Failure of Isentropic Limit

In the previous sections, the justification for isentropic limit has been proved for classical solutions of compressible Euler equations. It is well-known that shock waves may develope in finite time even for generic small smooth initial data. When shock forms, the justification of isentropic limit in previous sections breaks down. This can be seen easily by the following example.

Consider the full compressible Euler equations in one space dimension

$$\begin{cases}\rho_t + (\rho u)_x = 0, \ x \in \mathbb{R},\\ (\rho u)_t + (\rho u^2 + p(\rho, s))_x = 0,\\ (\rho E)_t + (\rho E u + p u)_x = 0\\ \rho(x,0) = \rho_0(x) \ge 0, \ u(x,0) = u_0(x), \ s(x,0) = \bar{s},\end{cases} \tag{5.1}$$

where $E = \frac{1}{2}u^2 + e$ and $\rho e = \frac{c_v}{R} p$ with two positive constants c_v and R, and its isentropic reduction

$$\begin{cases}\rho_t + (\rho u)_x = 0, \ x \in \mathbb{R},\\ (\rho u)_t + (\rho u^2 + p(\rho, \bar{s}))_x = 0,\\ \rho(x,0) = \rho_0(x) \ge 0, \ u(x,0) = u_0(x).\end{cases} \tag{5.2}$$

We remark here that unlike (1.1), we replaced the entropy equation by the energy conservation law in (5.1), since the entropy equation is no longer valid if singularity occurs in the solutions. It is clear that for any C^1 functions ρ_0 and u_0, both (5.1) and (5.2) share exactly the same C^1 solution $(\rho(x,t), u(x,t), \bar{s})$ up to a maximal existence time $T_1 > 0$. However, when this solution blows up at (x_1, T_1) for some $x_1 \in \mathbb{R}$, and shocks appear in the solution, the shock solution for (5.1) is different from that of (5.2). Indeed, the Riemann problems of (5.1) and (5.2) with the same Riemann data

$$\lim_{x\to x_1-}(\rho, u, s)(x, T_1) = (\rho_-, u_-, \bar{s}); \ \lim_{x\to x_1-}(\rho, u, s)(x, T_1) = (\rho_+, u_+, \bar{s}), \tag{5.3}$$

are different since the former one has a variable s (entropy s must increase across a shock wave, see [13]), while the latter has a constant s in the solution.

Therefore, the framework we used in justification the isentropic limit process in the previous sections is no longer valid when singularity occurs in the solutions of Euler equations. New insights and techniques are required to offer possible description of isentropic approximation for entropy weak solutions. Some research activities have been carried out in this direction, see [2,12] and [8], where the difference between solutions of isentropic and full Euler equations measured by BV-norm was shown to grow at most linearly in time.

Acknowledgments The authors are very grateful to valuable suggestions of the anonymous referees improving the presentations of this paper. J. Jia would like to thank China Scholarship Council that has provided a scholarship for supporting his research in the United States. J. Jia's research is supported partially by National Natural Science Foundation of China under the Grant No. 11131006, and by the National Basic Research Program of China under the Grant No. 2013CB329404. The research of R. Pan is supported by the National Science Foundation through the Grant DMS-1108994.

References

1. Bahouri, H., Chemin, J., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der mathematischen Wissenschaften, vol. 343. Springer, Berlin (2011)
2. Chen, S., Geng, J., Zhang, Y.: Isentropic approximation of quasi-one-dimensional unsteady nozzle flow. SIAM J. Math. Anal. **41**(4), 1693–1712 (2009)
3. Dafermos, C.M.: Hyperbolic Conservation Laws in Continuum Physics. Grundlehren der mathematischen Wissenschaflten, vol. 325. Springer, Berlin (2000)
4. Danchin, R.: Lecture, Notes: Fourier Analysis Methods for PDE's (2005). http://perso-math.univ-mlv.fr/users/danchin.raphael/courschine.pdf
5. Friedrichs, K.O.: Symmetric hyperbolic linear differential equations. Commun. Pure Appl. Math. **7**, 345–392 (1954)
6. Gavage, S.B., Serre, D.: Multi-dimensional Hyperbolic Partial Differential Equations: First-order Systems and Applications. Oxford Mathematical Monographs. Oxford Science Publications, Oxford (2007)
7. Kato, T.: The cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Ration. Mech. Anal. **58**(3), 181205 (1975)
8. Liu, C., Zhang, Y.: Isentropic approximation of the steady euler system in two space dimensions. Commun. Pure Appl. Anal **7**(2), 277291 (2008)
9. Majda, A.: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variable. Applied Mathematical Sciences, vol. 53. Springer, Berlin (1984)
10. Makino, T., Ukai, S., Kawashima, S.: Sur la solution à support compact de l'equation d'euler compressible. Japan J. Appl. Math. **3**, 249–257 (1986)
11. Peetre, J.: New Thoughts on Besov Spaces. Duke Univ. Math. Ser. I, Durham (1976)
12. Saint-Raymond, L.: Isentropic approximation of the compressible euler system in one space dimension. Arch. Ration. Mech. Anal. **155**(3), 171–199 (2000)
13. Smoller, J.A.: Shock Waves and Reaction-Diffusion Equations. Grundlehren der mathematischen Wissenschaflten, vol. 258, 2nd edn. Springer, Berlin (1994)