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Mechanical behavior analysis on electrostatically actuated rectangular microplates

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Abstract
Microplates are widely used in various MEMS devices based on electrostatic actuation such as MEMS switches, micro pumps and capacitive micromachined ultrasonic transducers (CMUTs). Accurate predictions for the mechanical behavior of the microplate under electrostatic force are important not only for the design and optimization of these electrostatic devices but also for their operation. This paper presents a novel reduced-order model for electrostatically actuated rectangular and square microplates with a new method to treat the nonlinear electrostatic force. The model was developed using Galerkin method which turned the partial-differential equation governing the microplates into an ordinary equation system. Using this model and cosine-like deflection functions, explicit expressions were established for the deflection and pull-in voltage of the rectangular and square microplates. The theoretical results were well validated with the finite element method simulations and experimental data of literature. The expressions for the deflection analysis are able to predict the deflection up to the pull-in position with an error less than 5.0%. The expressions for the pull-in voltage analysis can determine the pull-in voltages with errors less than 1.0%. Additionally, the method to calculate the capacitance variation of the electrostatically actuated microplates was proposed. These theoretical analyses are helpful for design and optimization of electrostatically actuated microdevices.

Keywords: electrostatically actuated rectangular microplates, static deflection analysis, pull-in voltage prediction, capacitance calculation, a new reduced-order model, explicit expressions (Some figures may appear in colour only in the online journal)

1. Introduction

Electrostatically actuated microplates have been widely used in various microelectromechanical system (MEMS) devices such as capacitive pressure sensors, MEMS switches, micro pumps and capacitive micromachined ultrasonic transducers (CMUTs) [1–6]. Applications of these microdevices have spread over pressure sensing, automatic control, biotechnology research, ultrasonic imaging and chemical detection [2–6]. In order to design and optimize these microplate-based electrostatic devices, clear knowledge of their mechanical behavior under electrostatic actuation is extremely necessary. To this end, numerous investigations have been directed towards understanding, predicting and controlling their mechanical response.
to an electrostatic force \[7-9\]. For instance, Yaralioglu et al \[7\] studied the capacitance change and then electromechanical coupling coefficient of CMUTs under a bias voltage. Chao et al \[9\] investigated the pull-in voltage of a clamped square microplate subjected to a distributed electrostatic force.

Generally, two parallel microplates are simultaneously employed to compose an electrostatic actuation configuration. One of them is a flexible conductor and capable of motion or deformation under the action of electrostatic force, while the other one is a stationary rigid conductor and remains fixed. The gap between the two microplates is usually air or a vacuum. When a dc voltage is applied across the two microplates, an electrostatic force is produced and forces the flexible microplate to deflect toward the fixed one. Accurate determination of this deflection is crucial to analyzing the performance of the electrostatic device. For example, for CMUTs, this deflection determines the capacitance change and thus the electro-mechanical coupling coefficient which characterizes the transducer bandwidth and sensitivity \[7, 10\]. For capacitive pressure sensors, the deflection of the flexible microplate as a result of both the electrostatic force and the pressure determines the capacitance variation and thus the sensor sensitivity and nonlinearity \[2, 11\]. If the applied voltage increases up to a critical value beyond which the mechanical restoring force of the microplate can no longer balance the electrostatic force, the flexible microplate will collapse and make contact with the rigid microplate. This phenomenon is called ‘pull-in’, and the corresponding critical voltage is called the ‘pull-in voltage’. This pull-in voltage is an important parameter for the operation of the microdevice. For example, the maximum dc bias voltage applied to the CMUT is dependent on the pull-in voltage \[12\]. The capacitive pressure sensors rely on the pull-in voltage to determine their working modes (normal or touch mode) \[2, 11\]. MEMS switches need to tune the bias voltage across the pull-in voltage back and forth to alternate switch-on and off. Overall, the deflection and pull-in voltage of the flexible microplate are two important parameters for the design and operation of these electrostatic microdevices, and need investigating thoroughly.

Analyses of the deflection and pull-in voltage are challenging because the classical structure dynamics methodology is not easily applicable to the nonlinear electrostatic force encountered in electrostatically actuated microdevices. Some researchers have used numerical simulation softwares such as ANSYS, COMSOL to simulate the mechanical behavior of the electrostatic microdevices \[3, 4\]. Though these numerical methods can provide accurate solutions, they suffer from the inherent flaws in computational modeling \[13\], such as time consuming, expensive in the design of feedback control laws, complex process of mapping the design space. In order to understand the significant characteristics of the mechanical behavior, researchers modeled the mechanical behavior theoretically and, then used the finite element method to validate their theoretical analyses. Initially, a simple 1D lumped mass-spring model was utilized to approximate the behavior of electrostatically actuated microplates \[7, 8, 14\]. However, this approximation considered the deformed plate (as shown in figure 1) as a rigid plate with piston-like motion. It neglects the static deflection of the microplate under electrostatic force. Furthermore, the famous prediction of pull-in position of the microplates as being, one-third of the air gap, was shown to underestimate the actual pull-in position \[9\]. Therefore, many researchers started to develop more sophisticated continuous models to analyze the behavior of the microplates. Vogl and Nayfeh \[15\] presented a continuous reduced-order model for an electrically actuated clamped circular microplate and studied its static equilibrium state under a general electric potential. Zhao et al \[16\] also developed a reduced-order model for rectangular microplates applicable to any classical boundary conditions. They further reported the mechanical behavior of a fully clamped rectangular microplate under electrostatic actuation using this model. Though these theoretical models can analyze the behavior of the microplates under electrostatic force, they cannot be used to establish simple and explicit expressions for the relationship between the static deflection and the applied voltage, as well as the relationship between the pull-in voltage and the structure parameters of the microplates. Therefore, Ahmad and Pratap \[12\] established a relationship for the static deflection of the clamped circular microplate and the applied voltage. Chao et al \[9\] and Liao et al \[17\] presented explicit expressions for the pull-in voltages for clamped square and circular microplates with uniform biaxial residual stress, respectively. However, Liao’s work did not provide explicit expression for the static deflection of the square microplate under electrostatic force. Furthermore, there have not been explicit expressions for static deflection and pull-in voltage of rectangular microplates. Inspired by these facts, the present study is dedicated to proposing a new analytical model, and then establishing explicit expressions which can accurately capture the deformation characteristics of rectangular and square microplates under electrostatic force, as well as explicit expressions for the pull-in voltages.

In this study, a reduced-order model was firstly established for the deflection of an electrostatically actuated rectangular microplate using a new method to treat the electrostatic force. Based on this reduced-order model, explicit expressions were established for the static deflections of the rectangular and square microplates under the electrostatic force. Subsequently, explicit expressions for the pull-in voltages of the rectangular and square microplates were derived using static deflection analysis. Then the capacitance variation of the parallel-plate configuration induced by the electrostatic force was calculated using the expressions for the static deflections. Finally, the results from these theoretical analyses were compared with those from other works and the finite element method (FEM) simulations. It is demonstrated that our analytical expressions can accurately predict the mechanical behavior of the electrostatically actuated microplates.

2. Theoretical analysis

2.1. Problem formulation

A schematic of an electrostatically actuated configuration composed of two parallel rectangular microplates is considered in
this theoretical analysis. Its schematic is shown in figure 1. The top rectangular microplate with its four edges clamped has a length $2a$, width $2b$ and thickness $h$. The microplate is assumed to be a thin, linear and isotropic microplate. It is parallel separated with the fully fixed bottom microplate with an effective electrode distance $d$. The space between the two microplates is assumed as a vacuum or air with a dielectric constant $\varepsilon_0$. The top and bottom microplates are electrically conductive, and work as the top and bottom electrodes of the electrostatic configuration, respectively. When a bias voltage is applied across the two microplates, the top microplate is deflected towards the bottom one by the resulting electrostatic force. The deflection of the top microplate is assumed to be small compared to its thickness $h$. The classical thin microplate theory is used for the deflection analysis, which is adequate when the thickness-to-length ratio $(h/2b)$ is relatively small (i.e. $(h/2b) < 1/20)$ [18]. It should be noted the plate referred to in the following sections is the top microplate unless stated otherwise.

The general equation governing the bending of an isotropic rectangular microplate is given in [19]. For the transverse deflection of the microplate under a distributed electrostatic force, the partial differential equation can be written as

$$
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = F_e(x, y),
$$

(1)

where $w = w(x, y)$ is the transverse deflection of the microplate; $x$ and $y$ are the coordinates originated at the center point of the microplate; $D$ is the flexural stiffness

$$
D = \frac{Eh^3}{12 (1 - \nu^2)},
$$

(2)

where $E$ and $\nu$ are the Young’s modulus and Poisson’s ratio, respectively; $F_e(x, y)$ is the electrostatic force per unit area acting on the microplate, and can be given by

$$
F_e(x, y) = \frac{\varepsilon_0 V^2}{2[d - w(x, y)]^2},
$$

(3)

where $V$ is the bias voltage applied to the microplate. Equation (3) is an approximation to the electrostatic force of a parallel microplate capacitor, which neglects the fringing field effects due to the very small ratio of the electrode distance to the area of the microplate.

For the rectangular microplate with its four edges clamped as shown in figure 1, the boundary conditions are

$$
w(x, y) = 0 \quad \text{and} \quad \frac{dw(x, y)}{dx} = 0 \quad \text{at} \quad x = \pm a, \quad \forall \quad y
$$

$$
w(x, y) = 0 \quad \text{and} \quad \frac{dw(x, y)}{dy} = 0 \quad \text{at} \quad y = \pm b, \quad \forall \quad x.
$$

(4)

When $2a = 2b$, the rectangular microplate becomes a square one. The boundary conditions in equation (4) are also applicable to the clamped square microplate [9].

![Figure 1. Schematic of a parallel microplate configuration under electrostatic actuation; (a) top view of the configuration; (b) cross-section view of the configuration.](image)

2.2. Reduced-order model

For equation (1), it is hard to establish an accurate analytical solution owing to the nonlinearity of the electrostatic-force term. So Galerkin method is employed to decompose the deflection governing equation into a coupled set of ordinary algebraic equations (a reduced-order model). Then, the mechanical behavior of the electrostatic actuated microplate can be simulated by solving these ordinary equation systems. This technique and its application to nonlinear system have been discussed in [20, 21].

In this study, a novel method to treat the electrostatic force given by equation (3) is proposed to generate the reduced-order model. The electrostatic force is written as the product of two same terms [$1/(d - w)$]. Then only one of the two terms is expanded using Taylor’s series around $w = 0$. Consequently, equation (3) can be rewritten as

$$
F_e(w) = \frac{\varepsilon_0 V^2}{2(d - w)(d - w)}
$$

$$
= \frac{\varepsilon_0 V^2}{2d(d - w)} \left( 1 + \frac{w}{d} + \frac{w^2}{d^2} + \frac{w^3}{d^3} + \ldots \right).
$$

(5)

This method to approximate the electrostatic force is different from the method presented by Ahmad and Pratap [12], Younis [13] and Liao [17], which expanded the whole expression of equation (3) using Taylor’s series around $w = 0$. As shown in figure 2, equation (5) shows a better approximation to the electrostatic force than equation (3) of [12] (the expression (3) of [12] is shown in appendix A). The first, second and third order electrostatic force expansions in equation (5) are much closer to the electrostatic force given by equation (3) than the same order expansions in equation (3) of [12]. Figure 2 also shows that the first order electrostatic force expansion using the new method exhibits a better approximation than the second order electrostatic force expansion using the previous method. This is because equation (5) has a similar term $(d - w)^{-1}$ to equation (3) [given in this study] and thus can better capture
the variation characteristic of the electrostatic force. Therefore, the proposed method by us can reduce the degree of \( w \) and the complexity of calculation process, which is more advantageous for achieving explicit expressions for the mechanical behavior of electrostatic configurations. In other words, the first order electrostatic force expansion is able to get analytical expressions with simple form and sufficient accuracy.

Subsequently, the corresponding function used to approximate the transverse deflection \( w(x,y) \) is assumed as

\[
w(x,y) = \sum_{i,j=0}^{N} k_{ij} \phi_{ij}(x,y),
\]

where \( k_{ij} \) is the unknown coefficient to be determined; \( \phi_{ij}(x,y) \) is the linearly independent coordinate functions (they are also called trial functions) that satisfy all the prescribed boundary conditions [22]; \( N \) is the largest value of the variables \( i \) and \( j \). When \( N \) approaches infinity, the approximation in equation (6) becomes exact if the chosen trial functions form a complete set. For the rectangular microplate, the cosine-like functions are chosen as the trial functions to approximate its deformed shape. The expression for the function is given by [23]

\[
\phi_{ij} = \cos^2 \left( \frac{2j \pi x}{2a} \right) \cos^2 \left( \frac{2i \pi y}{2b} \right).
\]

These trial functions satisfies all the boundary conditions given in equation (4), the lowest order \((i = 0, j = 0)\) of which has been widely used to simulate the deflection of silicon diaphragms under hydrostatic pressure [24, 25].

Having determined the treatment method of the nonlinear electrical-force and the trial functions, the process of establishing the reduced-order model can be started by substituting equations (5) and (6) into equation (1). Then, multiplying both sides of the resulting equation with the term \((d\omega)\) and functions \( \phi_{lm}(x, y) \), finally integrating the outcome over the area of the microplate, the ordinary algebraic equation system for equation (1) is obtained

\[
df V^4 w \times \phi_{lm} dx dy = \sum_{i,j=0}^{N} k_{ij} df \ V^4 w \times \phi_{ij} \phi_{lm} dx dy = Q \left( df \ V^4 w \times \phi_{ij} \phi_{lm} dx dy + \cdots \right)
\]

\[
= \frac{1}{d} \sum_{i,j=0}^{N} k_{ij} df \phi_{ij} \phi_{lm} dx dy + \sum_{i,j=0}^{N} k_{ij} \phi_{ij} \phi_{mg} \phi_{lm} \phi_{hg} dx dy + \cdots,
\]

\[
i,j,p,q,h,l,m = 0, 1, 2, 3 \cdots N,
\]

where

\[
\mathbb{V}^4 w = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}
\]

\[
= \sum_{i,j=0}^{N} k_{ij} \left[ \frac{\partial^4 \phi_{ij}(x,y)}{\partial x^4} + 2 \frac{\partial^2 \phi_{ij}(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_{ij}(x,y)}{\partial y^4} \right],
\]

and

\[
Q_i = \frac{\varepsilon_0 V^2}{2dD}.
\]

The integrals in equation (8) can be evaluated once the trial functions of equation (7) are substituted into (8) and the number \( N \) is chosen. Then, once all \( k_{ij} \) are determined by solving equation (8), an approximate solution to equation (1) can be established by equation (6).

2.3. Static deflection analysis

Owing to the closeness between the shape given by the trial function \( \phi_{i,j} = 0, j = 0 \) and that of a practical deformed microplate [24, 25], the lowest order \((N = 0)\) of the trial function in equation (6) is used. Additionally, as the complexity of both solving process and final solution increases with the order of the Taylor’s expansion of the electrostatic force, the first order electrostatic force expansion is used to establish explicit expressions for the static deflection. These simplifications can not only contribute to a simple theoretical expression for the relationship between the deflection and the structure parameters, but also are sufficient for predicting the behavior of the microplate under electrostatic actuation, which will be shown in section 3.

Substituting equation (7) into (6) and setting \( N = 0 \), the deflection function in equation (6) can be rewritten as

\[
w(x, y) = k_{00} \phi_{00}(x, y) = k_{00} \cos^2 \left( \frac{\pi x}{2a} \right) \cos^2 \left( \frac{\pi y}{2b} \right),
\]

where \( k_{00} \) in fact gives the largest deflection of the microplate (the deflection of the center point). Then substituting equation (11) into (8) and using the first order electrostatic force expansion, equation (8) can be simplified as

**Figure 2.** Comparison of different methods to treat the electrostatic force; the electrostatic force refers to equation (3); the previous method refers to equation (3) of [12]; the new method denotes equation (5).
\[ d \int_{-b}^{a} \nabla^4 w \times \phi_{00} \, dx \, dy - k_{00} \int_{-b}^{a} \nabla^4 w \times \phi_{00} \, dx \, dy = Q \left( \int_{-b}^{a} \int_{-a}^{b} \phi_{00} \, dx \, dy + k_{00} \int_{-b}^{a} \int_{-a}^{b} \phi_{00} \, dx \, dy \right). \]  
\tag{12}

Integrating the integrals and then solving the resulting equation, two possible solutions to the constant \( k_{00} \) are given by

\[ k_{00} = \frac{(3a^4 + 2a^2b^2 + 3b^4) d^2 \pi^4 - 9a^4b^4 Q_1 \pm M}{(5a^4 + 4a^2b^2 + 5b^4) d^4 \pi^4}, \]  
\tag{13}

where

\[ M = \left( \frac{(3a^4 + 2a^2b^2 + 3b^4) d^2 \pi^4 - 9a^4b^4 Q_1}{(5a^4 + 4a^2b^2 + 5b^4) d^4 \pi^4} \right)^2. \]  
\tag{14}

Actually, only one of the two possible solutions is consistent with the physical situation in which the deflection of the microplate increases with the applied voltage. So the correct expression for \( k_{00} \) is

\[ k_{00} = \frac{(3a^4 + 2a^2b^2 + 3b^4) d^2 \pi^4 - 9a^4b^4 Q_1 - M}{(5a^4 + 4a^2b^2 + 5b^4) d^4 \pi^4}. \]  
\tag{15}

Substituting equation (15) into (11), the complete expression for the transverse deflection of an electrostatically actuated rectangular microplate is obtained

\[ w(x, y) = \frac{(3a^4 + 2a^2b^2 + 3b^4) d^2 \pi^4 - 9a^4b^4 Q_1 - M}{(5a^4 + 4a^2b^2 + 5b^4) d^4 \pi^4} \times \cos^2 \left( \frac{\pi x}{2a} \right) \cos^2 \left( \frac{\pi y}{2b} \right). \]  
\tag{16}

When \( a = b \), the rectangular plate becomes a square one, which can also be understood as a special case of the rectangular microplate. Equation (16) can be simplified as

\[ w_{sq}(x, y) = \frac{8\pi^2 \pi^8 - 9a^4b^4 Q_1 - M_{sq}}{144d^4 \pi^4} \cos^2 \left( \frac{\pi x}{2a} \right) \cos^2 \left( \frac{\pi y}{2a} \right), \]  
\tag{17}

where \( w_{sq}(x, y) \) represents the transverse deflection of a clamped square microplate under electrostatic force; \( M_{sq} \) is determined by

\[ M_{sq} = \sqrt{64d^4 \pi^8 - 592a^4d^2 \pi^4 Q_1 + 81a^6Q_1^2}. \]  
\tag{18}

Equations (16) and (17) give the static deflections of the clamped rectangular and square microplates under electrostatic force. Furthermore, the two equations provide approaches to analyze the pull-in voltage and the capacitance changes of electrostatically actuated microplates.

2.4. Pull-in voltage prediction

Pull-in voltages of electrostatically actuated microplates can be determined from the two possible solutions in equation (13). It can be derived that there is a certain voltage at which the two solutions equal each other. This voltage is a critical point where the pull-in phenomenon occurs [12, 15, 26]. The pull-in voltage can be obtained using the similar method in [12]. Setting equation (14) equal to zero and solving the resulting equation, the expression for the pull-in voltage is obtained

\[ V_{pi} = \frac{1.549a^{3/2}D^{1/2}}{a^2 \sqrt{\varepsilon_0}} \left[ 107a^4 + 82a^2b^2 + 107b^4 + 4\sqrt{2(335a^6 + 518a^4b^2 + 870a^2b^4 + 518a^2b^6 + 335b^8)} \right]^2. \]  
\tag{19}

For the pull-in voltage of a clamped square microplate (that is, \( a = b \)), this expression can be simplified as

\[ V_{pi(sq)} = \frac{4.618a^{3/2}D^{1/2}}{a^2 \sqrt{\varepsilon_0}}. \]  
\tag{20}

This expression provides a simple method to predict the pull-in voltage of the clamped square microplate under electrostatic force.

2.5. Capacitance calculation

The capacitance of an electrostatically actuated configuration changes with the deformation of the upper flexible microplate. It can be evaluated by integrating over the area of the deformed microplate. The expression for the capacitance is given by [10]

\[ C = \int_{A} \frac{\varepsilon_0 \, dx \, dy}{(d - w(x, y))}. \]  
\tag{21}

where, \( A \) denotes the area of the integration. As equations (16) and (17) are cosine-like functions, the integration in equation (21) cannot be solved analytically [25]. Therefore, Gauss–Legendre numerical integration is used to evaluate equation (21). When the deflection functions (16) or (17) are determined, the capacitance between the fixed bottom microplate and the deformed upper microplate can be calculated.

3. Results and discussions

In order to validate the theoretical analyses obtained above, their results were compared with those from simulations and previous literature. The simulations were carried out using a commercially available FEM package (ANSYS12.0, ANSYS Inc., Canonsburg, PA). A 3D electromechanical coupling model was constructed for the electrostatically actuated configuration shown in figure 1. The top microplate was constructed using 3D structural elements (SOLID185) and its rim was constrained. The vacuum and the electrostatic effect between the upper flexible and bottom microplates were modeled using electromechanical coupling elements (TRANS126) which were generated using EMTGEN command. The electromechanical coupling elements can apply electrostatic attraction forces to the nodes to which they are attached. The top nodes of these elements were attached to the top microplate, and their bottom nodes were simply constrained representing the
Table 1. The material and geometry parameters for simulations (μMKSV units).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length 2a</td>
<td>$40 \mu m$</td>
</tr>
<tr>
<td>Width 2b</td>
<td>$20 \mu m$</td>
</tr>
<tr>
<td>Thickness h</td>
<td>$1 \mu m$</td>
</tr>
<tr>
<td>Poisson’s ratio μ</td>
<td>0.28</td>
</tr>
<tr>
<td>Separation distance d</td>
<td>$0.2 \mu m$</td>
</tr>
<tr>
<td>Young’s modulus E</td>
<td>$1.30 \times 10^5$ MPa</td>
</tr>
<tr>
<td>Dielectric constant ε₀</td>
<td>$8.854 \times 10^{-6}$ pF μm⁻¹</td>
</tr>
</tbody>
</table>

bottom electrode. Based on this model, the ANSYS Multi-Field Solver was employed to determine the static deflection of the flexible microplate under different voltages. Detailed material and geometry parameters used for simulations are shown in Table 1.

Figure 3 shows a comparison of the centre displacements of a clamped rectangular microplate under different voltages. The dotted line in figure 3 corresponds to the pull-in voltage of the microplate. Figure 3(a) shows the centre displacements of the microplate under a bias voltage range from 20V to 101V which nearly approaches the pull-in voltage, 102.2V. It can be seen that the calculated values agree with the simulated results over a large deflection range, from the undeformed state to that approaching the pull-in position. The relative errors of the calculated values with those simulated ones are shown in figure 3(b). The relative errors are less than 5% within the large deflection range. It is demonstrated that expression (16) has a high accuracy in predicting the centre deflection of a rectangular microplate.

Figure 4 shows the displacements of the rectangular microplate along axis directions under a bias voltage of 90V. Owing to the symmetry of the deformation of the microplate under a bias voltage, figure 4 only shows the deflection of the microplate along positive axis directions. Figure 4(a) shows a comparison of the displacements along x-axis direction calculated using equation (16) with those obtained by FEM simulations. Figure 4(b) shows a comparison between the displacements along y-axis direction calculated using equation (16) and those obtained by FEM simulations. As can be seen from the figures 4(a) and (b), both theoretical results are consistent with the simulated values. It is also shown that the calculated displacements along y-axis direction have a better agreement with the simulated values than those along x-axis direction. This may result from different symmetries of the rectangular plate about x-axis and y-axis, and the nonlinearity of the electrostatic force. Owing to the different lengths of the rectangular microplate in x-axis and y-axis directions, the symmetries of the electrostatic force distribution and the deformation about y-axis are not consistent with those about x-axis. So the difference between simulated and calculated deflections along y-axis is not consistent with that along x-axis. (This point can be derived from the figures shown in appendix B. When the length to width ratio approaches one, the symmetries along x-axis and y-axis approach consistent, and the differences along x-axis and y-axis approach similar.) The reason why the difference along y-axis is smaller than that along x-axis depends on the used trial function itself. The shape function used in equation (16) shows better performance to simulate the deformation of the rectangular microplate along its width direction. However, equation (16) is enough for predicting the deformation of the rectangular microplate under electrostatic force. For better simulating the deformation along x-axis, a mostly used method [23, 27] is to make a partial modification to the deflection function (16). For instance, a modified function for equation (16) was given in appendix C, which better approaches the deformation of the rectangular microplate along x-axis.

Figure 5 shows centre displacements of a square microplate under a voltage range from 20V to 66V. The dotted line along vertical direction in figure 5 denotes the pull-in voltage of the microplate as being, 67.2V. The edge length of the square microplate is 30μm, that is, $2a = 2b = 30 \mu m$. The other geometry and material parameters for the microplate are the same as those in table 1. Figure 5(a) shows a comparison of the calculated displacements using equation (17) with those obtained by FEM simulations. There is an agreement between the two. As shown in figure 5(b), the difference of the theoretical results with respect to those from FEM simulations is
less than 4.8% even if the deflection approaches the pull-in position of the microplate. These results demonstrate that the theoretical model (or equation (17)) has robustness, being able to predict the deflections up to the pull-in position.

A comparison of the displacements of the square microplate along axis directions calculated using equation (17) with those obtained by FEM simulations is shown in figure 6. Figures 6(a) and (b) show the displacements of the square microplate along x-axis and y-axis directions, respectively. Both theoretical results in figures 6(a) and (b) agree with those simulated values. These indicate that equation (17) can well predict the deformation of a clamped square microplate.

Furthermore, we also validated the theoretical model using the experimental data in other literature. Figure 7 shows a comparison of the centre displacements of a square microplate calculated using (17) with the experimental results given by Francais and Dufour [28]. It can be observed that there is an agreement between them. Figure 8 shows that the theoretical results using our reduced-order model are consistent with those calculated values using step by step linearized method given in [29]. These comparisons further demonstrate the robustness of our theoretical model in predicting the deflection of the microplate even if the deflection reaches up to the pull-in position. The geometry and material parameters for figures 7 and 8 are shown in table 2.

Figure 9 shows the results of another case study of a larger microplate with edge length of 500μm, that is, 2a = 2b = 500μm. The other geometry and material parameters are the same as given in table 2. The deflections were obtained under the voltages changing from 4V to 15.2V which very approaches the pull-in voltage, 15.7V. As shown in figure 9, the calculated deflections show good agreement with the results by nonlinear FEM simulations over the large deflection range, from the undeformed state to that approaching the pull-in position. As the cases of smaller microplates shown in figures 3 and 5, the maximum difference of the theoretical results with respect to those by the FEM simulations can also be controlled within the range of 5%. These results further demonstrated the availability of our theoretical model in predicting the static deflection of the electrostatically actuated microplates.

Having demonstrating the availability of the theoretical model for predicting the deflection of the microplate under
different voltages, we sequentially validated its ability to predict the pull-in voltage. Table 3 shows a comparison of the calculated pull-in voltage of a square microplate with that from FEM simulations, for which the parameters are the same as that used for figure 5. The difference between the calculated and simulated results is as low as 0.45%. Table 4 shows a comparison of the pull-in voltage calculated using equation (20) with the result given in [29]. The difference of our theoretical result with respect to that from Talebian’s
work is as low as 0.49%. Figure 10 shows a comparison of the pull-in voltages calculated using equation (19) with those obtained by FEM simulations under different ratios of electrode separation distance \(d\) to plate thickness \(h\). The parameters in table 1 were used for the comparison, in which only the separation distance \(d\) was changed to form the different ratios. The calculated results are consistent with those from the FEM simulations. The relative errors between them are less than 1.0% for different ratios of \(d\) to \(h\). These results demonstrate that the theoretical analyses can predict the pull-in voltages of clamped rectangular and square microplates with a very high accuracy.

Figure 11 shows capacitance of a rectangular microplate with parameters given in table 1 as a function of bias voltage. The theoretical value of the capacitance was calculated combining equation (21) and eight nodes Gauss–Legendre numerical integration. As can be seen from figure 11, the capacitance changes slow below about 80 V, and changes fast above that, especially when the bias voltage approaches the pull-in voltage, 102.2 V. This trend of the capacitance variation is consistent with that given in [12]. The capacitance of square microplates can also be calculated using equation (21) and Gauss–Legendre numerical integration. The accuracy of the quasi-analytical capacitance depends on both the deflection functions (equations (16) and (17)) and accuracy of the numerical integration.

The aforementioned results are based on the lowest-mode of the trial functions (that is, \(N = 0\)). To further validate performance of the established reduced-order model in predicting the behavior of a microplate under electrostatic force, higher modes were also studied. Figure 12 shows comparison of the deflection curves of the center point of a microplate for various numbers of modes \(N\). The parameters used for the microplate are given in table 2. As shown in the figure 12, when \(N = 1, 2\) and \(3\), the corresponding three deflection curves overlap each other. The deflection curves converge fast when the mode number \(N\) is more than zero. The modes with \(N = 1\) can be sufficient to model the static deflection of the microplate over its stable physical range. However, the deflection curve for \(N = 0\) very approaches that for \(N = 1\). The relative error of the pull-in voltages between \(N = 0\) and \(N = 1\) is as low as 1.15%. So it is accurate enough to use the lowest-mode based theoretical expression to predict the mechanical behavior of the electrostatically actuated microplate. They are more efficient and convenient for the design and optimization for the electrostatically actuated microdevices than those higher-mode functions based on numerical solutions.

As the stretching effects of the middle plane of the microplate, which is discussed in detail in [30, 31], is not considered in our theoretical model, the analytical solutions mentioned above are applicable to the cases where stretching effects are negligible (or the deflection of the plate is smaller than its thickness), and where the electrode distance \(d\) is equal to or smaller than the microplate thickness \(h\) (these cases have been widely used in the electrostatically actuated configurations [7, 12, 15, 17, 18]. Based on these assumptions, equations (16) and (17) can accurately predict the static deflection of rectangular and square microplates under electrostatic force even if the deflection is very close to the pull-in position. Equations (19) and (20) can accurately determine pull-in voltages of electrostatically actuated rectangular and square microplates, respectively. Equation (21) can be used to analyze capacitance variation and electromechanical coupling coefficient of electrostatic microdevices such as CMUTs.
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[7]. The theoretical model and these expressions are capable of widespread application to MEMS as a growing number of electrostatic microdevices require a high sensitivity, low energy consumption and low pull-in voltage, for which the electrostatic microdevices are often required to have a small electrode separation distance. Additionally, the method proposed in this study can be further expanded to analyze other mechanical behavior of the electrostatic microdevices such as dynamic pull-in voltages, resonant frequencies and large deflections if the strain of the middle plane of the microplate is taken into account.

4. Conclusions

In this study, we proposed a new method to approximate the electrostatic force, and then established a new reduced-order model for electrostatically actuated rectangular and square microplates using Galerkin method. Using this theoretical model and cosine-like deflection functions, we derived explicit expressions for the static deflection and pull-in voltage of the clamped rectangular and square microplates. The results from these theoretical expressions were validated using those from FEM simulations and literature. Excellent agreements were observed between them. As the thin plate theory is used in the model, these expressions are more applicable to the electrostatic configurations where the ratio of thickness to edge length of the microplate is smaller than 1/20; and where the deflection is very small compared with its thickness; and where the electrode distance is equal to or smaller than the microplate thickness. These theoretical analyses and expressions in our study are helpful for the design, optimization and operation of electrostatically actuated microdevices, such as CMUTs, capacitive pressure sensors, micro pumps and switches. The new method described in this paper could be expanded to analyze other mechanical behaviors of electrostatically actuated microdevices, as well as the mechanical behaviors of other electrostatically actuated structures such as beams.

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Appendix. A

The method to approximate the electrostatic force presented in [12], [13] and [17] was given as:

\[
F_e(w) = \frac{\varepsilon_0 V^2}{2d^2} \left( 1 + 2 \frac{w}{d} + 3 \frac{w^2}{d^2} + 4 \frac{w^3}{d^3} + \ldots \right). \tag{A.1}
\]
Appendix. B

For a fully clamped rectangular microplate under electrostatic force, the different lengths of the microplate along its length and width directions may lead to different symmetries of electrostatic force distribution and deformation about x-axis and y-axis. That is, the electrostatic force distribution and deformation shape along y-axis are not consistent with those along x-axis. This inconsistency decreases when the ratio of the length of the plate to its width approaches one. When the ratio equals one, the symmetries of electrostatic force distribution and deformation shape along x-axis and y-axis become almost consistent. In order to validate this point, we studied the deflections of rectangular microplates along x-axis and y-axis with different length to width ratios a/b.

Figures (B1)–(B3) show the comparison of displacements of the microplates with a/b = 2, 1.5 and 1, respectively. The parameters used for the microplates, except the length 2a, are as same as those in table 1. The deflection results were obtained using equation (16) and FEM simulations under bias voltages equal to 87% of the pull-in voltages. As shown in figures (B1)–(B3), the deflection differences between the calculated and simulated values along x-axis become more and more consistent with those along y-axis when the length to width ratio a/b changes from 2 to 1. When the length of the rectangular plate equals to its width, the electrostatic force distribution and deformation shape along x-axis and y-axis become consistent. So the deflection difference between the calculated and simulated results along x-axis and y-axis become similar.

Appendix. C

Given the different electrostatic force distribution and thus different deformation shapes of the rectangular plate along x-axis and y-axis, a modified function for the term \( \cos^2(\pi x/(2a)) \) of equation (16) can be used to better simulate the deformation along x-axis. The other terms of equation (16) keep unchanged for its advantages in pull-in voltage and center point deflection prediction. As shown in the figure 4, the calculated deflections are smaller than those obtained by FEM simulations. So we proposed a new type of term \( \cos(\pi x/(2a)) \) to substitute the term \( \cos^2(\pi x/(2a)) \) of equation (16). Then, a modified function for equation (16) can be given as
As shown in figure (C1), when using the modified function (C1), the agreement between calculated and simulated results along x-axis becomes better than that shown in figure 4. The deflection differences between calculated and simulated results along y-axis still keep same with those in figure 4. So, the modified function (C1) can better capture the deformations of the rectangular microplate in both of x-axis and y-axis directions.

Figure C1. Comparison of displacements of the rectangular microplate along axis directions using the modified function; (a) displacements along x-axis; (b) displacements along y-axis.

\[
w(x, y) = \frac{(3a^4 + 2a^2b^2 + 3b^4) d^2 - 9a^4b^4Q_1 - M}{(5a^4 + 4a^2b^2 + 5b^4) d^4} \cos\left(\frac{\pi x}{2a}\right) \cos^2\left(\frac{\pi y}{2b}\right).
\] (C.1)

Reference