An Improved Method for the Mechanical Behavior Analysis of Electrostatically Actuated Microplates Under Uniform Hydrostatic Pressure

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Abstract—Microplates are essential components of the most electrostatically actuated microdevices. Their mechanical behavior is influenced not only by electrostatic force, but by hydrostatic pressure of the environment. This paper presents an improved reduced-order model for an electrostatically actuated microplate under uniform hydrostatic pressure with a novel method for treating the electrostatic force. The model was developed using the Galerkin method and turns the partial-differential equation governing the plate into an ordinary equation system. Using an axisymmetric deflection function and the first-order Taylor series expansion of the electrostatic force, explicit expressions for the deflection and pull-in voltage of the microplate under the electrostatic force alone, and under both electrostatic force and hydrostatic pressure, were derived. The expressions with only the electrostatic force considered can predict the pull-in voltage with a higher accuracy and the deflection within a large range (from the undeformed state to the pull-in position) compared with literature. The expressions for both types of loadings show a better prediction accuracy when the pressure changes in the lower pressure range. The derived expressions are applicable to the electrostatically actuated configurations where the ratio of the plate diameter to its thickness varies from 100 to 40, and the electrode distance is smaller than or equal to the thickness. These theoretical analyses were validated with finite element method simulations and previous literature.

Index Terms—Capacitance variation, electrostatically actuated microplate, electrostatic force, pull-in voltage, static deflection, uniform hydrostatic pressure.

I. INTRODUCTION

MICROPLATES are essential components of electrostatically actuated devices such as capacitive pressure sensors, MEMS switches, micro pumps and capacitive micromachined ultrasonic transducers (CMUTs) [1]–[6]. Applications of these microdevices included pressure sensing, automatic control, biotechnology research, ultrasonic imaging and chemical detection [2]–[6]. In order to facilitate the design and optimization of these microplate-based electrostatic devices, numerous investigations have been directed towards understanding, predicting and controlling their mechanical responses to an electrostatic force [7]–[9]. Furthermore, in most cases, the effect of other additional loads such as hydrostatic pressure, residual stress, mass and temperature changes on the mechanical behavior are inevitable and are often investigated together with the electrostatic force [1], [3], [6], [9].

For instance, Goksen et al. [7] studied the capacitance change and then electromechanical coupling coefficient of a CMUT subjected to a bias voltage. Zhu and Espinosa [3] reported the effect of temperature on the reliability of RF MEMS switches.

Generally, two parallel microplates are simultaneously employed to provide an electrostatic actuation configuration for the microdevice. One of them is a flexible conductor and capable of motion or deformation, while the other one is a stationary rigid conductor and remains fixed under the action of electrostatic force. The gap between the two microplates is usually air or a vacuum. When a DC voltage is applied across the two microplates, an electrostatic force is produced and forces the flexible microplate to deflect toward the fixed one. Accurate determination of this deflection is crucial to analyzing the performance of the electrostatic device. For example, for CMUTs, this deflection determines the capacitance change and thus the electromechanical coupling coefficient which characterizes the transducer bandwidth and sensitivity [7], [10]. For capacitive pressure sensors, the deflection of the flexible microplate as a result of both the electrostatic force and the pressure determines the capacitance variation and thus the sensor sensitivity and nonlinearity [2], [11].

If the applied voltage increases up to a critical value beyond which the mechanical restoring force of the microplate can no longer balance the electrostatic force, the flexible microplate will collapse and make contact with the rigid microplate. This phenomenon is called “pull-in”, and the corresponding critical voltage is called the “pull-in voltage”. This pull-in voltage is extremely important for the operation of the microdevice. For example, the maximum DC bias voltage applied to the CMUT is dependent on the pull-in voltage [12]. The capacitive pressure sensors rely on the pull-in voltage to determine their working modes (normal or touch mode) [2], [11]. MEMS switches need to tune the bias voltage across
the pull-in voltage back and forth to alternate switch-on and off. Overall, the deflection and pull-in voltage of the flexible microplate are two important parameters for the operation of these electrostatic microdevices and need investigating thoroughly.

Analyses of the deflection and pull-in voltage are challenging because the classical structure dynamics methodology is not easily applicable to the nonlinear electrostatic force encountered in electrostatically actuated microdevices. Some researchers have used numerical simulation softwares such as ANSYS, COMSOL to simulate the mechanical behavior of the electrostatic microdevices [3], [4]. Though this numerical method can provide accurate solutions, it suffers from the inherent flaws in computational modelling [13]. In order to understand the significant characteristics of the mechanical behavior of these electrostatic microdevices, researchers modeled their behavior theoretically, and then used the finite element method to validate their theoretical analyses. Initially, a simple 1-D lumped mass-spring model was utilized to approximate the behavior of electrostatically actuated microplates [7], [8], [14]. However, this approximation considered the deformed plate as a rigid plate with piston-like motion. It is not applicable for the static deflection analysis. Furthermore, the famous prediction of pull-in position of the microplates as being, one-third of the air gap, was shown to underestimate the actual position [9]. Therefore, many researchers started to develop more sophisticated continuous models to analyze the response of the microplates. Vogl and Nayfeh [15] presented a continuous reduced-order model for an electrically actuated clamped circular microplate and studied its static equilibrium state under a general electric potential. Zhao et al. [16] also developed a reduced-order model for rectangular microplates applicable to any classical boundary conditions. They further reported the mechanical behavior of a fully clamped rectangular microplate under electrostatic actuation using this model.

Though these theoretical models can analyze the behavior of the microplates under electrostatic force, they cannot be used to establish simple and explicit expressions for the relationship between the static deflection and the applied voltage, as well as the relationship between the pull-in voltage and the structure parameters of the microplates. Therefore, Chao et al. [9] and Liao et al. [17] presented explicit expressions for the pull-in voltages for clamped rectangular and circular microplates with a uniform biaxial residual stress, respectively. However, their work did not develop an expression for the static deflection. Ahmad and Pratap [12] established a relationship for the static deflection of the microplate and the applied voltage using the linear term of the Taylor series expansions of the electrostatic force. However, the linearized term of the electrostatic force shows a poor fit to the actual value especially when the deflection increases [17], therefore, their relationship provides poor predictions as the deflection increases. So, it is necessary to establish an expression which can completely capture the deformation characteristics of the flexible microplate within its whole deflection range (from the undeformed state to the pull-in position).

Furthermore, electrostatically actuated microplates are often used in micro pressure sensors and micro pumps or in a hydrostatic pressure environment [1], [2], [4], [18], in which the hydrostatic pressure also has a significant effect on the mechanical behavior of the microplate. So, both effects of electrostatic force and hydrostatic pressure on the static deflection and pull-in voltage have to be taken into account. As a result, Nabian et al. [18] used a step-by-step linearization method to investigate the effect of uniform hydrostatic pressure on the static deflection and pull-in voltage of an electrostatically actuated microplate. However, they only obtained numerical solutions. Ahmad and Pratap [12] proposed an expression for the deflection of the microplate under both electrostatic force and hydrostatic pressure, which suffered from poor predictive capability when the deflection increases due to their use of the linearized electrostatic force. Therefore, it is necessary to establish an improved expression for the deflection of the microplate under electrostatic force and uniform hydrostatic pressure. Also, an expression for the pull-in voltage of the electrostatically actuated microplate under hydrostatic pressure needs establishing for electrostatic microdevices.

In this study, we presented a novel method to treat the electrostatic force and then generated a new type of continuous reduced-order model for the deflection governing equation of the microplate under both electrostatic force and uniform hydrostatic pressure. Based on the new model, we established explicit expressions for the static deflections of the microplate under only the electrostatic force and under both types of loadings. Subsequently, the pull-in voltage of the microplate under the two types of loadings was determined using static deflection analysis. Then the capacitance variation of the parallel-plate configuration induced by the two types of loadings was calculated using the expressions for the static deflections. Finally, we compared the results from these theoretical analyses with those from literature and the finite element method (FEM) simulations. It is demonstrated that our analytical expressions can better predict the mechanical behavior of the microplate subjected to electrostatic force and uniform hydrostatic pressure than those in previous work.

II. THEORETICAL ANALYSIS

A. Problem Formulation

Fig. 1 shows the schematic of an electrostatically actuated configuration for theoretical analysis. The upper flexible microplate has a radius $R$ and thickness $h$, and is clamped at its edge. It is assumed to be thin, homogenous and isotropic. The lower bottom microplate is parallel fixed below the upper one with an effective electrode distance $d_0$. The space between the upper and lower microplates is assumed as air or...
a vacuum with a dielectric permittivity $\varepsilon_0$. When subjected to an electrostatic force or a uniform hydrostatic pressure, the upper microplate is deflected towards the fixed bottom microplate. Kirchhoff plate theory is used for the deflection analysis, which can provide accurate results when the ratio of the thickness of the microplate to its diameter is very small ($< 1/20$) [18]. It should be noted that the plate referred to in the following sections is the upper flexible microplate unless stated otherwise.

For a circular plate with symmetrically distributed loads, the transverse deflection $w$ is symmetrical. The general equation that governs the symmetrical bending is given elsewhere [19]. For the clamped circular plate subjected to both axisymmetrical electrostatic force and hydrostatic pressure, the deflection governing equation can be expressed as

$$D \nabla^4 w = F_e(r) + P,$$  \hspace{1cm} (1)

where

$$\nabla^4 w = \frac{d^4 w}{dr^4} + \frac{2 d^3 w}{r^2 dr^2} - \frac{1}{r^2} \frac{d^2 w}{dr^2} + \frac{1}{r^2} \frac{dw}{dr},$$

$0 \leq r \leq R$, $w(r)$ is the transverse deflection; $D$ is the flexural rigidity; $F_e(r)$ is the electrostatic force per unit area; $P$ is the applied pressure from sources such as air or water. The flexural rigidity $D$ is defined as

$$D = Eh^3/[12(1 - v^2)],$$  \hspace{1cm} (2)

where $E$ is the Young’s modulus and, $v$ is the Poisson’s ratio. The electrostatic force per unit area $F_e(r)$ in (1) is given by

$$F_e(r) = \frac{\varepsilon_0 V^2}{2|d_0 - w(r)|^2},$$  \hspace{1cm} (3)

where $V$ is the applied DC voltage across the upper and lower plates [18].

The boundary conditions for the clamped circular plate with axisymmetrical deformation are [19]

$$w(R) = 0, \quad dw(R)/dr = 0, \quad dw(0)/dr = 0.$$  \hspace{1cm} (4)

B. Reduced-Order Model

Due to the nonlinear electrostatic force in the deflection governing equation, it is hard to establish an accurate analytical solution. So, many researchers searched for an approximation one [9], [12], [13], [15]–[17]. A common idea in their work was to use the well-known Gaderkin method to decompose the deflection governing equation into a coupled set of ordinary equations (the reduced-order model), through which the approximate solution can be obtained.

To this end, an initial assumption was made about the function used to approximate the transverse deflection $w$. For a circular plate with axisymmetrical deflection, it can be expressed as

$$w(r) = \sum_{m=1}^{N} \eta_m \varphi_m(r),$$  \hspace{1cm} (5)

where $\eta_m$ are the unknown coefficients to be determined, $\varphi_m(r)$ are the linearly independent coordinate functions (they are also called trial functions) that satisfy all the prescribed boundary conditions [20], and $N$ is the number of the chosen trial functions.

Subsequently, there are two methods to generate a reduced-order model for (1) according to the idea mentioned above. In the first method [9], [12], [13], [17], the electrostatic force is expanded using a Taylor series around $w = 0$. Thus, (3) can be written as

$$F_e(r) = \frac{\varepsilon_0 V^2}{2d_0^2} \left(1 + \frac{2w}{d_0} + \frac{w^2}{d_0^2} + \frac{4w^3}{d_0^3} + \frac{5w^4}{d_0^4} + \cdots \right).$$  \hspace{1cm} (6)

Substituting (5) and (6) into (1), multiplying both sides of the resulting equation by $\varphi_n(r)$, and integrating the outcome from $r = 0$ to $R$, a series of ordinary equations can be obtained

$$\int_0^R D \nabla^4 w \times \varphi_n dr = \frac{\varepsilon_0 V^2}{2d_0^2} \left(\int_0^R \varphi_n dr + \sum_{m=1}^N \eta_m \int_0^R \varphi_n \varphi_m \varphi_1 dr \right. \hspace{1cm} (7)

where

$$\left. + \frac{3}{d_0^3} \sum_{m, i = 1}^N \eta_m \eta_i \int_0^R \varphi_n \varphi_m \varphi_i \varphi_j dr \right)$$

$$+ \frac{4}{d_0^3} \sum_{m, i, j = 1}^N \eta_m \eta_i \eta_j \int_0^R \varphi_n \varphi_m \varphi_i \varphi_j \varphi_k dr + \cdots \right)$$

and integrating the outcome from $r = 0$ to $R$, another series of ordinary equations can be obtained as

$$d_0^2 \int_0^R D \nabla^4 w \times \varphi_n dr = 2d_0 \sum_{m=1}^N \eta_m \int_0^R D \nabla^4 w \times \varphi_n \varphi_m dr$$

$$+ \sum_{m, i = 1}^N \eta_m \eta_i \int_0^R D \nabla^4 w \times \varphi_n \varphi_m \varphi_i dr$$

$$= \frac{\varepsilon_0 V^2}{2} \int_0^R \varphi_n dr + Pd_0^2 \int_0^R \varphi_n dr$$

$$- 2Pd_0 \sum_{m=1}^N \eta_m \int_0^R \varphi_n \varphi_m dr$$

$$+ P \sum_{m, i = 1}^N \eta_m \eta_i \int_0^R \varphi_n \varphi_m \varphi_i dr$$

for $n = 1, 2, 3, \ldots, N.$  \hspace{1cm} (8)

The integrals in (7) and (8) can be evaluated once the trial functions $\varphi_n(r)$ in (5) and the number $N$ are chosen.
Then, once all $\eta_m$ are determined by solving (7) or (8), an approximate solution to (1) can be established by (5).

However, both methods mentioned above have inherent flaws for establishing explicit expressions which can be easily used to design and operate microdevices based on electrostatic actuation. For the first method, the complexity of (7) and the accuracy of its solution depend on the order of the electrostatic force expansion. The first and second Taylor series expansions have relatively simple forms; however, they yield large errors with respect to the electrostatic force term shown in (3) [17]. Though the higher order expansions can reduce the error, the degree of the term $\eta_m$ in (7) will be three or higher, which will greatly increase the complexity of establishing an explicit solution. For the second method, each of (8) is a standard cubic equation of $\eta_m$ which makes it complicated to establish an explicit and simple analytical solution to the equation system, especially when the number $N$ of trial functions increases. So, it is necessary to establish an improved reduced-order model to reduce the degree of $\eta_m$ and simultaneously obtain a solution with sufficient accuracy for engineering applications.

To this end, we propose a new method to treat the electrostatic force term. In this method, we expand one of two $(d_0-w)^{-1}$ terms as a Taylor series around $w = 0$. Thus, (3) can be expressed as

$$F_e(r) = \frac{\varepsilon_0 V^2}{2d_0(d_0-w)} (1 + \frac{w}{d_0} + \frac{w^2}{d_0^2} + \frac{w^3}{d_0^3} + \frac{w^4}{d_0^4} + \cdots). \quad (9)$$

Substituting (9) into (1), multiplying both sides of the resulting equation by $(d_0-w)$, substituting (5) into the result, then multiplying both sides of the resulting equation by $\varphi_n(r)$ and finally integrating the outcome from $r = 0$ to $R$, a new reduced-order equation system for (1) is obtained

$$d_0 \int_0^R D \nabla^4 w \times \varphi_n dr - \sum_{m=1}^N \eta_m \int_0^R D \nabla^4 w \times \varphi_n \varphi_m dr$$
$$= \frac{\varepsilon_0 V^2}{2d_0} \left( \int_0^R \varphi_n dr + \frac{1}{d_0} \sum_{m=1}^N \eta_m \int_0^R \varphi_n \varphi_m dr \right)$$
$$+ \frac{1}{d_0} \sum_{m,i=1}^N \eta_m \eta_i \int_0^R \varphi_n \varphi_m \varphi_i \varphi_j dr$$
$$+ \frac{1}{d_0} \sum_{m,i,j,k=1}^N \eta_m \eta_i \eta_j \eta_k \int_0^R \varphi_n \varphi_m \varphi_i \varphi_j \varphi_k dr + \cdots$$
$$+ P d_0 \int_0^R \varphi_n dr - P \sum_{m=1}^N \eta_m \int_0^R \varphi_n \varphi_m dr$$

for $m = 1, 2, 3, \cdots N$. \quad (10)

Similarly, (10) can be solved once the trial functions $\varphi_m(r)$ and the number $N$ in (5) are determined. An approximate solution can be established by equation (5) after all $\eta_m$ are calculated.

Equation (10) shows following advantages in establishing an approximate solution, especially an explicit expression. Firstly, for the same order Taylor series expansion of the electrostatic force term, the new method shows a higher accuracy than the first method. As can be seen in Fig. 2, a simple comparison of electrostatic forces was performed. The first, second and third order electrostatic force expansions in (9) are much closer to the electrostatic force calculated by (3) than the same order expansions in (6). Fig. 2 also shows that the first order electrostatic force expansion using the new method exhibits a better approximation than the second order electrostatic force expansion using the first method. That is because (9) has a similar term $(d_0-w)^{-1}$ to (3) and thus can better capture the variation characteristic of (3) than (6). This advantage suggests that a lower-order Taylor series expansion in (10) can reach the accuracy of a higher-order one in (7). Equation (10) can reduce the degree of term $\eta_m$ without any reduction in the accuracy, which can greatly simplify the calculation process. Secondly, according to the order of the electrostatic force expansion used in (10), the highest degree of $\eta_m$ in (10) can be set as low as two, which is less than the degree of $\eta_m$ in (8) (three). This advantage leads to an explicit and simple solution. In addition, although the second method is more accurate, it results in a much more complicated calculation than the first method [13]. So, the new method acts as a compromise between accuracy and complexity for the models given by the first and second methods.

C. Static Deflection Analysis

For a clamped circular plate with axisymmetric transverse deflection, the trial functions can be chosen as [12], [19]

$$\varphi_m = (R^2 - r^2)^{m+1}, \quad \text{for } m = 1, 2, 3 \cdots N. \quad (11)$$

These trial functions satisfy all the boundary conditions shown in (4). As this study aimed to establish an explicit expression to predict the deflection and pull-in voltage of the electrostatically actuated microplate, the number $N = 1$ is used. Substituting (11) into (5), it can be rewritten as

$$w(r) = \eta_1 \varphi_1 = \eta_1(R^2 - r^2). \quad (12)$$
This function is often used and is sufficient to approximate the axisymmetric deflection of circular plates [19], [20]. Substituting (11) and (12) into (10) and using the first order electrostatic force expansion, (10) can be simplified as

\[
d_0 \int_0^R D \nabla^4 w \times (R^2 - r^2)^2 dr - \eta_1 \int_0^R D \nabla^4 w \times (R^2 - r^2)^4 dr = Q \left( \int_0^R (R^2 - r^2)^2 dr + \frac{\eta_1}{d_0} \int_0^R (R^2 - r^2)^4 dr \right) + d_0 P \int_0^R (R^2 - r^2)^2 dr - \eta_1 P \int_0^R (R^2 - r^2)^4 dr
\]

where \( Q = \epsilon_0 V^2 / 2d_0 \). Evaluating the integrals and then solving the resulting equation, two possible solutions to the constant \( \eta_1 \) are given by

\[
\eta_{1a,b} = \left( 84 D d_0^2 + d_0 R^4 P - Q R^4 \pm B \right) / (128 D d_0 R^4),
\]

where

\[
B = \sqrt{(84 D d_0^2 - Q R^4 - d_0 R^4 P)^2 - 336 D d_0^2 R^4 Q}.
\]

As can be derived from (12), the term \( \eta_1 \) actually determines the maximum deflection of the microplate, which increases with the electrostatic force and hydrostatic pressure [18]. So the correct value is considered to be

\[
\eta_{1b} = \left( 84 D d_0^2 + d_0 R^4 P - Q R^4 - B \right) / (128 D d_0 R^4).
\]

Thus, replacing \( \eta_1 \) in (12) with \( \eta_{1b} \), the deflection \( w_{VP} \) of the clamped circular microplate under both electrostatic force and uniform hydrostatic pressure is obtained as

\[
w_{VP}(r) = \frac{84 D d_0^2 + d_0 R^4 P - Q R^4 - B}{128 D d_0 R^4} \times (R^2 - r^2)^2.
\]

When the electrostatic force equals zero, (17) can be simplified to

\[
w_P(r) = P (R^2 - r^2)^2 / (64 D).
\]

Equation (18) is the well-known expression for the deflection of the plate under uniform hydrostatic pressure only [19]. If only the effect of the electrostatic force on the deflection is considered, (17) can be simplified to

\[
w_V(r) = \frac{84 D d_0^2 - Q R^4 - B_1}{128 D d_0 R^4} \times (R^2 - r^2)^2,
\]

where, the term \( w_V \) in (19) denotes the deflection of the microplate subjected only to the electrostatic force, and \( B_1 \) is given by

\[
B_1 = \sqrt{7056 D^2 d_0^4 - 504 D d_0^2 R^4 Q + R^8 Q^2}.
\]

D. Pull-In Voltage Predication

The solution for the pull-in voltage of the clamped upper microplate can be obtained from (14). It can be shown that there is a voltage at which the value of \( \eta_{1a} \) equals that of \( \eta_{1b} \). This voltage is the critical point where the pull-in phenomenon occurs [12], [15], [21]. Thus, an explicit expression for the pull-in voltage of the electrostatically actuated plate under uniform hydrostatic pressure can be obtained using a similar method to that in [12]. Setting (15) equal to zero, four possible solutions for the pull-in voltage can be calculated (not given in this paper). As the pull-in voltage is positive and increases with the electrode distance \( d_0 \), but reduces with an increased hydrostatic pressure [12], [18], [22]. The correct expression is

\[
V = \frac{d_0}{\sqrt{\epsilon_0 R^2}} \sqrt{504 D d_0^2 + 2 R^4 P - 8 \sqrt{3528 D^2 d_0^4 + 42 D d_0 R^4 P}}.
\]

This equation provides a convenient method to predict the pull-in voltage of the electrostatically actuated plate under uniform hydrostatic pressure. If there is no hydrostatic pressure applied to the plate, the pull-in voltage is

\[
V_{pi} = \frac{5.369 d_0}{R^2} \sqrt{\frac{D d_0}{\epsilon_0}}
\]

where, \( V_{pi} \) represents the pull-in voltage of the electrostatically actuated plate without other loads or initial stress.

E. Capacitance Calculation

Determination of the capacitance change is crucial to the performance analysis of electrostatically actuated micro-devices such as CMUTs and capacitive pressure sensors [10], [11]. Based on the above static deflection analysis, the capacitance variation as a result of the plate deformation can be calculated using a surface integral over the plate area. The results is given by

\[
C = \int_0^R \int_0^{2\pi} \varepsilon_0 / (d_0 - w(r)) dr d\theta
\]

Substituting (17) into (23) and calculating the integral, the capacitance of the parallel-plate configuration subjected to electrostatic force as well as hydrostatic pressure is given by

\[
C_{VP} = \frac{\pi \varepsilon_0}{2\sqrt{d_0 \eta_{1b}}} \ln \frac{\sqrt{d_0} + \sqrt{\eta_{1b} R^2}}{\sqrt{d_0} - \sqrt{\eta_{1b} R^2}}.
\]

If only the effect of the electrostatic force on the capacitance is considered, the corresponding expression is

\[
C_V = \frac{\pi \varepsilon_0}{2\sqrt{d_0 H_1}} \ln \frac{\sqrt{d_0} + \sqrt{H_1 R^2}}{\sqrt{d_0} - \sqrt{H_1 R^2}}
\]

where

\[
H_1 = \left( 84 D d_0^2 - Q R^4 - B_1 \right) / (128 D d_0 R^4).
\]

The term \( C_V \) represents the capacitance variation induced solely by the electrostatic force. When both applied voltage and hydrostatic pressure equal zero, the capacitance of the parallel-plate configuration is the limit of equation (24). It is

\[
C_{V=0,P=0} = (\varepsilon_0 \pi R^2) / d_0.
\]

This equation is the most used capacitance approximation to an undeformed parallel-plate configuration [2].
III. VALIDATION OF THEORETICAL ANALYSIS

In order to validate the theoretical analyses obtained above, their results were compared with those from simulations and previous literature. The simulations were carried out using a commercially available FEM package (ANSYS12.0, ANSYS Inc., Canonsburg, PA). Due to the axial symmetry of the circular plates and applied loads shown in Fig. 1, a 2D-axisymmetrical model was constructed for a reduced computation time. The upper microplate was constructed using plane structural elements (PLANE182) and its rim was constrained. The vacuum and the electrostatic effect between the upper and bottom microplates were modeled using electromechanical coupling elements (TRANS126). The electromechanical coupling elements can apply electrostatic attraction forces to the nodes to which they are attached. The top nodes of these elements were attached to the upper microplate, and their bottom nodes were simply constrained representing the bottom microplate. Based on this model, the ANSYS Multi-Field Solver was employed to determine the static deflection of the upper flexible microplate under different voltages and uniform pressures. The simulated results are shown in Fig. 3–10. Detailed material and geometry parameters used for simulations are shown in Table I and II. Table II was only used for Fig. 7, and Table I was used for the others. The material parameters in Table II are as same as those in Table I.

Fig. 3 and 4 show a comparison of the deflection curves of the microplate obtained by the FEM simulation and (19). Fig. 3 shows the deformation profile of the plate under an applied voltage of 90 V with a maximum displacement of 0.11 μm, which is less than 30% of the thickness of the plate (shown in
Table I. Fig. 4 shows the deformation profile under an applied voltage of 136 V with a maximum displacement of 0.39 μm, which is larger than 30% of the thickness [23]. As can be seen from the two figures, the analytical deflection curves agree well with the simulated results for both small and relatively large deflections. The shape function given by (12) provides a good approximation to the deformation profile of the plate under the electrostatic force. It can be seen in Fig. 3 and 4 that a better deformation approximation may be achieved if the displacement of the center point (the maximum displacement) of the plate is determined accurately. Furthermore, as the shape function is sufficient to approximate the deformation profile of the clamped circular plate under uniform hydrostatic pressure [19], [20], it can be derived that (17) using the same shape function can closely approximate the shape of the deformed microplate subjected to both electrostatic force and uniform pressure, if the displacement of the center point of the microplate is predicted exactly.

Table II. Fig. 5 shows a comparison of the displacements of the center point (the maximum displacements) of the microplate under different voltages obtained by FEM simulations, (19) in our study and (14) in Ahmad’s research [12]. Fig. 5(a) shows the displacements as a function of the voltages within a range of 90 V to 138 V. It can be seen that the results from our and Ahmad’s analytical methods almost overlap with those from the FEM simulations when the applied voltage is below 120 V. When the applied voltage is above 120 V,
the displacements predicted by Ahmad’s method begin to deviate appreciably from the simulated results. However, our analytical values still show small difference from the simulated results even when the applied voltage approaches the pull-in voltage, 139 V (determined using FEM simulations). Fig. 5(b) shows the errors of the results predicted by our and Ahmad’s analytical methods compared with those obtained by the FEM simulations. The error from Ahmad’s method is smaller than 5% when the voltage is below 120 V, and increases to about 28% when the voltage rises up to 138 V. However, the error from our method is smaller than 5% within the whole voltage range (from 90 V to 138 V). In contrast, our solution (19) shows a better approximation to the deflection of the plate than (14) in [12]. Furthermore, a comparison of the maximum displacements predicted by (19) in our study and (19) in Ahmad’s research using the second order term of electrostatic force expansion was also carried out [12]. The maximum error of the results from Ahmad’s method with respect to those from the FEM simulations is 9.5% within the voltage range from 90 V to 138 V. However, as mentioned above, the maximum error from our analytical method is 4.7%. Equation (19) in our study also shows a better prediction for the deflection than the nonlinear solution (19) of Ahmad’s [12]. Therefore, our analytical method can predicted the deflection with a better accuracy within the voltage range from zero to the one approaching the pull-in voltage.

Fig. 6 presents the maximum displacements of the clamped circular microplate under different uniform hydrostatic pressures with a fixed applied voltage of 100 V. As shown in the Fig. 6, the results calculated using (17) are consistent with those obtained by FEM simulations when the applied pressure is below 50 kPa. The corresponding relative error with the simulated results is less than 5.2%. When the pressure increases, the relative error rises accordingly and reaches up to about 11% at a pressure of 80 kPa. Additionally, it can be observed that the analytical displacement is smaller than that from the FEM simulations. The reason for this phenomenon lies in the fact that, when the deflection of the plate under a fixed bias voltage increases with the hydrostatic pressure, a corresponding electrostatic force is produced which will further leads to an increase in the deflection. As a result, the hydrostatic pressure applied to the plate under the fixed voltage can induce a larger deflection than that caused by the pressure alone. Similar to the electrical spring softening in [24], the additional electrostatic force caused by the pressure can be interpreted as a pressure-induced electrical spring softening (PIESS). As (17) does not take the effect of the PIELSS into account, the displacements predicted by it are smaller than those simulated values. However, as mentioned above, (17) can provide an acceptable prediction for the deflection of the electrically actuated microplate under the pressure within a low range.

In order to further validate the availability of (17), a comparison of the center-point deflections calculated using (17) with the values reported by Nabian et al. [18] is shown in Table III. The deflections were calculated under different voltages with a constant uniform pressure of 200 kPa, for which the corresponding material and geometry parameters are shown in Table IV. As can be seen from Table III, the analytical results from (17) agree well with those reported by Nabian. This further demonstrated the validity of (17) in predicting the deflections of the microplate under both electrostatic force and uniform hydrostatic pressure.

Having demonstrated theavailabilities of (17) and (19) for static deflection analysis, we studied the performance of (21) and (22) for the pull-in voltage prediction in the following section. Equation (22) has a same form as (20) given by Ahmad [12], and with the equation derived by setting the residual stress in (27) reported by Liao [17] to zero. For a clamped circular plate with the material and geometry parameters shown in Table IV, we used our, Ahmad’s, and Liao’s equations to calculate the pull-in voltage $V_{pi}$. The results are shown in Table V along with the pull-in voltage from CoSolve-EM simulations reported by Osterberg [25]. The pull-in voltage predicted by (22) in our study deviates 0.7% from the full 3-D simulation value produced by CoSolve-EM, which is much smaller than errors from Ahmad’s and Liao’s method. Therefore, (22) in our study shows a higher accuracy in predicting the pull-in voltage than Ahmad’s and Liao’s methods.

Furthermore, a comparison of the pull-in voltages of the electrostatically actuated microplate under different uniform pressures obtained through (21) and FEM simulations is shown in Fig. 7. The FEM simulation is time-consuming because the analysis is nonlinear analysis, and needs a large number of elements for the large size structures. Therefore, the small-size parameters shown in Table II were used to simulate the
pull-in phenomenon in order to reduce the computation time. Additionally, the FEM simulation shows poor simulation performance at the pull-in point of the electrostatic plate; however it can accurately determine the static deflection when the applied voltage is below the pull-in voltage. A voltage making the maximum deflection of the plate equal to 50% of the air gap was used as the pull-in voltage in our FEM simulations, which is very close to the actual pull-in position being 53% of the air gap reported by Vogl et al. [9, 15]. Though the pull-in voltage obtained from our simulations is a little smaller than its actual value, it is sufficient to validate theoretical predictions. Results obtained from FEM simulations and (21) in Fig. 7 show the pull-in voltage decreases with the uniform hydrostatic pressure. The error of the calculated values with respect to those from the FEM simulations is below 4% within the pressure range from 0 to 200 kPa. When the pressure increases, the error rises accordingly up to 10.9% at the pressure of 400 kPa. As the simulated pull-in voltage was underestimated, the error of our prediction should be smaller than those mentioned above. The reason for the relatively large error in the pressure range from 200 kPa to 400 kPa is that the PIESS effect is not considered in (21). However, even with this relatively large error for high pressure, (21) can well determine the pull-in voltage of the electrostatic microplate under a uniform hydrostatic pressure in the low range.

Fig. 8 shows the capacitance variation of the parallel-plate configuration as a function of the applied voltage, which is achieved using (25). As shown in Fig. 8, the capacitance increases slowly with the voltage below 80 V, and rises fast when the voltage becomes larger, especially for the applied voltage close to the pull-in value, 139 V. This trend for the capacitance variation is similar to that in [12]. The accuracy of the capacitance variation depends on the static deflection analysis of the microplate under the applied voltage. An improved deflection analysis may contribute to a relatively accurate capacitance prediction.

Fig. 9 shows the capacitance of the parallel-plate configuration under different uniform hydrostatic pressures with a fixed voltage of 80 V, in which the value of the capacitance is obtained using (24). The capacitance increases almost linearly with the applied pressure in the range from 0 to 80 kPa. This linear relationship may be more obvious when the pressure changes in a lower range, such as in the pressure range from 0 to 20 kPa shown in Fig. 10. So (24) is helpful for the pressure-induced capacitance variation analysis of the electrostatic actuation structure, especially for designing and optimizing those capacitive pressure sensors.

The theoretical results mentioned above have high accuracy when compared with those from FEM simulations and literature. These demonstrate that the expressions, derived from the model based on thin plate theory and first order electrostatic force expansion, are accurate enough for the mechanical behavior analysis of the microplate under electrostatic force and uniform hydrostatic pressure. However, there are several points in our analytical model should be stated for better understanding and using it.

Firstly, the thin plate theory show poor accuracy when the deflection is larger than 30% of the thickness because it does not consider the stretch of the middle plane of the plate [19]. However, the theoretical analysis based on the thin plate theory in our study shows a small difference even for the deflection larger than 30% of the plate thickness, as shown in Fig. 5. In addition, the expression for the pull-in voltage prediction has a smaller error than those in literature, as shown in Table V. The reasons for these are as follows. The pull-in position of the electrostatically actuated configuration in our study occurs at about 54% of the electrode distance $d_0$, which very approximates to 53% of the electrode distance given in [15] and [17]. For the situations where $d_0 \leq h$, that is, the maximum deflection range in our study is expanded to 54% of the plate thickness, which is not a very large expansion with respect to the relatively accurate range (30% of the plate thickness) for the thin plate theory [19, 20]. Therefore, although the thin plate theory will overestimate the deflection in comparison with that taking the stretch of the middle plane of the plate into account (especially for the deflection larger than 30% of the plate thickness [19]), it does not generate a large deflection difference. Furthermore, the first order electrostatic force expansion is smaller than, but has a relatively accurate approximation to the actual value of the electrostatic force for $w/d_0 \leq 54\%$, as shown in Fig. 2. Therefore, the first order electrostatic force expansion proposed in our study does not generate a large difference either. As a result, the reduced-order model based on thin plate theory and first order electrostatic force expansion can contribute to analytical solutions with a high accuracy.

Secondly, although the second order electrostatic force expansion has a better approximation to (3) than the first order term, the reduced-order model using the second order term, similar to (13), does not contribute to more accurate solutions than the model (13) using the first order term. As mentioned before, the reduced-order model (13) based on the thin plate theory and the first order electrostatic force expansion can obtain relatively accurate solutions. Although the second order electrostatic force expansion is larger and more accurate than the first order term, yet the solution to the model including the second order electrostatic force term largely overestimates the deflection which has a larger difference with the results from FEM simulations and other literature than the solution to (13). Actually, as shown in Fig. 2, the first order electrostatic force expansion proposed by us even has a better approximation to the electrostatic force than the second order term in (6). Therefore, it has high approximation accuracy and can contribute to solutions with higher accuracy than those in literature, which is sufficient for the static mechanical behavior analysis of the electrostatically actuated microplates. The second order electrostatic force expansion or higher order terms given in (9) may be used for the model taking the stretch of the middle plane of the plate into account to obtain more accurate solutions.

Finally, the proposed reduced-order model is advantageous for establishing simple and explicit expressions for the mechanical behaviors of electrostatically actuated microplates. The model using the first order electrostatic force expansion can contribute to expressions with high accuracy, which are general and robust enough for different device dimensions.
Fig. 11. Comparison of deflections of the center point of the microplate with different radii $R$, fixed thickness $h$ and electrode distance $d_0$: (a) the deflections from small deformation to near pull-in positions obtained from FEM simulations and theoretical calculation; (b) the errors of the simulated results with those from FEM simulations.

Fig. 12. Comparison of deflections of the center point of the microplate with different thicknesses $h$, fixed radius $R$ and electrode distance $d_0$: (a) the deflections from small deformations to near pull-in positions obtained from FEM simulations and theoretical calculation; (b) the errors of the simulated results with those from FEM simulations.

To show the general performance of the model, a parametric study was carried out. The results are shown in Figs. 11 and 12, for which the basic parameters are as same as those in Table I. Fig. 11(a) shows the maximum deflections of the microplates with different radii $R$ (i.e., different ratios of plate radius to thickness $R/h$). As shown in the Fig. 11(b), the errors of the calculated values with respect to those from the FEM simulations are less than 5.0% within the deflection range from very small deformations to near pull-in positions for $R/h = 50, 40$ and $30$. However, the maximum error become as large as 9.9% for $R/h = 20$. Fig. 12(a) shows the maximum deflections of the microplates with different thicknesses $h$ (i.e., different ratios of plate radius to thickness $R/h$ and different ratios of thickness to electrode distance $h/d_0$). As shown in the Fig. 12(b), the errors between the calculated and simulated values are less than 5.0% within the large deflection range for $R/h = 40, 32$ and $25$. When the ratio $R/h = 20$, the maximum error is 6.0% which just has a very little increase in comparison with other cases. However, this error is much smaller than that shown in Fig. 11 for the same ratio $R/h = 20$. It is because that the thickness is larger than the electrode distance as being, $h/d_0 = 2$, in Fig. 12; yet the thickness is equal to the electrode distance as being, $h/d_0 = 1$, in Fig. 11. As the pull-in position occurs at 54% of the electrode distance $d_0$, for the same $d_0$, the larger ratio of thickness to electrode distance $h/d_0$ means the smaller ratio of the deflection at pull-in position to thickness $w_{pi}/h$, which will result in deflection analysis with a higher accuracy [19], [20]. Additionally, for the cases shown in Fig. 11, the relative errors between the calculated and simulated pull-in voltages change from 0.2% to 2.5% when the radius to thickness ratios vary from 50 to 20. However, for the cases shown in Fig. 12, the relative errors of the pull-in voltages changes from 0.7% to 1.7% when the radius to thickness ratios vary from 40 to 20 (i.e. the thickness to electrode distance ratios vary from 1 to 2). It can be found that the pull-in voltage with $h/d_0 = 2$ in Fig. 12 has a higher accuracy than that with $h/d_0 = 1$ in Fig. 11 for the same ratio $R/h = 20$. This suggests that a larger thickness to electrode distance ratio also leads to higher prediction accuracy for the pull-in voltage for the same radius to thickness ratio. However, even for the harsh case where $R/h = 20$ and $h/d_0 = 1$ shown in Fig. 11, the error of the predicted pull-in voltage is comparable to that given in [15] using a reduced-order with five modes. In conclusion, the case with $R/h = 20$ and $h/d_0 = 1$ may be a limit of device dimensions which the theoretical analysis is applicable to. The theoretical analysis has high accuracy when the ratio of radius to thickness varies from 50 to 20 (i.e. the diameter to thickness ratio varies from 100 to 40), which covers a very large part of the dimension range of thin plates [20].

In all, as the stretch of the middle plane of the flexible microplate is not considered in our theoretical model, the
derived analytical expressions are applicable to the situations in which the ratio of the diameter of the microplate to its thickness varies from 100 to 40, and the deflection of the microplate is smaller than its thickness, and in which the electrode distance is smaller than or equal to the thickness (these cases have been widely used in the electrostatically actuated configurations [7], [12], [15], [17], [18]). Based on these assumptions, (19) can predict the static deflection of the microplate under only an electrostatic force even if the deflection is close to the pull-in position. Equation (22) can provide a relatively accurate pull-in voltage prediction in comparison with the analytical methods in literature. Equations (17) and (21) are more suitable to analyzing static deflection and pull-in voltage of electrostatically actuated microplates subjected to hydrostatic pressure within a low range. Equation (25) can be used to analyze the capacitance variation and electromechanical coefficient of electrostatic microdevices such as CMUTs [7]. Equation (24) suggests that the capacitance of the parallel-plate configuration varies linearly with the pressure within a low range. These theoretical analyses are capable of widespread application to microelectromechanical system (MEMS) as a growing number of electrostatic microdevices require a high sensitivity, low energy consumption and low pull-in voltage, for which the electrostatic microdevices are often required to have a thin plate thickness and a small electrode separation distance. Additionally, the methods proposed in our study can be further expanded to analyze other mechanical behavior of the electrostatic microdevices such as dynamic pull-in voltages, resonant frequencies and large deflections if the strain of the middle plane of the microplate is taken into account.

IV. CONCLUSION

In this study, we proposed an improved reduced-order model for an electrostatically actuated microplate using a novel method to treat the electrostatic force. By using the Galerkin method and trial functions satisfying all the boundary conditions of the clamped microplate, we turned the distributed-parameter equation governing deflection of the plate into an ordinary equation system. Once the detailed form and number of the trial functions are given, the equation system can be solved and can produce a function $w(r)$ to approximate the static deflection and pull-in voltage of the microplate.

Based on this reduced-order model, we derived several explicit expressions for the static deflection and pull-in voltage of the microplate using the first order electrostatic force expansion and the most used trial function for the clamped circular plate. The expressions for static mechanical behavior of the plate under only electrostatic force can reliably determine the pull-in voltage and the deflection in a large deflection range, even if the deflection is close to the pull-in position. The expressions used to predict the deflection and pull-in voltage of the electrostatically actuated microplate under uniform hydrostatic pressure show high prediction accuracy when the applied pressure changes in a low range. Furthermore, the expressions for the capacitance variation of a parallel-plate configuration subjected to an electrostatic force and uniform pressure were derived using the static deflection analysis. As the stretch of the middle plane of the flexible microplate is not considered in our theoretical model, the derived analytical expressions are more applicable to the electrostatically actuated configurations in which the ratio of the diameter of the plate to its thickness varies from 100 to 40, and the deflection of the microplate is smaller than its thickness, and the electrode distance is smaller than or equal to the thickness. The limit case for our theoretical analysis is that the ratios of diameter to thickness and thickness to electrode distance are 40 and 1, respectively, which results in maximum errors of 9.9% in deflection and 2.5% in pull-in voltage. Additionally, for the same diameter to thickness ratio, a larger ratio of thickness to electrode distance will benefit a more accurate solution.

The theoretical analyses and expressions in our study show high accuracy within a large range of device dimensions when compared with FEM simulations and literature. This study is helpful for the design and optimization of electrostatically actuated microdevices, such as CMUTs, capacitive pressure sensors, micro pumps and switches. The new method described in this paper could be expanded to analyze other mechanical behaviors of the electrostatically actuated microdevices.

REFERENCES


### Biographies

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