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A comparative study of two modeling approaches in neural networks

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Abstract

The neuron state modeling and the local field modeling provides two fundamental modeling approaches to neural network research, based on which a neural network system can be called either as a static neural network model or as a local field neural network model. These two models are theoretically compared in terms of their trajectory transformation property, equilibrium correspondence property, nontrivial attractive manifold property, global convergence as well as stability in many different senses. The comparison reveals an important stability invariance property of the two models in the sense that the stability (in any sense) of the static model is equivalent to that of a subsystem deduced from the local field model when restricted to a specific manifold. Such stability invariance property lays a sound theoretical foundation of validity of a useful, cross-fertilization type stability analysis methodology for various neural network models.

Keywords: Static neural network modeling; Local field neural network modeling; Recurrent neural networks; Stability analysis; Asymptotic stability; Exponential stability; Global convergence; Globally attractive

1. Introduction

In the current neural network research, two fundamental modeling approaches are commonly adopted: either using the *neuron states* (the first approach) or using the *local field states* of neurons (the second approach) as basic variables to describe the dynamical evolution rule of a neural network. The recurrent back-propagation networks (Almeida, 1988; Pineda, 1987; Rohwer & Forrest, 1987), for instance, provide a typical example of the first approach. The networks, as in their standard form and as a direct generalization of the well-known back-propagation network (Hertz, Krogh, & Palmer, 1994), are modeled by

$$\tau\frac{\mathrm{d}v_i}{\mathrm{d}t} = -v_i + g_i\left(\sum_{j=1}^{N} w_{ij}v_j + \theta_i\right), \quad i=1,2,\ldots,N. \tag{1}$$

Here v_i is the state of neuron i with

$$u_i = \sum_{j=1}^{N} w_{ij}v_j + \theta_i$$

being its local field, g_i the activation function of neuron i, θ_i the external input imposed on neuron i, w_{ij} the synaptic connectivity value between neuron i and neuron j, and N the number of neurons in the networks. On the other hand, the famous Hopfield networks (Hopfield, 1982; Hopfield & Tank, 1986)are examples of the second approach and can be described in terms of the local field state u_i, $i=1,2,\ldots,N$, of neurons as

$$C_i\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{u_i}{R_i} + \sum_{j=1}^{N} w_{ij}g_j(u_j) + I_i, \quad i=1,2,\ldots,N, \tag{2}$$

where $v_i = g_i(u_i)$ gives the state of neuron i, and C_i, R_i and I_i are fixed physical parameters. We henceforth call the first approach the *static neural network modeling* and the second approach the *local field neural network modeling*. Correspondingly, a neural network modeled in terms of their neuron states such as Eq. (1) will be referred to as *a static neural network model*, whilst a network modeled in

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terms of local field states such as Eq. (2) will be referred to as *a local field neural network model*.

It is obvious that, given a neural network system, the two modeling approaches may both be applied to describe the system either from an external state point of view or from an internal state point of view. The questions then arise: would there exist any difference between these two modeling approaches? Which model would be more beneficial when a specific analysis purpose is concerned? Can these two different modeling approaches be cross-fertilized and complemented mutually (and if so how)? Answering such questions is clearly of great importance both theoretically and practically but, to the best of our knowledge, has not yet been given. The purpose of this paper is to answer some of these questions by establishing a comparison theory on dynamics of the two related models. Precisely, we will clarify the relationship between the dynamics of the static neural network model and the local field neural network model in terms of their trajectory transformation property, equilibrium state correspondence property, global convergence and stability in many different senses. The results obtained will not only uncover the consistency and inconsistency of the two modeling approaches, but also lay a sound theoretical foundation for validity of a useful, cross-fertilization type approach to neural network analysis developed recently by the authors (Qiao, Peng, Xu, & Zhang, 2003).

In this paper we consider the following generic static neural network model

$$\tau\frac{dx}{dt} = -x + G(Wx + q), \qquad x(0) = x_0 \in \mathbb{R}^N \tag{3}$$

and the local field neural network model

$$\tau\frac{dy}{dt} = -y + WG(y) + q, \qquad y(0) = y_0 \in \mathbb{R}^N \tag{4}$$

where $x = (x_1, x, \ldots, x_N)$ is the neuron states, $y = (y_1, y_2, \ldots, y_N)$ is the local fields, $W = (w_{ij})_{N\times N}$ is the synaptic weight matrix and $G : \mathbb{R}^N \to \Omega \subseteq \mathbb{R}^N$ is the nonlinear activation mapping with Ω being a convex subset of $\mathbb{R}^N$. Depending on the specific application, the nonlinear mapping G may be defined component wisely or otherwise. In the former case, G is of the form:

$$G(y) = (g_1(y_1), g_2(y_2), \ldots, g_N(y_N))^T,$$

$$y = (y_1, y_2, \ldots, y_N)^T \in \mathbb{R}^N$$

with g_i representing the activation function acted on neuron i, where A^T stands for the transpose of A.

It should be observed that the model (3) includes not only the recurrent back-propagation networks (Almeida, 1988; Pineda, 1987; Rohwer & Forrest, 1987), but also other extensively studied neural networks such as the optimization type networks proposed by Bouzerdoum and Pattison (1993), Forti and Tesi (1995), Friesz, Bernstein, Mehta, Tobin, and Ganjlizadeh (1994), Liang and Wang (2000a,b), Xia (1996), Xia and Wang (1998) and Xia and Wang (2000), the brain-state-in-a-box (BSB) type networks (Li, Michel, & Porod, 1989; Varga, Elek, & Zak, 1996). Similarly, the Eq. (4) models not only the Hopfield-type networks (Hopfield, 1982; Hopfield & Tank, 1986), but also the bidirectional associative memory (BAM) type networks (Kosko, 1988) as well as the cellular neural networks (CNNs) in Chua and Yang (1988), Park, Kim, Park, and Lee (2001) and Roska and Vandewalle (1995).

It should also be noticed that the static neural network model (3) and the local field model (4) can be correlated in a straightforward way if the matrix W is invertible (i.e. nonsingular). In this case, the two models can be transferred equivalently from one to the other. This trivial situation will not be considered here, so the nonsingularity of W will not be assumed in this paper.

2. Relationship between dynamics of the two models

In this section the relationships between the dynamics of the models (3) and (4) are systematically clarified in terms of their trajectory transformation property, equilibrium correspondence property, stability, superior nontrivial attractive manifold property, asymptotic stability and global convergence. From the relationships clarified, a stability-invariance property of the models will be concluded in a sense to be specified later. We also point out some open questions related to the equivalence of the stability between Eqs. (3) and (4).

2.1. Coherent systems

Besides the basic systems (3) and (4), we will also study the following four closely related systems:

$$\tau\frac{dx}{dt} = -x + G(Wx + q), \qquad x(0) = x_0 \in \overline{Co}\Re(G) \tag{3.1}$$

$$\tau\frac{dx}{dt} = -x + G(Wx + q), \qquad x(0) = x_0 \in \overline{Co}\Re(GW_q) \tag{3.2}$$

$$\tau\frac{dy}{dt} = -y + WG(y) + q, \qquad y(0) = y_0 \in \Re(W_q) \tag{4.1}$$

$$\tau\frac{dy}{dt} = -y + WG(y) + q, \qquad y(0) = y_0 \in \overline{\Re(WG)} + q \tag{4.2}$$

where W_q is the affine transformation defined by $W_q x = Wx + q$. These four systems are exactly the Eqs. (3) and (4) when restricted to the specific manifolds $\overline{Co}\Re(G)$, $\overline{Co}\Re(GW_q)$, $\Re(W_q)$ and $\overline{\Re(WG)} + q$, respectively, where $\Re(G)$ denotes the *range* of operator G and $\overline{Co}\Re(G)$ denotes the *closed, convex hull* of $\Re(G)$ (i.e. the smallest closed convex set containing $\Re(G)$). We will show below that these four systems can indeed be defined, in other words, $\overline{Co}\Re(G)$,

$\Re(W_q)$ and $\overline{\Re(WG)}+q$ do form invariant sets or manifolds of Eqs. (3) and (4), respectively.

We will assume throughout this paper the existence of a unique solution $x(t,x_0)$ to Eq. (3) and a unique solution $y(t,y_0)$ to Eq. (4) for any given initial states x_0, y_0 in $\mathbb{R}^N$ (which is the case, e.g. when G is locally Lipschitz continuous). As usual, the solution $x(t,x_0)$ is also called a *trajectory* of Eq. (3) through x_0 denoted henceforth by $\Gamma_1(x_0)$ (correspondingly, the trajectory $y(t,y_0)$ of Eq. (4) through y_0 is denoted by $\Gamma_2(y_0)$). A subset $D\subset\mathbb{R}^N$ is said to be an *invariant set* of the system (3) if $x_0\in D$ implies $\Gamma_1(x_0)\subseteq D$.

We first clarify the trajectory transformation relationship between systems (3) and (4.1).

Theorem 1. *Trajectory transformation relationship. If $\Gamma_1(x_0)$ is the trajectory of Eq. (3) through $x_0\in\mathbb{R}^N$, then $W\Gamma_1(x_0)+q$ is the trajectory of Eq. (4.1) through $y_0=Wx_0+q$, that is, $\Gamma_2(Wx_0+q)=W\Gamma_1(x_0)+q$.*

Conversely, if $y(t)=\Gamma_2(Wx_0+q)$ is the trajectory of Eq. (4.1) through $y_0=Wx_0+q$ for some $x_0\in\mathbb{R}^N$, then $x(t,x_0)$, defined by

$$x(t,x_0)=e^{-t/\tau}x_0+\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}G(y(s))ds, \tag{5}$$

is the trajectory of Eq. (3) through x_0, that is, $\Gamma_1(x_0)=x(t,x_0)$.

Proof. First, let $\Gamma_1(x_0)=x(t,x_0)$. Then $x(t,x_0)$ solves the Eq. (3), that is,

$$\tau\frac{dx(t,x_0)}{dt}=-x(t,x_0)+G(Wx(t,x_0)+q).$$

Multiplying both sides of this equation with W gives the result

$$\tau\frac{d[Wx(t,x_0)+q]}{dt}=-Wx(t,x_0)-q+WG(Wx(t,x_0)+q)+q,$$

which means that

$$\tau\frac{dy(t,y_0)}{dt}=-y(t,y_0)+WG(y(t,y_0))+q,$$

where $y(t,y_0)=Wx(t,x_0)+q$ and $y_0=Wx_0+q$. Thus $\Gamma_2(Wx_0+q)=W\Gamma_1(x_0)+q$ is the trajectory of Eq. (4.1) through Wx_0+q.
Now if $y(t)=\Gamma_2(Wx_0+q)$ is the trajectory of Eq. (4.1) through $y_0=Wx_0+q$, then we have

$$\tau\frac{dy(t)}{dt}=-y(t)+WG(y(t))+q.$$

This leads to

$$\begin{aligned}y(t)&=e^{-t/\tau}y(0)+\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}WG(y(s))ds+(1-e^{-t/\tau})q\\&=e^{-t/\tau}[y(0)-q]+\frac{1}{\tau}e^{-t/\tau}W\int_0^t e^{s/\tau}G(y(s))ds+q\\&=W\left(e^{-t/\tau}x_0+\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}G(y(s))ds\right)+q\\&=Wx(t,x_0)+q,\end{aligned} \tag{6}$$

where $x(t,x_0)$ is defined as in Eq. (5). On the other hand, a direct calculation using Eq. (5) shows that

$$\begin{aligned}\tau\frac{dx(t,x_0)}{dt}&=-e^{-t/\tau}x_0+G(y(t))-\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}G(y(s))ds\\&=-x(t,x_0)+G(y(t)).\end{aligned}$$

From Eq. (6) it then follows that

$$\tau\frac{dx(t,x_0)}{dt}=-x(t,x_0)+G(Wx(t,x_0)+q).$$

That is, $x(t,x_0)$ is the trajectory of Eq. (3) through x_0. The Proof of Theorem 1 is thus complete. □

Theorem 1 implies that the trajectories of systems (3) and (4.1) can be transferred from one to the other. However, it does not mean that the trajectories of Eq. (3) and (4) can be transferable unless W is invertible. This is because, in the case when W is not invertible, $\Re(W_q)$ is a strict subset of $\mathbb{R}^N$ so any trajectory $\Gamma_2(y_0)$ of Eq. (4) may not be expressible in terms of the solution of Eq. (3). This of course causes difficulty in our subsequent discussion. However, it is adequate for us to apply Theorem 1 to establish the following result which implies that systems (3.1), (3.2), (4.1) and (4.2) can indeed be well-defined.

Theorem 2. *Invariance-manifold property*

(i) *Any closed convex manifold Σ, containing $\Re(GW_q)$ (or $\Re(G)$), is an invariant manifold for system (3).*
(ii) *$\Re(W_q)$ is an invariant manifold for Eq. (4).*
(iii) *$\overline{\Re(WG)}+q$ is an invariant manifold for Eq. (4) provide $\Re(G)$ is bounded and convex.*

Proof. (i) Let $x=x(t,x_0)$ be a solution of Eq. (3) starting from $x_0\in\Sigma$ and let $L(x)=Wx+q$. Then, by the theory of differential equations (see, e.g. Verhulst, 1990), we have

$$\begin{aligned}x(t,x_0)&=e^{-t/\tau}x(0)+\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}G(L(x(s)))ds\\&=e^{-t/\tau}x_0+(1-e^{-t/\tau})\frac{e^{-t/\tau}\int_0^t e^{s/\tau}G(L(x(s)))ds}{(1-e^{-t/\tau})\tau}\end{aligned} \tag{7}$$

Write

$$I(t)=\frac{\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/\tau}G(L(x(s)))\mathrm{d}s}{(1-\mathrm{e}^{-t/\tau})\tau}$$

Then Eq. (7) implies that, since $\mathrm{e}^{-t/\tau}\in(0,1)$ and Σ is a convex set, $x(t,x_0)\in\Sigma$ if $I(t)\in\Sigma$. Thus in order to prove (i) it is enough to prove that $I(t)\in\Sigma$. Note that

$$(1-\mathrm{e}^{-t/\tau})\tau=\int_0^t \mathrm{e}^{-s/\tau}\mathrm{d}s=\lim_{1\to\infty}\sum_{k=1}^{n}\mathrm{e}^{-\xi_k/\tau}\Delta t_k$$

and

$$\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/\tau}G(L(x(s)))\mathrm{d}s=\int_0^t \mathrm{e}^{-(t-s)/\tau}G(L(x(s)))\mathrm{d}s$$

$$=\int_0^t \mathrm{e}^{-r/\tau}G(L(x(t-r)))\mathrm{d}r$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\mathrm{e}^{-\xi_k/\tau}G(L(x(t-\xi_k)))\Delta t_k.$$

So $I(t)=\lim_{n\to\infty}I_n(t)$ with

$$I_n(t)=\frac{\sum_{k=1}^{n}(\mathrm{e}^{-\xi_k/\tau}\Delta t_k)G(L(x(t-\xi_k)))}{\sum_{k=1}^{n}\mathrm{e}^{-\xi_k/\tau}\Delta t_k}.$$

Now let

$$\lambda_k=\mathrm{e}^{-\xi_k/\tau}\Delta t_k\Big/\sum_{k=1}^{n}\mathrm{e}^{-\xi_k/\tau}\Delta t_k.$$

Then $\lambda_k\in[0,1]$ and $\sum_{k=1}^{n}\lambda_k=1$ so, $I_n(t)\in\Sigma$ for any n since $G(L(x(t-\xi_k)))\in\Re(G)\subseteq\Sigma$ by the assumption and Σ is convex. Thus $I(t)\in\Sigma$ since Σ is closed, which completes the proof of (i).

(ii) Let $\Pi=\Re(W)+q$. Then for $y_0\in\Pi$ there is an $x_0\in\mathbb{R}^N$ such that $y_0=Wx_0+q$. Let $x(t,x_0)$ be the solution of Eq. (3) through x_0. Then it is easy to verify that $y(t,y_0):=Wx(t,x_0)+q$ is a solution of Eq. (4) through y_0. Clearly, $y(t,y_0)\in\Re(W)+q$. This proves (ii).

(iii) If $\Re(G)$ is a bounded convex set, then $\overline{Co}\Re(G)=\overline{\Re(G)}$. Taking $\Sigma=\overline{\Re(G)}$ and arguing as in the proof of (i) we obtain that $\Gamma_2(y_0)\in\overline{W(\overline{\Re(G)})+q}$ whenever $y_0\in\overline{W(\overline{\Re(G)})+q}$. To complete the proof we need to show that $W(\overline{\Re(G)})=\overline{\Re(WG)}$ by using the fact that W is linear and $\Re(G)$ is bounded. First, it is clear that $W(\overline{\Re(G)})\subseteq\overline{\Re(WG)}$. Now if $x\in\overline{\Re(WG)}$, then there is a sequence $\{WG(x_j)\}$ such that $\lim_{j\to\infty}WG(x_j)=x$. Since $\{G(x_j)\}$ is bounded, we may assume without loss of generality that $\lim_{j\to\infty}G(x_j)=x^*$ for some $x^*\in\overline{\Re(G)}$. From the linearity of W it follows that $x=\lim_{j\to\infty}WG(x_j)=Wx^*$, which implies that $x\in W(\overline{\Re(G)})$. So $\overline{\Re(WG)}\subseteq W(\overline{\Re(G)})$. Consequently, $W(\overline{\Re(G)})=\overline{\Re(WG)}$, which completes the proof of Theorem 2. □

Remark 1. (i) From the Proof of Theorem 2 it is easily seen that Theorem 2 (i) is actually true for any generic nonlinear operator L, not necessarily restricted to the affine transformation case as defined in Eqs. (3) and (4). This may manifest an exclusive feature of the static neural network model (3).
(ii) Theorem 2 says that any solutions $x(t,x_0)$ and $y(t,y_0)$ will remain in the manifolds $\overline{Co}\Re(G)$, $\overline{Co}\Re(GW_q)$, $\Re(W_q)$ and $\overline{W(\Re(G))+q}$ if they are initialized from them. It says, however, nothing about what would happen if the solutions are not starting from the manifolds. Theorem 3 provides us with some information in this case.

Theorem 3. *Attractive manifold property*
(i) *For any $x_0\in\mathbb{R}^N$ the trajectory $\Gamma_1(x_0)$ exponentially approaches to $\overline{Co}\Re(G)$ (or $\overline{Co}\Re(GW_q)$) in the sense that*

$$\mathrm{d}(x(t,x_0),\overline{Co}\Re(G))\le\mathrm{e}^{-t/\tau}\mathrm{d}(x_0,\overline{Co}\Re(G)),\tag{8}$$

where $\mathrm{d}(x,\Sigma)=\inf_{y\in\Sigma}\|x-y\|$ denotes the distance of x to the set Σ.
(ii) *For any $y_0\in\mathbb{R}^N$ the trajectory $\Gamma_2(y_0)$ exponentially approaches to $\Re(W_q)$ (or $\overline{\Re(WG)}+q$ in the case when $\Re(G)$ is bounded and convex) in the sense that*

$$\mathrm{d}(y(t,y_0),\Re(W)+q)\le\mathrm{e}^{-t/\tau}\mathrm{d}(y_0,\Re(W)+q).\tag{9}$$

Proof. (i) It follows from Eq. (7) that for any $x_1\in\overline{Co}\Re(G)$ (or $\overline{Co}\Re(GW_q)$),

$$x(t,x_0)=\mathrm{e}^{-t/\tau}(x_0-x_1)+\mathrm{e}^{-t/\tau}x_1+\frac{1}{\tau}\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/t}G(L(x(s)))\mathrm{d}s.$$

Let

$$z(t)=\mathrm{e}^{-t/\tau}x_1+\frac{1}{\tau}\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/t}G(L(x(s)))\mathrm{d}s.$$

Then argue as in the Proof of Theorem 2 to obtain that $z(t)\in\overline{Co}\Re(G)$ (or $\overline{Co}\Re(GW_q)$) for all $t\ge0$. Thus,

$$\mathrm{d}(x(t,x_0),\overline{Co}\Re(G))\le\|x(t,x_0)-z(t)\|\le\mathrm{e}^{-t/\tau}\|x_1-x_0\|$$

for any $x_1\in\overline{Co}\Re(G)$, which implies Eq. (8).
(ii) For any $y_1\in\Re(W)+q$ there is an $x_1\in\mathbb{R}^N$ such that $y_1=Wx_1+q$. Thus we have

$$y(t,x_0)=\mathrm{e}^{-t/\tau}(y_0-y_1)+\mathrm{e}^{-t/\tau}y_1+\frac{1}{\tau}\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/\tau}(WG(y(s))+q)\mathrm{d}s$$

$$=\mathrm{e}^{-t/\tau}(y_0-Wx_1-q)+\mathrm{e}^{-t/\tau}(Wx_1+q)$$

$$+\frac{1}{\tau}\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/\tau}(WG(y(s))+q)\mathrm{d}s$$

$$=:\mathrm{e}^{-t/\tau}(y_0-Wx_1-q)+z(t).$$

Similar argument as in the Proof of Theorem 2 together with the convexity and closedness of $\Re(W_q)$ (or $\overline{\Re(WG)}+q$) gives that $z(t)\in\Re(W_q)$ (or $\overline{\Re(WG)}+q$). Thus it follows that

$$d(y(t,y_0),\Re(W_q)) \le \|y(t,x_0)-z(t)\| \le e^{-t/\tau}\|y_0-y_1\|.$$

This implies Eq. (9) since $y_1 \in \Re(W_q)$ is arbitrary. The Proof of Theorem 3 is thus complete. □

Remark 2. Theorem 3 implies that $\overline{Co}\Re(G)$, $\overline{Co}\Re(GW_q)$ and $\Re(W_q)$ (or $\overline{\Re(WG)}+q$ in the case when $\Re(G)$ is bounded and convex) are, respectively, the superior nontrivial attractive manifold of systems (3) and (4).

2.2. Equilibrium relationships

In this section we will study the relationship among the equilibria of systems (3), (4), (3.1), (3.2), (4.1) and (4.2).

Note first that x^* is an equilibrium state of system (3) if and only if it is an equilibrium state of Eqs. (3.1) and (3.2) and that x^* solves the equation $F_1(x)=0$, that is, x^* is a zero of F_1, where F_1 is defined by

$$F_1(x) = -x + G(Wx+q), \qquad \forall x \in \mathbb{R}^N. \tag{10}$$

Similarly, y^* is an equilibrium state of Eq. (4) if and only if it is an equilibrium state of Eqs. (4.1) and (4.2) and further, y^* is a zero of F_2 with F_2 being defined by

$$F_2(y) = -y + WG(y) + q, \qquad \forall y \in \mathbb{R}^N. \tag{11}$$

Denote by $F_1^{-1}(0)$ the common equilibrium state set of Eqs. (3), (3.1) and (3.2) and by $F_2^{-1}(0)$ the equilibrium state set of Eqs. (4), (4.1) and (4.2).

Theorem 4. *Equilibrium relationship*
$x^* \in F_1^{-1}(0)$ *if and only if* $y^* = Wx^* + q \in F_2^{-1}(0)$. *Conversely*, $y^* \in F_2^{-1}(0)$ *if and only if* $x^* = G(y^*) \in F_1^{-1}(0)$.

Proof. If $x^* \in F_1^{-1}(0)$ is an equilibrium state of Eq. (3), then $x^* = G(Wx^*+q) = G(y^*)$. Multiplying both sides with W then gives $Wx^* = WG(y^*)$. So $y^* = WG(y^*)+q$, which means that $y^* \in F_2^{-1}(0)$ and is an equilibrium state of Eq. (4). On the other hand, if $y^* \in F_2^{-1}(0)$ is an equilibrium state of Eq. (4), then $y^* = WG(y^*)+q$. Let $x^* = G(y^*)$. Then $x^* = G(y^*) = G(WG(y^*)+q) = G(Wx^*+q)$. So $x^* = G(y^*) \in F_1^{-1}(0)$ and is an equilibrium state of Eq. (3). The proof is thus complete. □

From Theorem 4 it seems that systems (3) and (4) should have the same number of equilibria. This is, however, by no means evident if W is singular and G is not invertible. Theorem 5 ensures that such a conclusion is actually true being independent of the nonsingularity of W and the invertibility of G.

Theorem 5. *Identical cardinal number of equilibria. The numbers of equilibrium states of systems (3) and (4) are identical.*

Proof. Let $Card(F_1^{-1}(0))$ and $Card(F_2^{-1}(0))$ denote the cardinal numbers of the equilibrium state sets $F_1^{-1}(0)$ and $F_2^{-1}(0)$, respectively. We will show that $Card(F_1^{-1}(0)) = Card(F_2^{-1}(0))$ by proving that there exists a bijective mapping between $F_1^{-1}(0)$ and $F_2^{-1}(0)$. In fact, consider the mapping $P: F_2^{-1}(0) \to F_1^{-1}(0)$ defined by

$$F_2^{-1}(0) \ni y^* \mapsto G(y^*) = x^* \in F_1^{-1}(0).$$

By Theorem 4, G is well-defined. First, G is injective since, if $P(y_1^*) = P(y_2^*)$ for some $y_1^*, y_2^* \in F_2^{-1}(0)$ then, by the definition of equilibrium state of Eq. (4), $y_i^* = WG(y_i^*)+q$ $(i=1,2)$ so that $y_1^* = WG(y_1^*)+q = WG(y_2^*)+q = y_2^*$. Finally, P is surjective since, for any $x^* \in F_1^{-1}(0)$, we have $x^* = G(Wx^*+q)$ so, if $y^* = Wx^*+q$ then, by Theorem 4, $y^* \in F_2^{-1}(0)$, that is, $x^* = G(y^*)$. This proves the theorem. □

From the Proof of Theorem 5 it follows that the nonlinear activation mapping G, when restricted to the equilibrium state set $F_1^{-1}(0)$, is invertible no matter whether or not the activation mapping G itself is invertible. In this case, the inverse G^{-1} can be given by

$$G^{-1}(x^*) = Wx^* + q, \qquad \forall x^* \in F_1^{-1}(0). \tag{12}$$

Similarly, we can conclude that W, when restricted to $F_2^{-1}(0) - q$, is regular even though W itself may not be regular. Thus, $x^* \in F_1^{-1}(0)$ if and only if $y^* = G^{-1}(x^*) \in F_2^{-1}(0)$. Conversely, $y^* \in F_2^{-1}(0)$ if and only if $x^* = G(y^*) \in F_1^{-1}(0)$. A pair of equilibria x^* and y^* possessing such a property will be called as *a pair of mutually mapped equilibria* of system (3) and (4), denoted henceforth by (x^*, y^*). With such understanding, we state and prove a series of stability–invariance properties of systems (3) and (4.1) in Section 2.3.

2.3. Stability–invariance properties

We first recall some notion and notations from the theory of dynamical systems (taking Eq. (3) as an example). A point x^* is said to be a ω-limit point of $\Gamma_1(x_0)$ if there is a subsequence $\{t_i\}$ such that $x^* = \lim_{i\to\infty} x(t_i, x_0)$. All the ω-limit points of $\Gamma_1(x_0)$ constitute the ω-limit set of $\Gamma_1(x_0)$, denoted henceforth by $\omega(\Gamma_1(x_0))$. The ω-limit set is invariant under the dynamics. The equilibrium point x^* is said to be *stable* if any trajectory of Eq. (3) can stay within a small neighborhood of x^* whenever the initial state x_0 is close to x^*. The equilibrium point x^* is said to be *attractive* if there is a neighborhood $\Xi(x^*)$, called the *attraction basin* of x^*, such that any trajectory of Eq. (3) initialized from a state in $\Xi(x^*)$ will approach to x^* as time goes to infinity. An equilibrium state is said to be *asymptotically stable* if it is both stable and attractive. Further, the equilibrium state x^* is said to be *exponentially stable* if there exist a constant $\alpha > 0$ and a strictly increasing function M such that

$$\|x(t,x_0) - x^*\| \le M(\|x_0 - x^*\|)e^{-\alpha t}. \tag{13}$$

If, in addition, $M(s) = Ks$ (i.e. M is linear) for an absolute constant K in Eq. (13) then x^* is said to be *exponentially stable in the sense of Liapunov*. The equilibrium state x^* is *globally asymptotically stable* (*globally exponentially stable*) if it is asymptotically stable (exponentially stable) and $\Xi(x^*) = \mathbb{R}^N$.

Correspondingly to the global stability notion, we say that a system (say, Eq. (3)) is *globally convergent* if $x(t, x_0)$ converges to an equilibrium state of Eq. (3) for every initial point $x_0 \in \mathbb{R}^N$. Note that the limit of $x(t, x_0)$ may be different for different x_0. The system (3) is said to be *exponentially convergent* if it is globally convergent with the limiting state x^* satisfying Eq. (13). We need the following two simple lemmas.

Lemma 1. *Let W be an $N \times N$ matrix and let B be the unit open ball of $\mathbb{R}^N$ centered at zero and with radius 1, i.e. $B = \{x \mid \|x\| < 1\}$. Set $W(B) = \{Wx \mid x \in B\}$. Then there is a constant $\delta > 0$ such that*

$$\delta_1 B \cap \Re(W) \subseteq W(B), \qquad \text{whenever } \delta_1 \leq \delta.$$

Proof. It is clear that $W(B)$ is a neighborhood of 0 in the subspace $\Re(W) \subset \mathbb{R}^N$. Thus there is an open set U in the space $\Re(W)$ such that $0 \in U \subseteq W(B)$. Since U is a relatively open set of $\mathbb{R}^N$, then there must be an open set V in $\mathbb{R}^N$ such that $U = V \cap \Re(W)$. Clearly, V is a neighborhood of 0 in $\mathbb{R}^N$. Thus there is a constant $\delta > 0$ such that, whenever $\delta_1 \leq \delta$, we have $\delta_1 B \cap \Re(W) \subseteq V \cap \Re(W) \subseteq W(B)$. This proves the lemma. □

Lemma 2. *If $\Re(G)$ is bounded and convex, then $\overline{\Re(WG)} + q = W(\overline{\Re(G)}) + q$.*

Proof. The inclusion $W(\overline{\Re(G)}) + q \subseteq \overline{\Re(WG)} + q$ is trivial. To prove the inverse inclusion, take any fixed $y \in \overline{\Re(WG)} + q$ and assume that $\{y_i\} \subseteq \mathbb{R}^N$ is a sequence such that $y = \lim_{i\to\infty} WG(y_i) + q$. Let $x_i = G(y_i)$. Then $x_i \in \Re(G)$. Since $\Re(G)$ is bounded, we may assume without loss of generality that $\lim_{i\to\infty} x_i = \lim_{i\to\infty} G(y_i) = x$ for some $x \in \overline{\Re(G)}$. It follows that $y = \lim_{i\to\infty} WG(y_i) + q = \lim_{i\to\infty} Wx_i + q = Wx + q$. This means that $y \in W(\overline{\Re(G)})$ so $\overline{\Re(WG)} \subseteq W(\overline{\Re(G)})$. Consequently, $W(\overline{\Re(G)}) + q = \overline{\Re(WG)} + q$. The lemma is thus proved. □

With the above lemmas, we are now in a position to examine the relationship among the stability of models (3), (4), (3.1), (3.2), (4.1) and (4.2).

Theorem 6. *Stability. Let $(x^*, y^*) \in F_1^{-1}(0) \times F_2^{-1}(0)$ be any pair of mutually mapped equilibria of systems (3) and (4). Assume that G is a Lipschitzian, that is, there is a positive constant $L(G)$ such that*

$$\|G(y_1) - G(y_2)\| \leq L(G)\|y_1 - y_2\|, \qquad \forall y_1, y_2 \in \mathbb{R}^N.$$

Then x^*, as an equilibrium state of Eq. (3), is stable if and only if y^*, as an equilibrium state of Eq. (4.1), is stable.

Proof. '$\Rightarrow$:' If $x^* \in F_1^{-1}(0)$ is stable, then for any $\varepsilon > 0$ there is $\delta_1(\varepsilon) > 0$ such that, whenever $\|x_0 - x^*\| < \delta_1(\epsilon) < 1$, we have

$$\|x(t, x_0) - x^*\| < \|W\|^{-1}\varepsilon \qquad \forall t \geq 0. \tag{14}$$

Let $\delta < \delta_1(\varepsilon)$ be so small that $\delta B \cap \Re(W) \subseteq \delta_1(\epsilon)W(B)$. (By Lemma 1 this is possible.) We now prove that for $y_0 \in \Re(W) + q$,

$$\|y(t, y_0) - y^*\| < \varepsilon$$

whenever $\|y_0 - y^*\| < \delta(\varepsilon)$, which together with Lemma 2 implies that y^*, as an equilibrium state of system (4.1), is stable.

For any $y_0 \in \Re(W) + q$ and $y^* \in F_2^{-1}(0)$ we have $y_0 - y^* \in \Re(W)$, so $y_0 - y^* \in \delta_1(\varepsilon)W(B)$ whenever $\|y_0 - y^*\| < \delta(\varepsilon)$. Thus there is a $z_0 \in \delta_1(\varepsilon)W(B)$ so that we can write $y_0 - y^* = Wz_0$ or equivalently, $y_0 = W(z_0 + x^*) + q$. From Theorem 1 it follows that if $x(t)$ is the solution of system (3) through $x_0 = z_0 + x^*$ then $y(t, y_0) = Wx(t) + q$ is the unique solution of Eq. (4.1) passing through y_0. So, for any $t > 0$, we have

$$\|y(t, y_0) - y^*\| = \|Wx(t) - Wx^*\| \leq \|W\|\|x(t) - x^*\|.$$

This, together with Eq. (14), and on noting that $\|x_0 - x^*\| = \|z_0\| < \delta_1(\varepsilon)$, implies that $\|y(t, y_0) - y^*\| < \varepsilon$, as expected.

'$\Leftarrow$:' Given any $\varepsilon > 0$, by assumption there is a $\delta_1(\varepsilon) > 0$ such that for $y_0 \in \Re(W) + q$, we have $\|y(t, y_0) - y^*\| < \varepsilon/[2L(G)]$ for $t \geq 0$ whenever $\|y_0 - y^*\| < \delta_1(\varepsilon)$. Letting

$$\delta(\varepsilon) = \min\left\{\frac{\delta_1(\varepsilon)}{\|W\|}, \frac{\varepsilon}{2}\right\},$$

we can prove that for $x_0 \in \mathbb{R}^N$, $\|x(t, x_0) - x^*\| < \epsilon$ provided $\|x_0 - x^*\| < \delta(\varepsilon)$. In fact, let $y_0 = Wx_0 + q$. Then $y_0 \in \Re(W) + q$ and satisfies that $\|y_0 - y^*\| < \delta_1(\varepsilon)$ whenever $\|x_0 - x^*\| < \delta(\varepsilon)$. So, by Theorem 1, we can write $y(t, y_0) = Wx(t, x_0) + q$. Further, we have

$$\|y(t, y_0) - y^*\| < \frac{\varepsilon}{2L(G)}$$

whenever $\|x_0 - x^*\| < \delta(\varepsilon)$. From Eq. (5) it thus follows that

$$\begin{aligned}\|x(t,x_0) - x^*\| &\leq e^{-t/\tau}\|x_0 - x^*\| \\ &\quad + \left\|\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}G(y(s))ds - (1 - e^{-t/\tau})x^*\right\| \\ &= e^{-t/\tau}\|x_0 - x^*\| + \frac{1}{\tau}\left\|e^{-t/\tau}\int_0^t e^{s/\tau}[G(y(s)) - G(y^*)]ds\right\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\frac{1}{\tau}e^{-t/\tau}\int_0^t e^{s/\tau}ds < \varepsilon.\end{aligned}$$

This implies the stability of x^* since ε is arbitrary. The theorem is thus proved. □

Theorem 7. *Attractivity and global convergence*
Let the conditions of Theorem 6 be satisfied.

(i) x^* *is globally attractive for the system (3) if and only if* y^* *is globally attractive (in the manifold* $\Re(\mathrm{W_q})$*) for the system (4.1).*
(ii) *The system (3) is globally convergent if and only if the system (4.1) is globally convergent.*

Proof. Under the assumption of the theorem, systems (3) and (4) have a unique solution, denoted by $x(t,x_0)$ and $y(t,y_0)$, respectively. Note that $y(t,y_0)$ is also the unique solution of Eq. (4.1) when restricted to the manifold $\Re(W_q)$. If $x(t,x_0)$ converges to an equilibrium state x^* of Eq. (3) for $x_0 \in \mathbb{R}^N$, then $y(t,Wx_0+q) = Wx(t,x_0)+q$ is the unique solution of Eq. (4) through $y_0 = Wx_0+q$ (Theorem 1). Since W is a matrix, it follows directly that $y(t,Wx_0+q)$ converges to the state $y^* = Wx^*+q$ (clearly, $y^* \in F_2^{-1}(0)$ iff $x^* \in F_1^{-1}(0)$). We now prove that the convergence of $y(t,Wx_0+q)$ to y^* implies the convergence of $x(t,x_0)$ to x^*. Consider the unique solution $x(t,x_0)$ of Eq. (3) initialized from x_0. Let us assume that $y(t,y_0)$ converges to an equilibrium state y^* of Eq. (4) with $y_0 = Wx_0+q$. Then, by the uniqueness of solution to Eq. (4), $y(t,Wx_0+q) = Wx(t,x_0)+q$. It also follows by Theorem 4 that $y^* = Wx^*+q$. Thus, we have $y(t,Wx_0+q)-y^* = W(x(t,x_0)-x^*) \to 0$ as $t\to\infty$. So, for any given $\varepsilon>0$ there is a positive constant T_ε such that

$$\|y(t,Wx_0+q)-y^*\|_2 \le \varepsilon/L(G) \qquad \text{whenever } t>T_\varepsilon.$$

On the other hand, we have from Eq. (3) that

$$\begin{aligned}\tau\frac{\mathrm{d}(\|x(t,x_0)-x^*\|)}{\mathrm{d}t} &\le -\|x(t,x_0)-x^*\| + \|G(Wx(t,x_0)+q)-G(Wx^*+q)\| \\ &\le -\|x(t,x_0)-x^*\| + L(G)\|W(x(t,x_0)-x^*)\|.\end{aligned}$$

Solving this inequality gives

$$\|x(t,x_0)-x^*\| \le \mathrm{e}^{-(t-s)/\tau}\|x(s,x_0)-x^*\| + \frac{1}{\tau}L(G)\mathrm{e}^{-t/\tau} \times \int_s^t \mathrm{e}^{r/\tau}\|y(r,Wx_0+q)-y^*\|\mathrm{d}r,$$

which holds for any $t>s>0$. Taking $t>s>T_\varepsilon$ in this inequality, we then obtain that

$$\begin{aligned}\|x(t,x_0)-x^*\| &\le \mathrm{e}^{-(t-s)/\tau}\|x(s,x_0)-x^*\| + \frac{\varepsilon}{\tau}\mathrm{e}^{-t/\tau}\int_s^t \mathrm{e}^{r/\tau}\,\mathrm{d}r \\ &= \mathrm{e}^{-(t-s)/\tau}\|x(s,x_0)-x^*\| + \varepsilon(1-\mathrm{e}^{-(t-s)/\tau}),\end{aligned} \tag{15}$$

which gives that

$$\lim_{t\to\infty}\|x(t,x_0)-x^*\| \le \varepsilon.$$

Since ε is arbitrary, we conclude that $\lim_{t\to\infty}\|x(t,x_0)-x^*\| = 0$. That is, $x(t,x_0)$ converges to the equilibrium state x^*. This completes the proof of the theorem. □

Theorem 8 extends the equivalence between the global convergence of systems (3) and (4.1) to the case of exponential convergence.

Theorem 8. *Exponential convergence*
Assume that the activation mapping G is a Lipschitzian. Then Eq. (3) is exponentially convergent iff Eq. (4.1) is exponentially convergent.

Proof. '$\Rightarrow$': If system (3) is exponentially convergent, then there is an $x^* \in F_1^{-1}(0)$ such that the solution $x(t,x_0)$ exponentially converges to x^* in the sense of Eq. (13). Let $y_0 = Wx_0+q$. Then, by Theorems 1 and 4, $y(t,y_0) = Wx(t,x_0)+q$ is the unique solution of Eq. (4) through y_0 and $y^* = G^{-1}(x^*) = Wx^*+q \in F_2^{-1}(0)$. Since $y_0 - y^* = W(x_0-x^*)$, then by making use of the generalize inverse of W (say, the Moore–Penrose inverse W^+ (see, e.g. Rao & Mitra, 1971) we can write $x_0 - x^* = W^+(y_0-y^*)$. Thus we obtain from Eq. (13) that

$$\begin{aligned}\|y(t,y_0)-y^*\| &= \|W(x(t,x_0)-x^*)\| \\ &\le \|W\|M(\|x_0-x^*\|)\mathrm{e}^{-\alpha t} \le \|W\|M(\|W^+\|\|y_0-y^*\|)\mathrm{e}^{-\alpha t} \\ &= M_1(\|y_0-y^*\|)\mathrm{e}^{-\alpha t},\end{aligned} \tag{16}$$

where $M_1(s) = \|W\|M(\|W^+\|s)$. That is, the system (4.1) is exponentially convergent.
'$\Leftarrow$': Suppose that for any fixed $y_0 \in \Re(W)+q$, the solution $y(t,y_0)$ exponentially converges to y^* in the sense of Eq. (16), where $y^* \in F_2^{-1}(0)$. Then it follows from Theorem 4 that $x^* = G(y^*)$ is an equilibrium state of Eq. (3). Further we have from Eq. (12) that $y^* = G^{-1}(x^*) = Wx^*+q$. Now if $x(t,x_0)$ is the unique solution of Eq. (3) through x_0 then $y(t,Wx_0+q) = Wx(t,x_0)+q$ is the solution of Eq. (4) through $y_0 = Wx_0+q$. From Eq. (16) it follows that

$$\begin{aligned}\|W(x(t,x_0)-x^*)\| &= \|y(t,Wx_0+q)-y^*\| \\ &\le M_1(\|Wx_0+q-y^*\|)\mathrm{e}^{-\beta t} = M_1(\|W(x_0-x^*)\|)\mathrm{e}^{-\beta t} \\ &\le M_1(\|W\|\|x_0-x^*\|)\mathrm{e}^{-\beta t}.\end{aligned} \tag{17}$$

Let $V(t,x_0) = x(t,x_0)-x^*$. Then

$$\tau\frac{\mathrm{d}V(t,x_0)}{\mathrm{d}t} = -V(t,x_0) + G(Wx(t,x_0)+q) - G(Wx^*+q),$$

from which we obtain that

$$\begin{aligned}\tau\frac{\mathrm{d}\|V(t,x_0)\|}{\mathrm{d}t} &\le -\|V(t,x_0)\| + \|G(Wx(t,x_0)+q)-G(Wx^*+q)\| \\ &\le -\|V(t,x_0)\| + L(G)\|W(x(t,x_0)-x^*)\|\end{aligned}$$

This combined with Eq. (17) implies that

$$\begin{aligned}\|V(t,x_0)\| &\le e^{-t/\tau}\|V(0,x_0)\| + \frac{1}{\tau}L(G)e^{-t/\tau} \\ &\quad\times\int_0^t e^{s/\tau}\|W(x(s,x_0)-x^*)\|ds \\ &\le e^{-t/\tau}\|x_0-x^*\| + \frac{L(G)}{\tau}e^{-t/\tau} \\ &\quad\times\int_0^t e^{(1/\tau-\beta)s}M_1(\|W\|\|x_0-x^*\|)ds \\ &\le f(t)\max\{M_1(\|W\|\|x_0-x^*\|),\|x_0-x^*\|\}, \end{aligned} \tag{18}$$

where

$$f(t) = e^{-t/\tau} + \frac{1}{\tau}L(G)e^{-t/\tau}\int_0^t e^{(1/\tau-\beta)s}ds.$$

We now estimate the function $f(t)$. First, if $\tau\beta = 1$ then we have

$$f(t) = e^{-t/\tau} + \frac{t}{\tau}L(G)e^{-t/\tau} \le \left(1+\frac{L(G)}{\varepsilon}\right)e^{-(1-\varepsilon)t/\tau}.$$

Now if $\tau\beta \ne 1$ then it follows that

$$f(t) = e^{-t/\tau} + L(G)\frac{e^{-\beta t}-e^{-t/\tau}}{1-\tau\beta} \le \left(1+\frac{L(G)}{|1-\tau\beta|}\right)e^{-\min\{1/\tau,\beta\}t}$$

Thus and from Eq. (18) we derive Eq. (13) with

$$M(\|x_0-x^*\|) = K\max\{M_1(\|W\|\|x_0-x^*\|),\|x_0-x^*\|\},$$

where

$$K = \begin{cases} 1+\dfrac{L(G)}{|1-\tau\beta|}, & \tau\beta\ne 1 \\ 1+\dfrac{L(G)}{\varepsilon}, & \tau\beta = 1 \end{cases}$$

and

$$\alpha = \begin{cases} \min\left\{\dfrac{1}{\tau},\beta\right\}, & \tau\beta\ne 1 \\ \dfrac{1-\varepsilon}{\tau}, & \tau\beta = 1 \end{cases}$$

This means that the system (3) is globally exponentially convergent, which completes the proof of the theorem. □

As a direct consequence of Theorems 6–8, we immediately arrive at the following basic stability–invariance results on models (3) and (4.1).

Theorem 9. *Stability–invariance between Eqs. (3) and (4.1)*
Assume that G is a Lipschitz continuous mapping and that (x^*, y^*) is a pair of mutually mapped equilibrium states of Eqs. (3) and (4). Then the stability of systems (3) and (4.1) is invariant in the following sense.

(i) *x^* is stable (asymptotically stable/exponentially stable) iff y^* is stable (asymptotically stable/exponentially stable).*
(ii) *x^* is globally asymptotically stable (globally exponentially stable) iff y^* is globally asymptotically stable (globally exponentially stable).*
(iii) *System (3) is globally convergent (globally exponentially convergent) iff system (4.1) is globally convergent (globally exponentially convergent).*

Remark 3. The stability–invariance property stated in Theorem 9 should be precisely understood. It means that x^* is stable in a sense (say, globally or asymptotically) iff y^* is so in exactly the same sense (namely, globally or asymptotically). However, it should be carefully discriminated that 'x^* is stable in system (3)' makes sense in the topology of $\mathbb{R}^N$ since Eq. (3) is a dynamical system defined on the whole space $\mathbb{R}^N$, whereas 'x^* is stable in system (3.1)' then makes sense in the topology of $\overline{Co}\Re(G)$, which is, of course, a relative topology of $\mathbb{R}^N$, since Eq. (3.1) is a dynamical system defined on the manifold $\overline{Co}\Re(G)$.
It is natural to ask the question: can such types of stability–invariance be extended to other pair of systems such as Eqs. (3) and (4), (3.1) and (4.1), (3.1) and (4.2), or (3.2) and (4.2)?. We cannot answer such a question in general, but an easy observation shows that such stability invariance property can indeed be extended to Eqs. (3.1) and (4.2), as stated in Theorem 10.

Theorem 10. *Stability–invariance between Eqs. (3.1) and (4.2)*
Assume that G is a Lipschitz continuous mapping and that (x^*, y^*) is any pair of mutually mapped equilibrium states of Eqs. (3.1) and (4.2). Then the stability of systems (3.1) and (4.2) is invariant in the sense that

(i) *x^* is stable (asymptotically stable or exponentially stable) iff y^* is stable (asymptotically stable or exponentially stable)*;
(ii) *x^* is globally asymptotically stable (globally exponentially stable) iff y^* is globally asymptotically stable (globally exponentially stable)*;
(iii) *system (3.1) is globally convergent (globally exponentially convergent) iff system (4.2) is globally convergent (globally exponentially convergent).*

Proof. From the Proof of Theorems 7–8 we know that to prove the theorem it suffices to prove $\overline{\Re(WG)} + q = W\overline{\Re(G)} + q$ in the case when $\Re(G)$ is bounded and convex. This is, however, the result of Lemma 2. So the theorem is thus proved. □

2.4. An inequitable relation

We now go further with an attempt to explore the relationship between the stability of the original systems (3) and (4).

Theorem 11. *Assume that G is a Lipschitz continuous mapping and that* (x^*, y^*) *is any pair of mutually mapped equilibrium states of Eqs. (3) and (4). Then*

(i) y^* *is exponentially attractive (globally exponentially attractive) if* x^* *is exponentially stable (globally exponentially stable) in the sense of Liapunov*;
(ii) *system (4) is globally convergent if system (3) is globally exponentially convergent in the sense of Liapunov*.

Proof. We only prove (i). The proof of (ii) is similar.
Denote by $B(x,r)$ the open ball in $\mathbb{R}^N$ centered at x and with radius r. Assume that x^* is an exponentially stable equilibrium state of system (3). Then there exist positive constants C, α, r such that, whenever $x_0 \in B(x^*, r)$,

$$\|x(t,x_0) - x^*\| \le C\mathrm{e}^{-\alpha t}\|x_0 - x^*\|, \qquad \forall t \ge 0. \tag{19}$$

We now verify that there exist two positive constants δ, β and a function $M(s) > 0$ such that for any $y_0 \in B(y^*, \delta)$,

$$\|y(t,y_0) - y^*\| \le M(\|y_0 - y^*\|)\mathrm{e}^{-\beta t}, \qquad \forall t \ge 0. \tag{20}$$

To this end, let $F(y) = G(y + y^*) - G(y^*)$ and take the transformations $u(t) = x(t,x_0) - x^*$ and $v(t) = y(t,y_0) - y^*$ to transform systems (3) and (4) into

$$\tau\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -u(t) + F(Wu(t)), \qquad u(0) = x_0 - x^* \tag{21}$$

$$\tau\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -v(t) + WF(v(t)), \qquad v(0) = y_0 - y^* \tag{22}$$

Under such transformations, the equilibria pair (x^*, y^*) of Eqs. (3) and (4) is changed into the equilibria pair $(u^*, v^*) = (0,0)$ of systems (21) and (22). Further, $u^* = 0$ is exponentially stable in the sense that for any $u_0 \in B(0,r)$,

$$\|u(t)\| \le C\mathrm{e}^{-\alpha t}\|u_0\|, \qquad t \ge 0, \tag{23}$$

where $u(t)$ is the unique solution of Eq. (21) with the initial state $u(0) = x_0 - x^*$. This standard global stability property of system (21) ensures the validity of an in-depth inverse Liapunov function theorem (see, e.g. Theorem 4.5 in (Khalil, 1992, pp. 180)). This theorem implies that for system (21) there is an energy function

$$E : B(0,r) \longrightarrow \mathbb{R}^N \tag{24}$$

such that

(*P1*) E is equivalent to the norm of $\mathbb{R}^N$, that is, there are positive constants C_1 and C_2 such that

$$C_1\|u\|^2 \le E(u) \le C_2\|u\|^2;$$

(*P2*) E is differentiable (almost everywhere) and $\|\nabla E(u)\| \le C_3\|u\|$ for some positive constant C_3; and

(*P3*) E is a strict energy function of Eq. (21) in the sense that

$$\langle \nabla E(u), -u + F(Wu)\rangle \le -C_4\|u\|^2$$

for some positive constant $C_4 (\neq 2C_2)$.

We are now in a position to prove the exponential attractivity of the equilibrium state $v^* = 0$ for system (22). This we do in the following four steps.
Step 1. Note first that using the matrix W, $\mathbb{R}^N$ can be decomposed into the direct sum $\mathbb{R}^N = \Re(W) \oplus Ker(W^{\mathrm{T}})$, where $Ker(W^{\mathrm{T}})$ is the kernel space of W^{T} (i.e. $Ker(W^{\mathrm{T}}) = \{x : W^{\mathrm{T}}x = 0\}$) which is orthogonal to $\Re(W)$. Thus there are two orthogonal projections P_1 (from $\mathbb{R}^N$ onto $\Re(W)$) and P_2 (from $\mathbb{R}^N$ onto $Ker(W^{\mathrm{T}})$) such that for any $x \in \mathbb{R}^N$, we can write $x = P_1x + P_2x$. In view of the orthogonal property of the two projections, the system (22) can be rewritten as

$$\begin{cases} \tau\dfrac{\mathrm{d}(P_1v(t))}{\mathrm{d}t} = -P_1v(t) + WF(v(t)), \\ \tau\dfrac{\mathrm{d}(P_2v(t))}{\mathrm{d}t} = -P_2v(t), \qquad v(0) = y_0 - y^* \end{cases}$$

or in the integral form as

$$\begin{cases} P_1v(t) = \mathrm{e}^{-t/\tau}P_1v(0) + \dfrac{1}{\tau}\mathrm{e}^{-t/\tau}\displaystyle\int_0^t \mathrm{e}^{s/\tau}WF(v(s))\mathrm{d}s \\ P_2v(t) = \mathrm{e}^{-t/\tau}P_2v(0), \qquad v(0) = y_0 - y^*. \end{cases} \tag{25}$$

Now $P_1v(0) \in \Re(W)$ so $P_1v(0) = Wx_0$ for some $x_0 \in \mathbb{R}^N$. It then follows from Eq. (25) that

$$P_1v(t) = W\left(\mathrm{e}^{-t/\tau}x_0 + \frac{1}{\tau}\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/\tau}F(v(s))\mathrm{d}s\right).$$

Let

$$z(t) = \mathrm{e}^{-t/\tau}x_0 + \frac{1}{\tau}\mathrm{e}^{-t/\tau}\int_0^t \mathrm{e}^{s/\tau}F(v(s))\mathrm{d}s. \tag{26}$$

Then Eq. (25) can be written in the form

$$\begin{cases} P_1v(t) = Wz(t) \\ \tau\dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = -z(t) + F(v(t)) = -z(t) + F(Wz(t) + \mathrm{e}^{-t/\tau}P_2v(0)). \end{cases} \tag{27}$$

Step 2. To prove that there is a $T > 0$ such that $z(t) \in B(0,r)$ for $t \in [0,T)$ (so E in Eq. (24) can be applied to $z(t)$) and further that for $t \in (0,T)$,

$$\|z(t)\| \le \mathrm{e}^{-k_1t/\tau}\left[\sqrt{\frac{C_2}{C_1}}\|x_0\| + \frac{k_2}{|1 - k_1|\sqrt{C_1}}\right] < r\mathrm{e}^{-k_1t/\tau} \tag{28}$$

whenever $v(0) \in B(0,\delta)$ for some sufficiently small $\delta > 0$, where k_1 and k_2 are positive constants to be specified later. First, by Lemma 1 we can take a positive constant $\delta < |2C_2 - C_4|C_1r/(2C_2C_3L(G))$ such that $\delta B(0,1) \cap \Re(W) \subseteq$

$\gamma WB(0,1)$, where $L(G)$ is the Lipschitz constant of G and

$$\gamma = \min\left\{\frac{r}{2}\sqrt{\frac{C_1}{C_2}}, r\right\}.$$

For any $v \in \delta B(0,1) = B(0,\delta)$ one has $P_1 v \in B(0,\delta)$ so there exists an $x \in B(0,\gamma)$ such that $P_1 v = Wx$ and

$$\sqrt{\frac{C_2}{C_1}}\|x\| + \frac{C_2 C_3 L(G)}{|2C_2 - C_4|C_1}\delta < r.$$

Thus, for $v_0 = v(0) \in B(0,\delta)$ we have $v_0 = P_1 v_0 + P_2 v_0 = Wx_0 + P_2 v_0$ with $x_0 \in B(0,\gamma)$ satisfying the above inequality. Since, from Eq. (26), we have $z(0) = x_0 \in B(0,r)$, then the openness of $B(0,r)$ together with the continuity of G ensure that there is a $T > 0$ such that $z(t) \in B(0,r)$ for all $t \in (0,T)$.

Now differentiate E along the curve $z(t)$ in the interval $(0,T)$ and make use of Eq. (27) and $(P1)$–$(P3)$ to obtain that

$$\begin{aligned}\tau\frac{\mathrm{d}E(z(t))}{\mathrm{d}t} &= \langle \nabla E(z(t)), \tau\frac{\mathrm{d}z(t)}{\mathrm{d}t}\rangle = \langle \nabla E(z(t)), -z(t) \\ &\quad + F(Wz(t))\rangle + \langle \nabla E(z(t)), F(Wz(t) + e^{-t/\tau}P_2 v_0) \\ &\quad - F(Wz(t))\rangle \\ &\le -C_4\|z(t)\|^2 + L(G)\|P_2 v_0\|C_3\|z(t)\|\mathrm{e}^{-t/\tau} \\ &\le -\frac{C_4}{C_2}E(z(t)) + \frac{C_3 L(G)\delta}{\sqrt{C_1}}\sqrt{E(z(t))}\mathrm{e}^{-t/\tau}.\end{aligned}$$

Let $V(t) = \sqrt{E(z(t))}$. Then the above inequality implies that

$$\begin{aligned}\tau\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le -\frac{C_4}{2C_2}V(t) + \frac{C_3 L(G)\delta}{2\sqrt{C_1}}\mathrm{e}^{-t/\tau} \\ &= -k_1 V(t) + k_2\mathrm{e}^{-t/\tau},\end{aligned} \tag{29}$$

where

$$k_1 = \frac{C_4}{2C_2}, \qquad k_2 = \frac{C_3 L(G)\delta}{2\sqrt{C_1}}.$$

We may assume without loss of generality that $k_1 < 1$. From Eq. (29) it follows that

$$\begin{aligned}V(t) &\le \mathrm{e}^{-k_1 t/\tau}V(0) + \frac{k_2}{1-k_1}(\mathrm{e}^{-k_1 t/\tau} - \mathrm{e}^{-t/\tau}) \\ &\le \mathrm{e}^{-k_1 t/\tau}\left(V(0) + \frac{k_2}{|1-k_1|}\right).\end{aligned}$$

This together with *(P2)* and Eq. (29) implies Eq. (28).

Step 3. We show that $z(t) \in B(0,r)$ and satisfies Eq. (28) for all $t \ge 0$.

Suppose this is not true. Then there would be a T_1 such that

$$z(t) \in B(0,r), \qquad \forall t \in [0,T_1), \qquad \text{but } z(T_1) \notin B(0,r).$$

Then the same argument as in deriving Eq. (28) gives $\|z(t)\| < r\mathrm{e}^{-k_1 t/\tau}$ for $t \in [0,T_1)$. Letting $t \to T_1$ leads to $\|z(T_1)\| \le r\mathrm{e}^{-k_1 T_1/\tau} < r$, which contradicts to the fact that $z(T_1) \notin B(0,r)$. The claim is thus true.

Step 4. It follows from Eqs. (27) and (28) that, whenever $v_0 \in B(0,\delta)$,

$$\begin{aligned}\|v(t)\| &= \|P_1 v(t)\| + \|P_2 v(t)\| = \|Wz(t)\| + \|e^{-t/\tau}P_2(v_0)\| \\ &\le \|W\|\|z(t)\| + \mathrm{e}^{-t/\tau}\|P_2(v_0)\| \\ &\le \|W\|r\mathrm{e}^{-k_1 t/\tau} + \mathrm{e}^{-t/\tau}\|P_2(y_0 - y^*)\| \\ &\le M(\|y_0 - y^*\|)\mathrm{e}^{-\beta t/\tau},\end{aligned}$$

where $M(s) = s + \|W\|r$ and $\beta = \min\{k_1, 1\}$. Since $v(t) = y(t,y_0) - y^*$ then this implies that

$$\|y(t,y_0) - y^*\| \le M(\|y_0 - y^*\|)\mathrm{e}^{-\beta t/\tau}$$

for all $t > 0$ as long as $y_0 \in B(y^*,\delta)$. This means that y^*, as an equilibrium state of Eq. (4), is exponentially attractive. The theorem is thus proved. □

2.5. Open questions

From Theorems 7–11 the relationship among dynamics of the six models (3), (4), (3.1), (3.2), (4.1) and (4.2) can be summarized in Fig. 1.

From Fig. 1 it is seen that, as far as stability is concerned, the models (3) and (4.1) as well as (3.1) and (4.2) are equivalent. This may be of great benefit in conducting stability analysis of various dynamical neural networks. However, the relationship between stability of the models (3) and (4) may not be equivalent, as hinted in Theorem 11. In particular, the following open questions may deserve further investigation:

(*Q1*) Is there any equivalent stability property (P) between the models (3) and (4)? Precisely, is the stability, asymptotical stability or exponential stability of the equilibrium state x^* of Eq. (3) equivalent to the stability, asymptotical stability or exponential stability of the equilibrium state y^* of Eq. (4)?

(*Q2*) In the statement (i) of Theorem 11 can the exponential (or globally exponential) attractivity of y^* be replaced

$$\begin{array}{ccccccc} & & \boxed{\text{Model (3)}} & \Longrightarrow & \boxed{\text{Model (3-1)}} & \Longrightarrow & \boxed{\text{Model (3-2)}} \\ & & \Updownarrow & & \Updownarrow & & \\ \boxed{\text{Model (4)}} & \Longrightarrow & \boxed{\text{Model (4-1)}} & \Longrightarrow & \boxed{\text{Model (4-2)}} & & \end{array}$$

Fig. 1. Relationships among dynamics of the six models (3), (3.1), (3.2), (4), (4.1), and (4.2). Here ‘ ⇒ ’ means that if the former model has the property (P) then the latter model has the same property, ‘ ⇔ ’ means that the former model has the property (P) iff the latter model has the property (P). The property (P) may be any stability concept such as local stability, asymptotic stability, exponential stability, attractivity, convergence, or any of their global counterparts.

by the exponential (or globally exponential) stability of y^* in the Liapunov sense? or can the exponential (or globally exponential) stability of x^* in the Liapunov sense be replaced with the exponential (or globally exponential) stability of x^* in the normal sense as in Eq. (13)? (Note that in the Proof of Theorem 11, we have actually verified every fact of the exponential stability of y^* except that $M(s)$ is linear.)

($Q3$) Is the stability condition of system (4.1) intrinsically weaker than that of system (4)? Or to be more specific, does there exist a set of conditions that guarantees the stability for system (4.1) in certain sense but not for Eq. (4)?

We strongly believe that any positive solution to one of these questions will yield significant impact on the stability analysis of nonlinear dynamical systems in general and of neural networks in particular.

3. An application example

In this section we give an example to illustrate how the comparison theory developed in Section 2 can be efficiently applied to derive new results on stability of neural networks. A more extensive and systematical application of the theory will be presented in (Qiao et al., 2003).

Consider the following BCOp-type neural networks:

$$\frac{dx_i}{dt} = -x_i + g_i\left(x_i - \alpha_i \sum_{j=1}^{N} Q_{ij}x_j + q_i\right), \tag{30}$$

$i = 1, 2, \ldots, N,$

where $x = (x_1, x_2, \ldots, x_N)^T$, $q = (q_1, q_2, \ldots, q_N)^T$ with constant q_i $(i = 1, \ldots, N)$, and $Q = (Q_{ij})_{N\times N}$ and $\Lambda = diag\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ are $N\times N$ matrices with α_i $(i = 1, \ldots, N)$ being positive parameters. The function g_i in Eq. (30) is the one-dimensional nearest point projection defined by

$$g_i(s) = \begin{cases} a_i & s \le a_i, \\ s & s \in [a_i, b_i], \\ b_i & s \ge b_i. \end{cases} \tag{31}$$

The BCOp-type neural networks (30) have been studied in recent years by many authors (see, e.g. Arik & Tavsanoglu, 2000; Bouzerdoum & Pattison, 1993; Forti & Tesi, 1995; Friesz et al., 1994; Liang & Wang, 2000a,b; Xia, 1996; Xia & Wang, 1998; Xia & Wang, 2000 and the references quoted there). The application aim of such networks is to solve a bound-constraint quadratic optimization problem:

$$\min E(x) = x^T Qx + x^T q + c \qquad \text{s.t. } a_i \le x_i \le b_i, \tag{32}$$

$i = 1, 2, \ldots N$

For such an application it is known that the global convergence of the networks is a prerequisite. The latest result on global convergence of the networks states that the trajectory of Eq. (30) will globally and exponentially converge to the unique solution of Eq. (30) when Q is positive definite and $(I - \Lambda Q)$ is invertible (Liang & Si, 2001; Liang & Wang, 2000a,b). We now apply the comparison theory established in Section 2 to improve and generalize this convergence result of Eq. (30).

The BCOp-type neural network (30) is clearly of the type of static neural network model (3). So, by the stability invariance between systems (3) and (4.1) (Theorem 9), the network (30) is globally (or globally exponentially) convergent iff the following system, when restricted to the invariant manifold $\Re(W_q)$ with $W = I - \Lambda Q$, is so:

$$\frac{dy_i}{dt} = -y_i + \left(g_i(y_i) - \alpha_i \sum_{j=1}^{N} Q_{ij} g_j(y_j)\right) + q_i, \tag{33}$$

$i = 1, 2, \ldots, N.$

The system (33) is a special case of the Hopfield-type neural networks (4) of which the stability and convergence have been throughly studied (see, e.g. Chen & Amari, 2001a; Fang & Kincaid, 1996; Forti & Tesi, 1995; Guan, Chen, & Qin, 2000; Liang & Si, 2001; Matsuoka, 1991; Qiao, Peng, & Xu, 2001; Yang & Dillon, 1994; Zhang, Heng, & Fu, 1999). The fundamental results on globally exponential stability and convergence for Eq. (4) can be summarized, for instance, in Theorem 12.

Theorem 12. *Assume that $G = (g_1, g_2, \ldots, g_N)^T$ in Eq. (4) is diagonally nonlinear with each g_i being Lipschitz continuous (that is, $|g_i(s) - g_i(t)| \le L_i|s - t|$ for any $s, t \in \mathbb{R}$). For any diagonal matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ we write*

$$M = L^{-1} - W$$

$$M(\Gamma) = L^{-1}\Gamma - \frac{\Gamma W + W^T \Gamma}{2},$$

where $L = diag\{L_1, L_2, \ldots, L_N\}$. Then the system (4) has a unique equilibrium state y^*. Further, y^* is globally exponentially stable if there is a positive definite diagonal matrix Γ such that, for any $i \in \{1, 2, \cdots, N\}$, one of the following conditions (C) and (C1)–(C4) is satisfied:

(C) *$M(\Gamma)$ is positive definite,*
($C1$) $L_i^{-1}\xi_i - \xi_i w_{ii} > \sum_{j\ne i}^{N} \xi_j |w_{ij}|$,
($C2$) $L_i^{-1}\xi_i - \xi_i w_{ii} > \sum_{j\ne i}^{N} \xi_j |w_{ji}|$,
($C3$) $L_i^{-1}\xi_i - \xi_i w_{ii} > \frac{1}{2}\sum_{j\ne i}^{N} |\xi_j w_{ji} + \xi_i w_{ij}|$,
($C4$) $\lambda_{\min}(M(\Gamma)) > 0$ *(i.e. the matrix measure $\mu(-\Gamma M) < 0$), where $\lambda_{\min}(A)$ stands for the smallest eigenvalue of the matrix A.*

Applying Theorem 12 to the system (33) with $W = I - \Lambda Q$ and g_i being defined as in Eq. (31) (so $L_i = 1$ in this case) and then making use of Theorem 9 we obtain the following result on the exponential convergence of the system (30), where $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$.

Theorem 13. *Convergence of BCOp-type NNs*
For any $x_0 \in \Omega$ there is a unique solution $x(t, x_0)$ to the BCOp-type neural networks (30). The solution converges exponentially to the unique equilibrium state of Eq. (30) if there is a positive definite diagonal matrix $\Gamma = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that $\Gamma Q + Q^{\mathrm{T}}\Gamma$ is positive definite. In particular, this is true when one of the following conditions (D1)–(D4) is satisfied:

(*D1*) $\xi_i Q_{ii} > \sum_{j \neq i}^{N} \xi_j |Q_{ij}|$,
(*D2*) $\xi_i Q_{ii} > \sum_{j \neq i}^{N} \xi_j |Q_{ji}|$,
(*D3*) $\xi_i Q_{ii} > \frac{1}{2} \sum_{j \neq i}^{N} |\xi_j Q_{ji} + \xi_i Q_{ij}|$,
(*D4*) $\lambda_{\min}(\Gamma Q) > 0$.

Theorem 13 improved and generalized the latest-known convergence result for the BCOp-type neural networks in the sense that the invertibility of $(I - \Lambda Q)$ as well as the positive definiteness of Q itself are no longer required. This application example illustrates the potential usefulness of the comparsion theory developed in the present paper.

4. Concluding remarks

We have developed a comparison theory for two fundamental modeling approaches, the neuron state modeling and the local field modeling approaches, in the current neural network research. Two representative models (3) and (4), as well as their coherent models (3.1), (3.2), (4.1) and (4.2), have been studied in detail in terms of their dynamics including the trajectory transfer property, the equilibrium state correspondence property, the nontrivial attractive manifold property, the global convergence and stability in various senses. From the analysis it has been concluded that (i) the trajectories of systems (3) and (4.1) can be transferred from one to the other (Theorem 1), (ii) the equilibria of systems (3) and (4) are mutually corresponded in a specific one-to-one manner (more precisely, via the nonlinear mapping G) (Theorems 4 and 5), (iii) the nontrivial attractive manifolds of systems (3) and (4) are $\overline{Co\Re(G)}$, $\overline{Co\Re(GW_q)}$, $\Re(W_q)$ and $\overline{\Re(WG)} + q$, respectively (Theorem 3), and (iv) the stability of systems (3) and (4.1), as well as (3.1) and (4.2), is exactly the same in the local, global, asymptotic or exponential sense. However, such an equivalent stability property is not yet clear for the models (3) and (4) though some preliminary result is available (Theorem 11). The comparison theory has been applied to a BCOp-type neural network to illustrate its usefulness.

Our results obtained have shed light on consistency and inconsistency of the neuron state and local field modeling approaches. Further, it is observed that in the current neural network research, the neuron state and local field modeling are applied very often in a mutually irrelated manner so that the two approaches have hardly ever been cross-fertilized. Certain types of neural networks such as the Hopfield-type models have been studied extensively, and thus many deep results have been obtained for the models so far (see, e.g. Arik & Tavsanoglu, 2000; Chen & Amari, 2001a,b; Cohen & Grossberg, 1983; Guan et al., 2000; Hirsch, 1989; Juang, 1999; Liang & Si, 2001; Liang & Wu, 1999; Qiao et al., 2001; Zhang et al., 1999). In contrast, some other types of neural networks such as the recurrent back-propagation type networks have attracted only little attention and thus fall short of a systematic and in-depth theoretical analysis (Almeida, 1988; Haykin, 1994; Hertz et al., 1994; Pineda, 1987; Rohwer & Forrest, 1987). The relationships obtained in this paper among the dynamics of Eqs. (3), (3.1), (3.2), (4), (4.1), and (4.2) may contribute to the formalization of certain new, powerful, cross-fertilizing type approaches for neural network analysis, leading to new and deep stability results for less studied networks, as demonstrated in Section 3 for BCOp-type neural networks. Moreover, for example, based on the stability invariance property of Eqs. (3) and (4.1) (Theorem 9) as well as the implication relation between Eqs. (4) and (4.1) (see Fig. 1 in Section 2.5), we can naturally propose to study the recurrent back-propagation networks (1) by means of the Hopfield-type neural networks (2). This may form a new methodology for stability analysis of the recurrent back-propagation type neural networks. Such a cross-fertilizing approach will be explored in detail in (Qiao et al., 2003).

Finally, we would like to call for a positive or negative solution to any of the open questions proposed in Section 2.5. Answering these questions is by no means only for the purpose of clarifying models (3) and (4). It can also make a significant impact on the general dynamical system theory. In view of Theorem 11 as well as the stability equivalence between Eqs. (3) and (4.1) (note that Eq. (4.1) may be viewed as a subsystem of Eq. (4)), we conjecture that models (3) and (4) may not be equivalent as far as the stability is concerned. If so, some counterexamples have to be constructed.

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