# A New Approach for Estimating the Attraction Domain for Hopfield-Type Neural Networks

**Dequan Jin**
*dqjin@yahoo.cn*
**Jigen Peng**
*jgpeng@mail.xjtu.edu.cn*
*Department of Mathematics, Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, China*

**In this letter, using methods proposed by E. Kaslik, St. Balint, and their colleagues, we develop a new method, expansion approach, for estimating the attraction domain of asymptotically stable equilibrium points of Hopfield-type neural networks. We prove theoretically and demonstrate numerically that the proposed approach is feasible and efficient. The numerical results that obtained in the application examples, including the network system considered by E. Kaslik, L. Brăescu, and St. Balint, indicate that the proposed approach is able to achieve better attraction domain estimation.**

## 1 Introduction

The Hopfield-type neural networks that we consider in this letter are described by the following differential equations (Haykin, 1994):

$$\frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} \omega_{ij} g_j(u_j(t)) + I_i \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where $u_i(t)$ are the neural voltages, $R_i$ the resistances, $\omega_{ij}$ the connection weights, $g_i$ the input–output connection functions, and $I_i$ the external inputs.

Hopfield-type neural networks are mainly applied as either associative memories (pattern recognition) or optimization solvers (Morita, 1993; Haykin, 1994; Xu & Kwong, 1997). When applied as associative memories, the equilibrium points of the neural networks represent the stored patterns. The attraction domain of each equilibrium point coincides with the region from which the corresponding stored pattern can be retrieved even in the existence of noise; that is, the attraction domain of a stable equilibrium point characterizes the error-correction capability of the corresponding stored pattern. When applied as an optimization solver, the equilibrium points of the neural networks characterize all possible optimal solutions of the optimization problem. The attraction domain of each equilibrium point then coincides with the region that the network, starting from any initial guess in it, will evolve to the optimal solution. Therefore, identifying the attraction domain is important in the application of neural networks (Burshtein, 1998; Chesi, 2004; Driessche & Zou, 1998).

The approaches extensively used in the existing investigation into this field of neural networks are mainly based on Lyapunov direct method and so depend on the construction of Lyapunov function. However, there is no general rule guiding us to construct an optimal Lyapunov function for a given system; that is, constructing a Lyapunov function requires skill. Between 1985 and 1995, L. Gruyitch obtained many interesting results concerning the determination of attraction domain (Gruyitch, Richard, Borne, & Gentina, 2004). However, in those results, the Lyapunov functions used to characterize the attraction domain are constructed by the method of characteristics, which strongly depends on the solutions of system.

Important work about the determination of attraction domain has been done by St. Balint, E. Kaslik, A. M. Balint, and their colleagues since 1985. In 1985, St. Balint developed a novel Lyapunov function (named the optimal Lyapunov function in Balint, Balint, & Negru, 1986) and proved that the determination of attraction domain can be reduced to the determination of the analytic domain of the vector field when the field is $\mathbb{R}$-analytic (Balint, 1985). In this case, the optimal Lyapunov function can be found by solving linear systems of algebraic equations whose solutions are the coefficients of the expansion of the optimal Lyapunov function at the equilibrium point. In 1986, the special case when the Jacobian matrix is diagonalizable at the equilibrium point was considered in Balint et al. (1986), where a recurrence formula for finding the coefficients of the expansion of the optimal Lyapunov function has been established. When this formula was used, the optimal Lyapunov functions and attraction domains were found for several two-dimensional systems. The hypothesis that the Jacobian matrix is diagonalizable is used only for finding the recurrence formula; otherwise, it is possible to find the coefficients as developed in Balint (1985) by solving some linear algebraic equations but impossible to find a recurrence formula. In 1987, a method for approximating the attraction domain by the region of convergence of the Taylor series of the optimal Lyapunov function was established by Balint, Negru, Balint, and Simiantu (1987) under the assumption that the Jacobian matrix is diagonalizable at the equilibrium point. The region of convergence of the Taylor series, obtained by the recurrence formula given in Balint et al. (1986), has been shown to be part of attraction domain, and its boundary in some directions touches the boundary of attraction domain. Hence, by using the recurrence formula, it is possible to find the radius of the convergence in any direction so that the region of convergence of the Taylor's series can be found. However, the weakness of the approximation method is that part of the attraction domain may be lost when the domain is not symmetric with respect to the equilibrium points.

In 2003, a new approach for approximating the attraction domain by gradually extending the Lyapunov function (called the "embryo" of the optimal Lyapunov function) obtained in a neighborhood of the equilibrium point was presented by Kaslik, Balint, and Balint (2003), which shows that it is possible to improve the approximation obtained by the region of convergence of the Taylor series of the optimal Lyapunov function at the equilibrium point by expanding the obtained "embryo" of the optimal Lyapunov function at a point apart from the equilibrium point but in the region of convergence of the Taylor series and near the boundary of the region. (For the gradual approximation of the attraction domain for both the continuous and discrete analytical systems by using "embryo," we also refer Kaslik, Balint, Grigis, & Balint, 2005.)

In 2005, another important step in approximation of the attraction domain was made in Kaslik, Balint, and Balint (2005), in the case when the Jacobian matrix is diagonalizable at the equilibrium point. That article discusses the capacity of the Taylor polynomials of the optimal Lyapunov function to provide good approximations of the attraction domain for autonomous and $\mathbb{R}$-analytical systems and presents some very good approximations of the attraction domain for some systems. The attraction domain problem for the discrete systems has been considered in Kaslik, Balint, Birauas, and Balint (2003), and Balint, Kaslik, Balint, and Grigis (2006). In 2005 and 2006, the method that Kaslik, Balint, and Balint (2005) established was applied to evaluate the attraction domains of Hopfield-type neural networks (Kaslik, Brăescu, & Balint, 2005; Kaslik & Balint, 2006) where some good results were achieved.

The approximation methods developed by St. Balint, E. Kaslik, and their colleagues are good approaches to determining the attraction domain and also provide a way to construct Lyapunov functions by using a practical recurrence formula. In the approximation methods presented in Kaslik, Balint, and Balint (2005), the Taylor polynomials of the optimal Lyapunov function can approximate the optimal Lyapunov function with better accuracy as the order $p$ of the polynomial increases, and so better approximations of the attraction domain can be achieved with greater recurrence, though only in the region of convergence of the Taylor series. The meaning is twofold. On one hand, if the region of convergence of the Taylor series is the whole attraction domain, the approximation accuracy to the attraction domain increases with the order $p$. On the other hand, if the region of convergence is not the whole attraction domain, the approximation accuracy may not increase monotonically with the order $p$ (see example 1 in this letter). In this case, it cannot be expected that the approximation accuracy would be better after more recurrences. Although it would be possible to find a better approximation by lower-order polynomials, how to choose the best finite order $p$ that can give the best approximation is an unsolved theoretical problem.

The purpose of this letter is to develop a new approach to better estimating the attraction domain of Hopfield-type neural networks. In section 2, the methods based on optimal Lyapunov function are briefly described, and then an illustrative example of Hopfield-type neural networks is given to show that the approximation achieved by the method established in Kaslik, Balint, and Balint (2005) does not improve with the order of the Taylor expansion. Section 3 is devoted to establishing a new approach, the iterative expansion approach, to estimate the attraction domain of Hopfield-type neural networks, and several comparison numerical examples are given to show the feasibility and efficiency of this approach. The last section is the conclusion.

## 2 Approximation Techniques for Attraction Domain

In this section, the methods based on the optimal Lyapunov function for approximating the attraction domain of the asymptotically stable equilibrium points of nonlinear systems are stated briefly. The details refer to Kaslik, Balint, and Balint (2005) and Kaslik, Balint, Grigis, and Balint (2005).

Consider the following nonlinear system,

$$\dot{x}(t) = f(x(t)), \quad t \geq 0, \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$. The field function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous (assumed for the existence and uniqueness of solutions). Without losing generality, we assume that $x = 0$ is an asymptotically stable equilibrium point of system 2.1.

**Proposition 1** *(Balint, 1985). Suppose that the function $f$ is an $\mathbb{R}$-analytic function and the real parts of the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}(0)$ are negative. Then the attraction domain $D_a(0)$ of the equilibrium point $x = 0$ of system 2.1 coincides with the natural domain of analyticity of the $\mathbb{R}$-analytic function $V$ defined by*

$$\begin{cases} \langle \nabla V(x), f(x) \rangle = -||x||^2 \\ V(0) = 0 \end{cases} . \tag{2.2}$$

*The function $V$ is strictly positive on $D_a(0) \setminus \{0\}$ and $\lim_{x\to y} V(x) = \infty$ for all $y \in \partial D_a(0)$ or for $||x|| \to \infty$.*

The $\mathbb{R}$-analytic function $V$ satisfying equation 2.2 is named the *optimal Lyapunov function* with respect to the equilibrium point $x = 0$ (Balint et al., 1986). Several approximation techniques have been developed based on optimal Lyapunov function. We briefly state them.

Assume that $f$ is $\mathbb{R}$-analytic and the Jacobian matrix $\frac{\partial f}{\partial x}(0)$ is diagonalizable. Let $S:\mathbb{C}^n \to \mathbb{C}^n$ be such an linear isomorphism that $S^{-1}\frac{\partial f}{\partial x}(0)S = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and let $g = S^{-1} \circ f \circ S$ and $W = V \circ S$, where $V$ is the optimal Lyapunov function. Write the Taylor series of $W$ at 0 as

$$W(x) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1 j_2 \cdots j_n} x_1^{j_1} x_2^{j_2}, \ldots, x_n^{j_n} \tag{2.3}$$

and the Taylor series of each components $g_i$ of $g$ at 0 as

$$g_i(x) = \lambda_i x_i + \sum_{m=2}^{\infty} \sum_{|j|=m} b^i_{j_1 j_2 \cdots j_n} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}, \quad i = 1, 2, \ldots, n, \tag{2.4}$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $j = \{j_1, j_2, \ldots, j_n\}$ with $j_i$ being nonnegative integers and $|j| = \sum_{i=1}^{n} j_i$. Then the coefficients $B_{j_1 j_2, \ldots, j_n}$ of series 2.3 are determined by the coefficients $b^i_{j_1 j_2, \ldots, j_n}$ of equation 2.4 in accordance with the following recurrence rule:

$$B_{j_1 j_2 \cdots j_n} = \begin{cases} -\dfrac{1}{2\lambda_{i_0}} \displaystyle\sum_{i=1}^{n} s_{ii_0}^2 & \text{if } |j| = j_{i_0} = 2, \\ -\dfrac{2}{\lambda_p + \lambda_q} \displaystyle\sum_{i=1}^{n} s_{ip} s_{iq} & \text{if } |j| = 2,\ j_p = j_q = 1, \\ -\dfrac{1}{\sum_{i=1}^{n} j_i \lambda_i} \displaystyle\sum_{p=2}^{|j|-1} \sum_{|k|=p, k_i \le j_i} \sum_{i=1}^{n} \big[ (j_i - k_i + 1) & \\ \quad \times b^j_{k_1 k_2, \ldots, k_n} B_{j_1 - k_1, \ldots, j_i - k_i + 1, \ldots, j_n - k_n} \big] & \text{if } |j| \ge 3, \end{cases} \tag{2.5}$$

where $s_{ij}$ are the elements of the linear isomorphism $S$.

In general, it is not practical to get the precise presentation of the optimal Lyapunov function using the recurrence formula 2.5, since for most nonlinear systems, it needs to iterate infinite times. Therefore, techniques for the gradual approximation of attraction domain have been developed (Balint et al., 1987; Kaslik, Balint, & Balint, 2003; Kaslik, Balint, et al., 2005) for the case when the Jacobian matrix is diagonalizable. We state them briefly below.

The region of convergence $D^0$ of expansion 2.3 is given by

$$D^0 = \left\{ x \in C^n : \overline{\lim_m} \sqrt[m]{\sum_{|j|=m} |B_{j_1 j_2,\ldots,j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}|} < 1 \right\}, \tag{2.6}$$

using the part of the attraction domain that can be found (Balint et al., 1986). However, in practice, since the coefficients $B_{j_1 j_2,\ldots,j_n}$ can be computed only up to a finite order, the region

$$D_p^0 = \left\{ x \in C^n : \sqrt[p]{\sum_{|j|=p} |B_{j_1 j_2,\ldots,j_n} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}|} < 1 \right\} \tag{2.7}$$

and the "embryo" of $W$,

$$W_p^0(x_1, x_2, \ldots, x_n) = \sum_{m=2}^{p} \sum_{|j|=m} B_{j_1 j_2,\ldots,j_n} x_1^{j_1} x_2^{j_2}, \ldots, x_n^{j_n}, \tag{2.8}$$

are taken into consideration when the order $p$ is big enough. Correspondingly, the "embryo" of $V$ is $V_p^0 = W_p^0 \circ S^{-1}$ defined on $S(D_p^0) \subset \mathbb{R}^n$. Denote $L[V_p^0] \subset S(D_p^0)$, the region on which $V_p^0$ satisfies Lyapunov conditions:

$$\begin{cases} V_p^0(x) \geq 0, \\ \langle \nabla V_p^0, f(x) \rangle < 0, \quad x \neq 0. \end{cases} \tag{2.9}$$

Then $D_{a\,p}^{\,0} := L[V_p^0]$ is part of the attraction domain.

In order to extend the first approximation, $D_{a\,p}^{\,0}$, $W_p^0$ is expanded at a point $x^0$ close to $\partial S^{-1}(D_{a\,p}^{\,0})$ that

$$W_p^1(x_1, x_2, \ldots, x_n) = \sum_{m=2}^{p} \sum_{|j|=m} B^1_{j_1 j_2 \cdots j_n} (x_1 - x^0)^{j_1} (x_2 - x^0)^{j_2} \cdots (x_n - x^0)^{j_n}. \tag{2.10}$$

Let $D_p^1$ be defined as in equation 2.7 with respect to $W_p^1$, and let $V_p^1 = W_p^1 \circ S^{-1}$ be defined on $S(D_p^1)$. Then the region $D_{a\,p}^{\,1} := L[V_p^1] \subset S(D_p^1)$, where $V_p^1$ satisfies equation 2.9, is a new part of the attraction domain. Continuing this procedure, one can get a better estimation $D_{a\,p}^{\,0} \bigcup D_{a\,p}^{\,1} \bigcup \cdots \bigcup D_{a\,p}^{\,k}$ of the attraction domain $D_a(0)$. However, this procedure cannot be continued if in the region $D_{a\,p}^{\,0} \bigcup D_{a\,p}^{\,1} \bigcup \cdots \bigcup D_{a\,p}^{\,k}$ obtained at step $p$, there is not a point that has a neighborhood where $V$ is locally bounded.

In 2005, a new technique for approximation of the attraction domain via the Taylor polynomial of optimal Lyapunov function up to finite degree was proposed in Kaslik, Balint, et al. (2005)

Let $p \geq 2$ be an integer, and let

$$V_p(x) = \sum_{m=2}^{p} \sum_{|j|=m} A_{j_1 j_2,\ldots,j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}, \quad x = (x_1, x_2, \ldots, x_n)^T \tag{2.11}$$

be the Taylor polynomial of order $p$ of the optimal Lyapunov function $V$, where the coefficients $A_{j_1 j_2,\ldots,j_n}$ are determined by the recurrence formula 2.5 with $V = W \circ S^{-1}$. Then, by Kaslik, Balint, and Balint (2005, theorem 13) there exists $r_p > 0$ such that $V_p(x) > 0$ and $\langle \nabla V_p(x), f(x) \rangle < 0$ for all $x \in \{x \in \mathbb{R}^n : ||x|| < r_p\} \backslash \{0\}$. That is, $V_p$ is a Lyapunov function of system 2.1.

Let $G_{V_p}(0) = \{0, 0 \neq x \in \mathbb{R}^n : V_p(x) > 0, \langle \nabla V_p(x), f(x) \rangle < 0\}$. Then by theorems 16 to 31 and their corollaries in Kaslik, Balint, and Balint (2005), there exist a constant $c > 0$ and a unique invariant set $N_{V_p}^c$ of system 2.1 such that $0 \in N_{V_p}^c \subset G_{V_p}(0)$, $V_p(x) < c$ in the interior $int(N_{V_p}^c)$ and $V_p(x) = c$ on the boundary $\partial N_{V_p}^c$. Then $N_{V_p}^c$ is part of the attraction domain $D_a(0)$, and the range of $N_{V_p}^c$ increases with $c$ in its existence interval. There exists an upper bound $c_p$ of $c$ satisfying that for every $c \in (0, c_p)$, $V_p$ is radially increasing in $N_{V_p}^c$. Obviously, $N_{V_p}^{c_p} = \bigcup_c N_{V_p}^c$ is the largest part of $D_a(0)$ that can be achieved in this manner (Kaslik, Balint, & Balint, 2005).

It is suggested that $N_{V_p}^{c_p}$ can serve as an approximation of the attraction domain of asymptotically stable equilibrium point 0 with appropriately selected $p \geq 2$ (Kaslik, Balint, & Balint, 2005). In the examples considered in Kaslik, Balint, and Balint (2005), the sets $N_{V_p}^{c_p}$ are good approximations to attraction domains. Kaslik, Brăescu, et al. (2005) explained how to apply the approximation technique to Hopfield-type neural networks. According to this method, when the Taylor series of the optimal Lyapunov function at the equilibrium point converges in the attraction domain, the approximation accuracy increases with the order $p$. This means that the approximation increasingly approaches the attraction domain as $p$ increases. But this is not true when the region of convergence is not the whole attraction domain. The following example of neural networks shows that the set $N_{V_p}^{c_p}$ does not become larger when $p$ increases.

**Example 1.** Consider the Hopfield-type neural networks:

$$\begin{cases} \dfrac{dx}{dt} = -x + \dfrac{1}{1+e^{-x}} - \dfrac{1}{1+e^{-y}} \\ \dfrac{dy}{dt} = -y - \dfrac{1}{1+e^{-x}} + \dfrac{1}{1+e^{-y}} \end{cases}. \tag{2.12}$$

It is clear that the origin $(0, 0)^T$ is an equilibrium point of equation 2.12, at which the Jacobian matrix $\begin{pmatrix} -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{3}{4} \end{pmatrix}$ is symmetric and stable, and so is diagonalizable. The above approximation technique is applied to this system for $p = 8$, 20, and 36, respectively. It is evaluated that $c_8 = 5.9513$, $c_{20} = 4.4402$, and $c_{36} = 4.1185$. The sets $N_{V_8}^{5.9513}$, $N_{V_{20}}^{4.4402}$, and $N_{V_{36}}^{4.1185}$ are portrayed, respectively, in Figure 1.

It is clear that the set $N_{V_p}^{c_p}$ does not become larger as $p$ increases; contrarily, it becomes smaller. This implies that the range of the set $N_{V_p}^{c_p}$, as part of the attraction domain, strongly depends on the selection of the degree $p$, and so we cannot get a more accurate approximation result simply by increasing the order of the polynomial function $V_p$. To the best of our knowledge, no theoretical result guides the selection of the "best" $p$ (see Kaslik, Balint, & Balint, 2005, conjecture 33).

## 3 A New Iterative Expansion Approach

In this section, we assume that $V \in C^2(\mathbb{R}^n)$, for which there exist an open region $G_V(0)$ containing the origin and a constant $r_v > 0$ with $B(r_v) \subset G_V(0)$ such that

$$V(x) > 0 \quad \text{and} \quad \dot{V}(x) =: \langle \nabla V(x), f(x) \rangle < 0 \tag{3.1}$$

in $G_V(0) \setminus B(r_v)$, where $B(r_v)$ is the ball with radius $r_v$ centered at 0.

**Proposition 2.** *Suppose that the field function $f \in C^2(\mathbb{R}^n)$. Let $K$ be a compact subset of $G_V(0)$ satisfying that $\partial G_V(0) \bigcap K = \Phi$ and $\bar{B}(r_v) \subset int K$. Then for any given $r > r_v$ that $\bar{B}(r) \subset int K$, there exists a constant $d^* > 0$ such that for every $d \in (0, d^*)$, the function $V^d$ defined by $V^d(x) = V(x + df(x))$ satisfies that*

$$V^d(x) > 0 \quad and \quad \dot{V}^d(x) = \langle \nabla V^d(x), f(x) \rangle < 0 \tag{3.2}$$

*in $K \setminus B(r)$, where $\bar{B}(r)$ is the closure of $B(r)$.*

**Proof.** By definition, the derivative of $V^d$ along the solution of system 2.1 for every $d > 0$ is given by

$$\dot{V}^d(x) = \left\langle \nabla V(x + df(x)), f(x) + d\left[\frac{\partial f}{\partial x}\right] f(x) \right\rangle, \tag{3.3}$$

where $[\frac{\partial f(x)}{\partial x}]$ is the Jacobian matrix of $f(x)$ (Zhao, 2006).

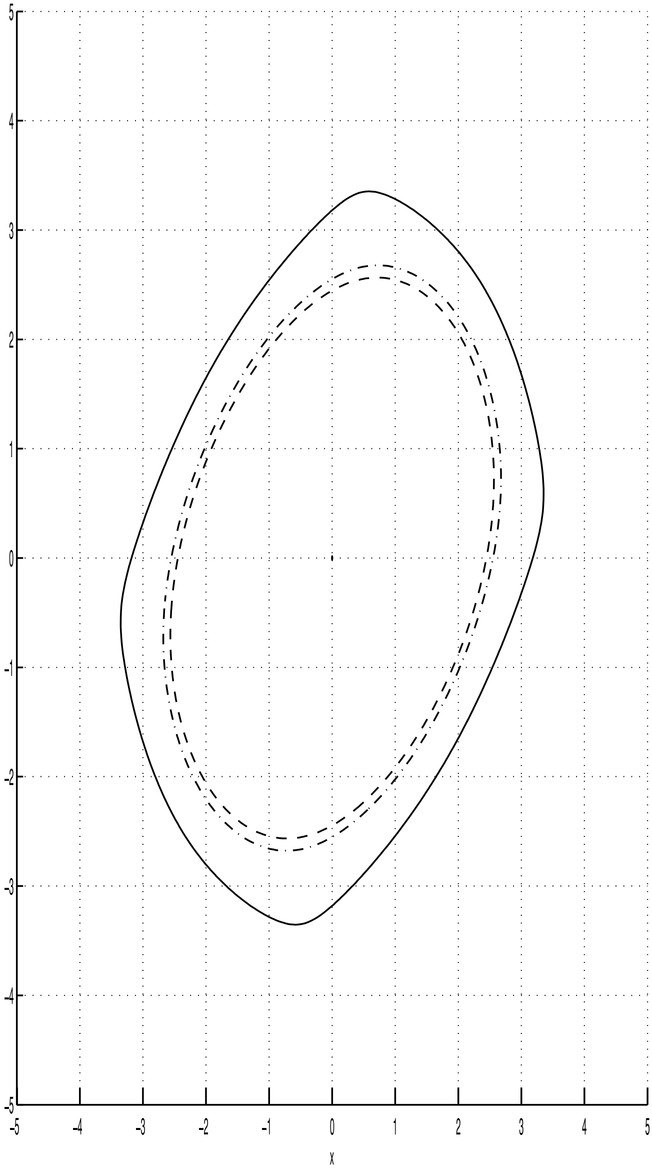


Figure 1: Boundaries of the sets $N_{V_{36}}^{4.1185}$ (dash curve), $N_{V_{20}}^{4.4402}$ (dash-dot curve), and $N_{V_8}^{5.9513}$ (solid curve).

For any $r > r_v$ satisfying that $\bar{B}(r) \subset intK$, since $G_V(0)$ is open and $K \subset G_V(0)$ is compact, it can be shown that $B((r + r_v)/2)$ is a proper subset of the open region $U = \{x \in G_V(0) : dist(x, \partial G_V(0)) > dist(\partial G_V(0), K)\}$. Therefore, the continuity of $f$ and $\nabla V(x)$ implies that for some $\delta > 0$,

$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle < -2\delta, \quad \forall x \in \bar{U} \backslash B((r + r_v)/2). \tag{3.4}$$

Moreover, it is clear that $K \backslash B(r) \subset \bar{U} \backslash B((r + r_v)/2)$. Hence, a constant $d_1 > 0$ can be found such that $\forall d \in [0, d_1]$ and $\forall x \in K \setminus B(r)$, $x + df(x) \in \bar{U} \backslash B((r + r_v)/2)$, which implies that $V^d(x) > 0$ and

$$\langle \nabla V(x + df(x)), f(x + df(x)) \rangle < -2\delta. \tag{3.5}$$

Let $K_1 = \{y = x + df(x) : x \in K \backslash B(r), d \in [0, d_1]\}$. Then the compactness of $K \backslash B(r)$, associated with the continuity of $f$, guarantees that $K_1$ is a compact set and hence $f$ is bounded in $K_1$ (i.e., $f(x + df(x))$ is bounded in $K \backslash B(r)$ for any $d \in (0, d_1)$). Now we consider the second-order Taylor expansion of $f(x + df(x))$,

$$f(x + df(x)) = f(x) + d\left[\frac{\partial f}{\partial x}\right] f(x) + d^2 h(x, d), \tag{3.6}$$

with respect to $d$ at 0, where $h(x, d) = \frac{1}{2}\langle Hessef(x + \xi f(x)) \cdot f(x), f(x) \rangle$ for some $\xi \in (0, d)$, $Hessef$ denoting the Hessian functional of $f$. Obviously $h(x, d)$ is bounded in $[K \backslash B(r)] \times [0, d_1]$; that is, there exists $M_1 > 0$ such that for all $x \in K \backslash B(r)$ and $0 < d < d_1$, $\|h(x, d)\| < M_1$. Therefore, letting $d_2 = \min\{d_1, 1/M_1\}$, we have that for all $x \in K \backslash B(r)$ and $0 < d < d_2$,

$$\left\| f(x + df(x)) - f(x) - d\left[\frac{\partial f}{\partial x}\right] f(x) \right\| = \|d^2 h(x, d)\| < d, \tag{3.7}$$

and then

$$\left\langle \nabla V(x + df(x)), f(x) + d\left[\frac{\partial f}{\partial x}\right] f(x) - f(x + df(x)) \right\rangle < M_2 d \tag{3.8}$$

with $M_2 = \sup_{x \in K \backslash B(r), d \in [0, d_2]} \|\nabla V(x + df(x))\|$.

Let $d^* = \min\{d_2, \delta/M\}$ with $M = \max\{M_1, M_2\}$. Then, when $0 < d < d^*$, there holds for all $x \in K\backslash B(r)$ that

$$\dot{V}^d(x) = \left\langle \nabla V(x + df(x)),\ f(x) + d\left[\frac{\partial f}{\partial x}\right] f(x) \right\rangle$$
$$< \left\langle \nabla V(x + df(x)),\ f(x + df(x)) \right\rangle + Md$$
$$< -2\delta + Md < -\delta. \tag{3.9}$$

Following proposition 2, if $V$ satisfies equation 3.1 ($V$ is a Lyapunov function), then the composed function $V^d$ satisfies the same properties in some region apart from the origin. It can be further proved that the new function $V^d$ holds some important properties.

**Proposition 3.** *Let $K$ be defined as in proposition 2. Then for any $r > r_v$ that $\bar{B}(r) \subset int K$, there exists a constant $d^{**} \in (0, d^*)$ such that when $0 < d < d^{**}$,*

$$V^d(x) < V(x), \quad \forall x \in K\backslash B(r), \tag{3.10}$$

*where $d^*$ is the constant determined in proposition 2.*

**Proof.** For every $x \in K\backslash B(r)$, expanding the real function $y(d) = V(x + df(x))$ at 0, we obtain that

$$V^d(x) = V(x + df(x)) = V(x) + d\langle \nabla V(x), f(x)\rangle + d^2 h_v(x, d), \tag{3.11}$$

where $h_v(x, d) = \frac{1}{2} f^T(x) Hesse V(x + \xi f(x)) f(x)$ for some $\xi \in (0, d)$, $Hesse V$ denoting the Hessian matrix of $V$. Since $V \in C^2(\mathbb{R}^n)$ and $f$ is continuous, it follows that $h_v(x, d)$ is bounded for $x \in K\backslash B(r)$ and $d \in (0, d^*)$.

Similar to equation 3.4, there holds for some $\delta_v > 0$ that for all $x \in K\backslash B(r)$,

$$\dot{V}(x) = \langle \nabla V(x), f(x)\rangle < -\delta_v. \tag{3.12}$$

So, letting $M_v = \sup\{\|h_v(x, d)\| : x \in K\backslash B(r), 0 < d < d^*\}$, we get from equation 3.11 that for all $x \in K \setminus B(r)$ and $0 < d < d^*$,

$$V^d(x) - V(x) = d\langle \nabla V(x), f(x)\rangle + d^2 h_v(x, d) < -\delta_v d + M_v d^2. \tag{3.13}$$

Hence, for $d^{**} = \min\{d^*, \delta_v/M_v\}$, there holds that

$$V^d(x) < V(x), \quad \forall x \in K\backslash B(r) \tag{3.14}$$

whenever $0 < d < d^{**}$, as required.

With the above propositions in hand, an expansion approach to approximation of the attraction domain can be established. For this, we make a further assumption on $V$ that $\partial B(r_v) \subset N_V^{c_\rho} := \{x \in B(r_\rho) : V(x) \leq c_\rho\}$ for some $r_\rho > r_v$ that $B(r_\rho) \subset G_V(0)$ and for a certain $c_\rho < \inf_{\|x\|=r_\rho}\{V(x)\}$. Obviously, $N_V^{c_\rho}$ is a compact subset of $G_V(0)$. Let $r_0 > r_v$ satisfy that $\partial B(r_0) \subset int N_V^{c_\rho}$. Then, by proposition 3, there corresponds a constant $d^{**} > 0$ to $r_0$ such that $V^d(x) > 0$, $\langle \nabla V^d(x), f(x)\rangle < 0$ and $V^d(x) < V(x)$ in $N_V^{c_\rho} \backslash B(r_0)$ when $0 < d < d^{**}$. For each $d \in (0, d^{**})$, letting

$$G_{V^d}(0) := \{x \in \mathbb{R}^n : V^d(x) > 0, \langle \nabla V^d(x), f(x)\rangle < 0\} \bigcup B(r_0), \tag{3.15}$$

it follows that

$$\partial B(r_0) \subset \{x \in \mathbb{R}^n : V^d(x) > 0, \langle \nabla V^d(x), f(x)\rangle < 0\}. \tag{3.16}$$

Then, following propositions 2 and 3, we give another proposition:

**Proposition 4.** *For a given $d \in (0, d^{**})$ satisfying that $r_\rho < dist(0, \partial G_{V^d}(0))$, if there exists an $r_d$ between $r_\rho$ and $dist(0, \partial G_{V^d}(0))$ such that $c_d := \inf_{\|x\|=r_d}\{V^d(x)\} > c_\rho$, then $N_V^{c_\rho} \subset D_{V^d}^{c_\rho} \bigcup B(r_0)$, and for any solution $x(t)$ of system 2.1 starting in $D_{V^d}^{c_\rho} \backslash B(r_0)$, there exists a constant $T \geq 0$ such that $x(t)$ goes into $N_V^{c_\rho}$ at $t = T$, where $D_{V^d}^{c_\rho} = \{x \in B(r_d) : V^d(x) \leq c_\rho\}$.*

**Proof.** Since $V^d(x) < V(x)$ in $N_V^{c_\rho} \backslash B(r_0)$, with the definition of $N_V^{c_\rho}$, it follows that $V^d(x) < c_\rho$ for all $x \in N_V^{c_\rho} \backslash B(r_0)$. Hence, as $N_V^{c_\rho} \subset B(r_\rho) \subset B(r_d)$, $N_V^{c_\rho} \backslash B(r_0) \subset D_{V^d}^{c_\rho}$. Moreover, since $\partial B(r_0) \subset int N_V^{c_\rho}$, it follows that $\partial B(r_0) \subset D_{V^d}^{c_\rho}$ and thus $N_V^{c_\rho} \subset D_{V^d}^{c_\rho} \bigcup B(r_0)$.

It can be shown that $D_{V^d}^{c_\rho} \bigcup B(r_0)$ is a positive invariant set. In fact, for any solution $x(t)$ of system 2.1 starting in $D_{V^d}^{c_\rho}$, since $\dot{V}_d(x(t)) < 0$ whenever $x(t)$ in $D_{V^d}^{c_\rho}$, it is easy to prove that

$$V^d(x(t)) \leq V^d(x(0)) < c_\rho, \quad \forall t \geq 0, \tag{3.17}$$

which, together with the fact that $c_\rho < c_d$ and $D_{V^d}^{c_\rho} \subset B(r_d) \subset G_{V^d}(0)$, implies that $x(t)$ stays in $D_{V^d}^{c_\rho} \bigcup B(r_0)$ for all $t > 0$. In the same way, it can be proved that any solution $x(t)$ of system 2.1, starting in $B(r_0)$ stays in $D_{V^d}^{c_\rho} \bigcup B(r_0)$.

With the definition of $D_{V^d}^{c_\rho}$, it is clear to see that $D_{V^d}^{c_\rho}$ is compact. Therefore, since $\dot{V}^d(x) < 0$ for all $x \in D_{V^d}^{c_\rho} \backslash B(r_0)$, then $-\gamma := \max_{x \in D_{V^d}^{c_\rho} \backslash B(r_0)} \dot{V}_d(x)$ is negative. Hence, for any solution $x(t)$ of system 2.1 starting in $D_{V^d}^{c_\rho} \backslash B(r_0)$, there holds that

$$V^d(x(t)) = V^d(x(0)) + \int_0^t \dot{V}^d(x(\tau))d\tau \le V^d(x(0)) - \gamma t, \tag{3.18}$$

as long as $x(t)$ remains in $D_{V^d}^{c_\rho} \backslash B(r_0)$. Obviously, $x(t)$ is unable to stay in $D_{V^d}^{c_\rho} \setminus B(r_0)$ for all $t > 0$; otherwise, $V^d(x(t)) < 0$ when $t$ is sufficiently large, which contradicts that $V^d(x) > 0$ for all $x \in D_{V^d}^{c_\rho} \backslash B(r_0)$. Therefore, since $D_{V^d}^{c_\rho} \bigcup B(r_0)$ is positive invariant, it follows that $x(T) \in \partial B(r_0) \subset N_V^{c_\rho}$ for some $T > 0$.

**Remark 1.** It is clear to see in the proof that the step length $d$ is not necessarily restricted in $(0, d^{**})$. Indeed, it is sufficient for a constant $d$ if $V^d(x) > 0$, $\langle \nabla V^d(x), f(x) \rangle < 0$, and $V^d(x) < V(x)$ in $N_V^{c_\rho} \setminus B(r_0)$ for some $r_0 > r_v$.

It should be noted that we use $B(r_\rho)$ and $B(r_d)$ in proposition 4 simply to make the statement brief and clear. In fact, they can be replaced by any two neighborhoods $S_\rho$ and $S_d$ of the equilibrium point 0 that are contained in $G_{V^d}(0)$ and satisfy $S_\rho \subset S_d$. In this case, $c_\rho$ and $c_d$ are redefined as the minimal values of $V$ and $V^d$ on the boundaries of $S_\rho$ and $S_d$, respectively.

**Remark 2.** It should be noted in proposition 4 that for sufficiently small $d \in (0, d^{**})$, there must exist such an $r_d$ between $r_\rho$ and $dist(0, \partial G_{V^d}(0))$ that $c_d := \inf_{\|x\|=r_d}\{V^d(x)\} > c_\rho$. In fact, since $c_\rho < \inf_{\|x\|=r_\rho} V(x)$, there exists a constant $d \in (0, d^{**})$ such that $c_\rho < \inf_{\|x\|=r_\rho} V^d(x)$. So the continuity of $V^d(x)$ gives that $c_\rho < \inf_{\|x\|=r_d} V^d(x)$ as long as $r_d$ is sufficiently close to $r_\rho$.

The following corollary is immediately derived from proposition 4:

**Corollary 1.** *If $N_V^{c_\rho}$ is part of the attraction domain of the equilibrium point 0, so is $D_{V^d}^{c_\rho} \bigcup B(r_0)$.*

Now let $V_p$ be the Lyapunov function determined by formula 2.11. It is known that the set $N_{V_p}^{c_\rho}$ is part of the attraction domain. Hence, by proposition 4, the larger set $D_{V_p^d}^{c_\rho} \bigcup B(r_0)$ is also part of the attraction domain. Accordingly, proposition 4 suggests an expansion approach to attraction domain determination on the basis of the approximation methods introduced in section 2. The expansion approach consists of the following steps:

1. Compute the Lyapunov function $V_p(x)$ for a certain $p \geq 2$ by using the recurrence formula 2.5. Then evaluate $r_p$, $c_p$, and the set $N_{V_p}^{c_p}$ using the methods shown in Kaslik, Balint, and Balint (2005). Let

$V(x) = V_p(x)$, $r_\rho = r_p$, $c_\rho = c_p$, and $N_V^{c_\rho} = N_{V_p}^{c_p}$. Select an $r_0 > 0$ such that $\partial B(r_0) \subset int N_V^{c_\rho}$.

2. Given an initial $d \in (0, 1)$, a minimal step length $\epsilon$, and a termination time $n$ and given a positive constant $\delta < 1$, let $i = 1$ and $\Omega_d^{i-1} = N_V^{c_\rho}$.
3. Compute $V^d(x) = V(x + df(x))$. If $V^d$ satisfies that $V^d(x) > 0$, $\langle \nabla V^d(x), f(x) \rangle < 0$, and $V^d(x) < V(x)$ in $N_V^{c_\rho} \backslash B(r_0)$, go to step 4; otherwise, go to step 6.
4. Compute $G_{V^d}(0)$. If $B(r_\rho) \nsubseteq G_{V^d}(0)$, go to step 6; otherwise, compute $r_m = dist(0, \partial G_{V^d}(0))$ and $c_d = \sup_{r_\rho < r < r_m} \inf_{\|x\|=r} \{V^d(x)\}$, and solve $r_d = \max\{\|x\| : V^d(x) = c_\rho,\ r_\rho < \|x\| < r_m\}$. If $c_d > c_\rho$, go to step 5; otherwise, go to step 6.
5. Compute the region $D_{V^d}^{c_\rho} = \{x \in B(r_d) : V^d(x) \le c_\rho\}$. Let $\Omega_d^i = D_{V^d}^{c_\rho} \bigcup B(r_0)$, $V(x) = V^d$, $r_v = r_0$, $r_\rho = r_d$, and $N_V^{c_\rho} = D_{V^d}^{c_\rho}$, $i = i + 1$. If $i > n$, stop and output $\Omega_d^i$. Otherwise, select an $r_0 > r_v$ such that $\partial B(r_0) \subset int N_V^{c_\rho}$ and go to step 3;
6. Let $d = \delta d$. If $d \le \epsilon$, stop, and output $\Omega_d^i$; otherwise, go to step 3.

**Remark 3.** Since the Lyapunov function $V_p$ used in the the first step is the Taylor polynomials of the optimal Lyapunov function $V$, the field function $f$ here is required to be analytic. However, according to proposition 2, it is worth pointing out that $V_p$ can be replaced by any other Lyapunov function satisfying $C^2(\mathbb{R}^n)$ condition, and, correspondingly, the analyticity requirement on field function $f$ reduces to $C^2(\mathbb{R}^n)$.

**Example 2.** As an application example of the expansion approach, we consider the Hopfield-type neural network system described in example 1:

$$\begin{cases} \dfrac{dx}{dt} = -x + \dfrac{1}{1+e^{-x}} - \dfrac{1}{1+e^{-y}} \\ \dfrac{dy}{dt} = -y - \dfrac{1}{1+e^{-x}} + \dfrac{1}{1+e^{-y}} \end{cases}. \tag{3.19}$$

On the basis of the numerical experiment of example 1, we apply the expansion approach to system 3.19 based on $V_{20}$, with the initial $d = 0.1$ and the minimal step length $\epsilon = 10^{-8}$. The regions $\Omega_{0.1}^1$ and $\Omega_{0.1}^3$ are portrayed in Figure 2, where the $N_{V_8}^{5.9513}$ and $N_{V_{20}}^{4.4402}$ obtained in example 1 are also displayed for comparison.

It is clear in Figure 2 that through one-time iteratively circulating, the resulting set $\Omega_{0.1}^1$ is larger than $N_{V_{20}}^{4.4402}$, and after iteratively circulating three times, the resulting set $\Omega_{0.1}^3$ contains the region $N_{V_8}^{5.9513}$, which is the largest one attained in example 1.

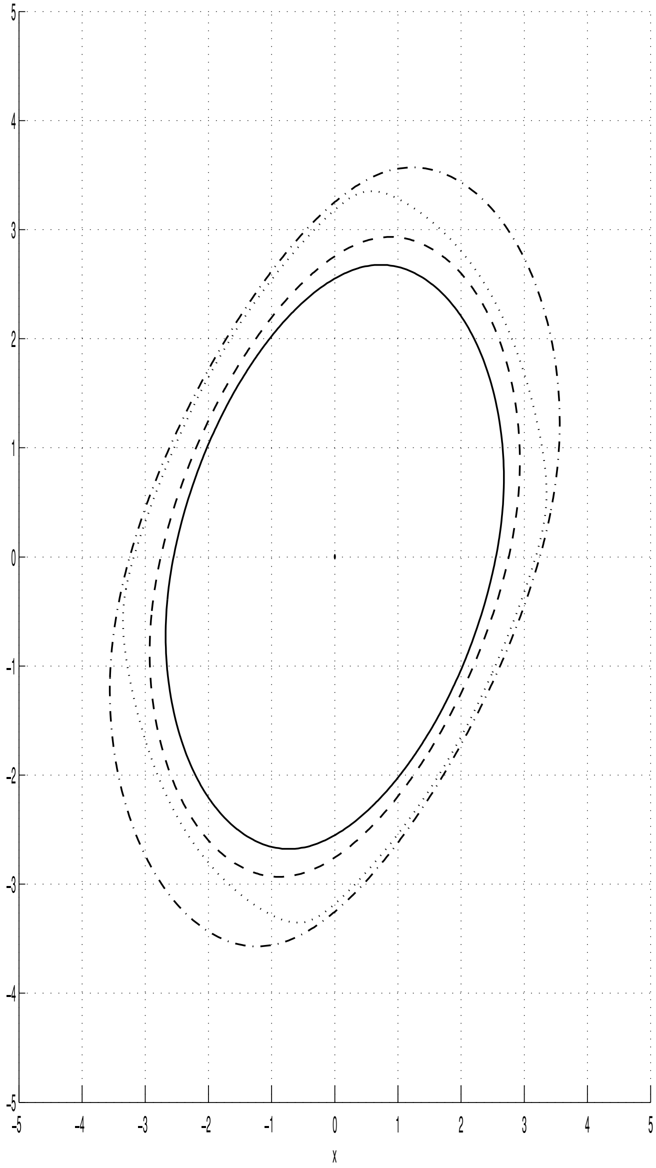


Figure 2: Boundaries of the sets $N_{V_{20}}^{4.4402}$ (solid curve), $\Omega_{0.1}^{1}$ (dash curve), $N_{V_8}^{5.9513}$ (dot curve), and $\Omega_{0.1}^{3}$ (dash-dot curve).

Below we further apply the expansion approach to the Hopfield-type neural networks considered in Kaslik, Brăescu, et al. (2005, example 2).

**Example 3.** The neural networks that we consider are

$$\begin{cases} \dfrac{dx}{dt} = -x + \dfrac{17\ln 4}{15}\tanh(y) \\ \dfrac{dy}{dt} = -y + \dfrac{17\ln 4}{15}\tanh(x) \end{cases}. \tag{3.20}$$

It is easy to check that this system has three equilibrium points: the unstable equilibrium point $(0,0)^T$ and the locally exponentially stable equilibrium points $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$. In Kaslik, Brăescu, et al. (2005, example 2), using the methods shown in Kaslik, Balint, and Balint (2005), the authors computed the sets $N_6^8$ for the equilibrium points $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, respectively, and portrayed them in a figure. For comparison, here their numerical experiments are repeated. The resulting sets $N_6^8$ for $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$ are the same as those obtained in Kaslik, Brăescu, et al. (2005). We apply the expansion approach to system 3.20, based on the Lyapunov function $V_6$, with the initial $d = 0.05$ and the minimal step length $\epsilon = 10^{-8}$. The sets $\Omega_{0.05}^1$ and $\Omega_{0.05}^2$ for the equilibrium points $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, respectively, are displayed in Figure 3. In Figure 3, the set $N_6^8$ is properly contained in the sets $\Omega_{0.05}^1$ and $\Omega_{0.05}^2$ for both equilibrium points $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, and $\Omega_{0.05}^2$ is larger than $\Omega_{0.05}^1$, which shows that the set $\Omega_{0.05}^n$ becomes larger as the iterative circulation times $n$ increases.

In the rest of the letter, we compare this iterative expansion approach with the gradual approximation method using the "embryo" stated in Kaslik, Balint, Grigis, et al. (2005) which is also based on the optimal Lyapunov function. Consider the exponentially stable equilibrium points $(\ln 4, \ln 4)^T$ of the system given in example 3. The union of the three sets whose boundaries are the dash curve and the dot curve, respectively, is the approximation of the attraction domain obtained by the gradual approximation method after two steps. The thin solid curve is the boundary of $N_6^8$. The thick solid curve is the boundary of $\Omega_{0.05}^2$ obtained by the iterative expansion approach. As shown in Figure 4, the iterative expansion approach can enlarge $N_{V_6}^8$ derived by the approximation method presented in Kaslik, Balint, and Balint (2005) in all directions, and the gradual approximation method can make a better enlargement of its first-step estimation significantly in the selected directions.

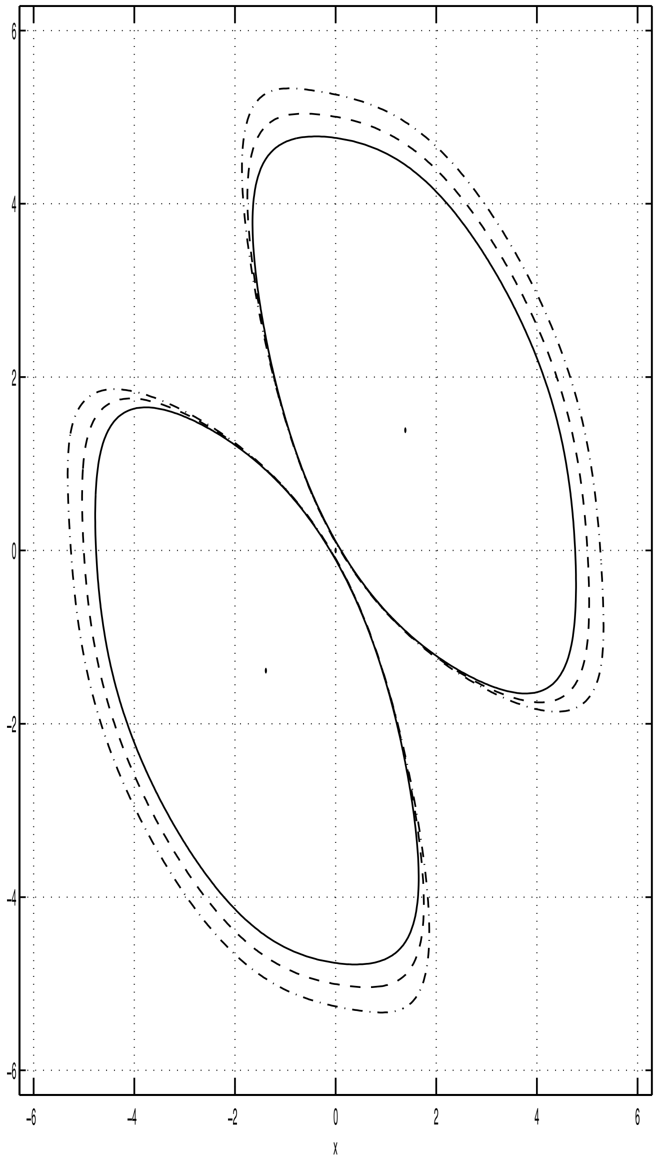


Figure 3: The boundaries of the sets $N^8_{V_6}$ (solid curve), $\Omega^1_{0.05}$ (dash curve), and $\Omega^2_{0.05}$ (dash-dot curve) for $(\ln 4, \ln 4)^T$ and $(-\ln 4, -\ln 4)^T$, respectively.

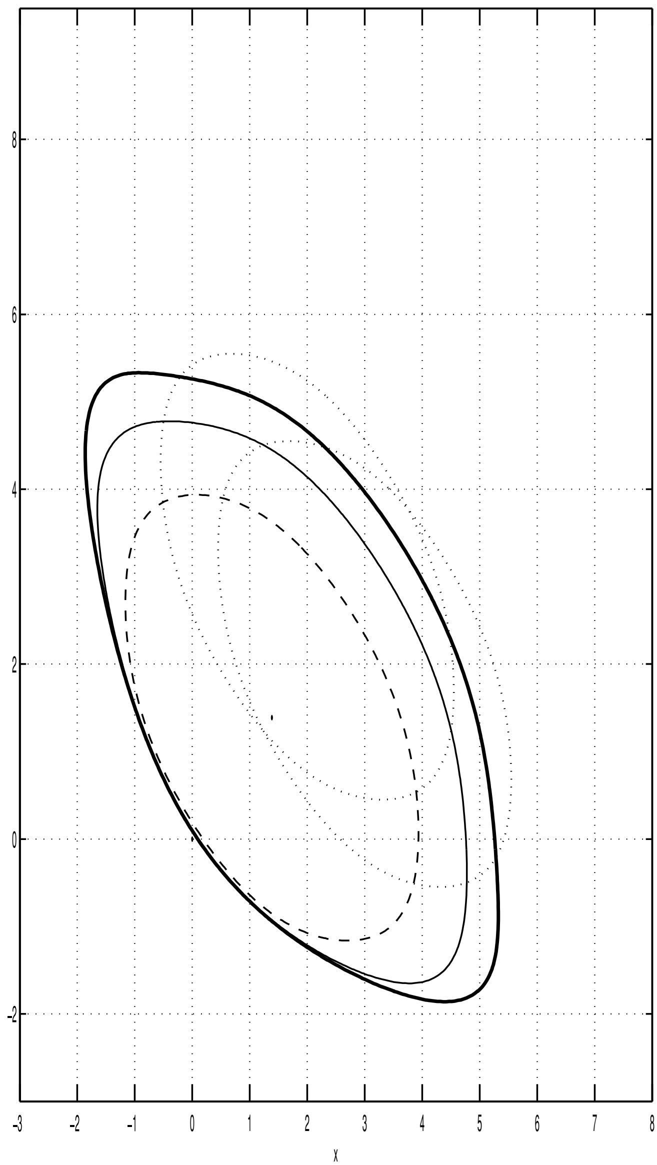


Figure 4: The boundaries of the sets $N^8_{V_6}$ (thin solid curve), the gradual approximation after two steps (dash curve and dot curve), and $\Omega^2_{0.05}$ (thick solid curve) for $(\ln 4, \ln 4)^T$.

## 4 Conclusion

In this letter, on the basis of the approximation methods developed by E. Kaslik, A. M. Balint, and St. Balint and their colleagues, we propose a new method, expansion approach, for estimating the of attraction domain of an asymptotically stable equilibrium point of nonlinear systems that can be numerically performed in several steps. The feasibility and efficiency of this approach have been not only theoretically proved but also numerically demonstrated. The numerical results obtained in applications to the Hopfield-type neural networks, including that considered in Kaslik, Brăescu et al. (2005), have demonstrated that the expansion approach is able to achieve a better attraction domain estimation. Moreover, in step 1 of the iterative expansion approach, the Lyapunov function $V_p$ can be replaced by any other Lyapunov function satisfying the $C(\mathbb{R}^n)$ condition. Therefore, the expansion approach can be extended to more general nonlinear systems, since constructing such $V_p$ requires the real analyticity of the field function $f$ of the considered systems.

Nevertheless, the proposed approach is an attempt in the direction of estimating the attraction domain of the asymptotically stable equilibrium points of neural networks. Many aspects need be improved; for example, the approach depends on the selection of the initial step length, it needs to continuously solve several optimization problems, and it consumes much computer time. We invite suggestions on related aspects of our work.

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