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A new approach to stability of neural networks with time-varying delays

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Abstract

The stability of neural networks is a prerequisite for successful applications of the networks as either associative memories or optimization solvers. Because the integration and communication delays are ubiquitous, the stability of neural networks with delays has received extensive attention. However, the approach used in the previous investigation is mainly based on Liapunov's direct method. Since the construction of Liapunov function is very skilful, there is little compatibility among the existing results. In this paper, we develop a new approach to stability analysis of Hopfield-type neural networks with time-varying delays by defining two novel quantities of nonlinear function similar to the matrix norm and the matrix measure, respectively. With the new approach, we present sufficient conditions of the stability, which are either the generalization of those existing or new. The developed approach may be also applied for any general system with time delays rather than Hopfield-type neural networks.

Keywords: Hopfield-type neural networks; Time-varying delay; Exponential stability; Minimal Lipschitz constant; Nonlinear Lipschitz measure

1. Introduction

Hopfield-type neural networks are mainly applied either as associative memories (or pattern recognition) or as optimization solvers. In both applications, the stability of the networks is prerequisite. Indeed, when they are applied as associative memories, the equilibrium points (stable states) of networks represent the stored patterns, and their stability means that the stored patterns can be retrieved even in the presence of noise. While when applied as optimization solvers, the equilibrium points of networks characterize all possible optimal solutions of the optimization problem, and stability of the networks then ensures the convergence to the optimal solutions (in particular, the global stability ensures the convergence to an optimal solution starting from any initial guess). In addition, the stability is fundamental for the network design. Because of this, the stability analysis of the Hopfield-type networks has received extensive attention (for example, Belair, Campbell & Driessche, 1996; Fang & Kincaid, 1996; Gopalsamy & He, 1994a,b; Grujic & Michel, 1991; Matsuoka, 1992; Xu & Kwong, 1995).

The early study on stability of Hopfield-type networks mainly dealt with the model (Hopfield, 1984) in which the updating and propagation are assumed to occur instantaneously (i.e. no time delay is considered). Since the integration and communication delays are ubiquitous both in biological and in artificial neural systems, the investigation on stability of the resulting delay-networks has attracted considerable interest in recent years (e.g. Baldi & Atiya, 1994; Belair, 1993; Belair et al., 1996; Gopalsamy & He, 1994a,b; Hou & Qian, 1998; Marcus & Westervelt, 1989; Pakdaman & Malta, 1998).

The approach applied to stability analysis of the neural networks with time delays in previous publications is basically the Liapunov's direct method, that is, based on construction of proper Liapunov function. It is known, however, that there exists no general rule to guide how a proper Liapunov function can be constructed for a given system. So, the construction of the Liapunov function frequently becomes very skilful, and hence, there is little compatibility among all of the stability criteria obtained so far. In this paper, we propose to develop a new stability analysis approach to the stability of the networks with time delays.

The Hopfield-type neural networks we study in this paper are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}u_i(t) = -\frac{1}{R_i}u_i(t) + \sum_{j=1}^{n} w_{ij}f_j(u_j(t-\tau_{ij}(t))) + I_i \qquad i = 1, 2, \ldots, n, \qquad (1)$$

where R_i are time constants, w_{ij} are the connection strengths,

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f_i are the input–output transfer functions, $\tau_{ij}(t)$ are the time-varying transmission delays, and I_i are the signals from outside.

Model (1) is the most popular and typical neural network model. Some other models, such as the continuous bi-directional associative memory networks, can be deduced from a special form of Eq. (1). For instance, the bi-directional associative memory networks considered by Gopalsamy and He (1994b)

$$\begin{cases} \dfrac{\mathrm{d}w_i(t)}{\mathrm{d}t} = -w_i(t) + \sum_{j=1}^{k} a_{ij} S(\lambda_j v_j(t - \sigma_{ij})) + I_i \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1,\ 2,\ \dots, k, \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = -v_i(t) + \sum_{j=1}^{k} b_{ij} S(\mu_j w_j(t - \tau_{ij})) + J_i \end{cases} \tag{2}$$

where $S(u) = \tanh(u)$, can be deduced from the model of the form (1) with $n = 2k$, $u_i(t) = w_i(t)$ for $i = 1,\ 2,\ \dots,\ k$ and $u_i(t) = v_{i-k}(t)$ for $i = k+1,\ k+2, \dots, 2k$, and $R_i = 1$ for all $i = 1,\ 2,\ \dots n$, the connection strengths w_{ij} specified by

$$\begin{cases} w_{ij} = a_{i(j-k)}, & 1 \le i < k, k < j \le n; \\ w_{ij} = b_{(i-k)j}, & k < i \le n, 1 \le j \le k; \\ w_{ij} = 0, & 1 \le i,j \le k, \text{ or } k < i,j \le n; \end{cases} \tag{3}$$

and, $f_i(u) = S(\lambda_i u)$ for $i = 1,\ 2,\ \dots,\ k$ and $f_i(u) = S(\mu_{i-k} u)$ for $i = k+1,\ k+2, \dots,\ n$.

Throughout the present paper, we will use the following assumptions.

(H1). Each transfer function f_i is monotonically increasing, and satisfies the Lipschitz condition:

$$|f_i(u) - f_i(v)| \le M_i |u - v|,$$

for some $M_i > 0$ and all real numbers u, v.

(H2). The delays $r_{ij}(t)$ are bounded, that is, there exists a constant b such that $0 \le \tau_{ij} \le b$ for all time t and $i, j = 1, 2, \dots, n$.

In neural network applications, the transfer function f_i is frequently chosen as a sigmoidal function. That is, f_i is monotonically increasing, odd and with the saturation properties that $lim_{u \to \pm\infty} f_i(u) = \pm 1$ and $f_i'(0) \ge f_i'(u)$ for all real number u. A typical such sigmoidal function, as used in dominant existing publications (Baldi & Atiya, 1994; Gopalsamy & He, 1994b; Hou & Qian, 1998; etc.) is as follows

$$f_i(u) = \tanh(\beta_i u), \tag{4}$$

where $\beta_i > 0$ is a parameter. Such a function normally satisfies assumption (H1). It should be also observed that assumption (H2) is actually very natural from the application point of view.

In Section 2, we develop a new approach to the stability analysis of an abstract system with time-varying delays. This approach then is applied to prove our main theorems on existence and stability of the equilibrium point of Eq. (1) in Sections 3 and 4, respectively.

2. The new approach to stability analysis

In this section, we develop a new approach for the stability analysis of a general time-delay system of the type

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(u(t)) + G(u_\tau(t)), \tag{5}$$

where F and G both are mappings from an open subset Ω of R^n into R^n, and $G(u_\tau(t))$ is defined as

$$G_i(u(t)) = G_i((u_1(t - \tau_{i1}(t)), u_2(t - \tau_{i2}(t))), \dots, u(t - \tau_{in}(t)))^T),$$

where $G(u) = (G_1(u), G_2(u), \dots, G_n(u))^T$.

Let R^n be the n-dimensional real vector space with vector norm $\|\cdot\|$. The vector norm that will be used may be 1-norm $\|\cdot\|_1$ and ∞-norm $\|\cdot\|_\infty$, which are defined, respectively, by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \text{ and } \|x\|_\infty = \max_{1 \le i \le n} |x_i|,$$

where $x = (x_1, x_2, \dots, x_n)^T \in R^n$. We also use the so called matrix-deduced norm $\|\cdot\|_P$, which is defined, given a non-singular matrix P and a specific vector norm $\|\cdot\|$, by $\|x\|_p = \|Px\|$. In particular, if the given matrix $P = diag(d_1, d_2, \dots, d_n)$ $(d_i \ne 0, i = 1, 2, \dots, n)$ and the vector norm is $\|\cdot\|_1$, then the matrix-deduced norm, denoted by $\|\cdot\|_{1,P}$, is given by

$$\|x\|_{1,P} = \sum_{i=1}^{n} |d_i x_i| \tag{6}$$

For a given $n \times n$ matrix A, denote by $\|A\|$ and $\mu(A)$ its matrix norm induced by the given vector norm $\|\cdot\|$ and its corresponding matrix measure, respectively. By definition (Horn & Johnson, 1991)

$$\|A\| = \sup_{x \ne 0} \frac{\|A\|}{\|x\|} \text{ and } \mu(A) = \lim_{\lambda \to 0^+} \frac{\|I + \lambda A\| - 1}{\lambda} \tag{7}$$

where I denotes the identity matrix. The matrix measure depends on the given vector norm of R^n. For example, corresponding to 1-norm and ∞-norm, the matrix measures of A, denoted by $\mu_i(A)$ and $\mu_\infty(A)$, respectively, are specified by

$$\mu_i(A) = \max_{1 \le j \le n} \left\{ a_{jj} + \sum_{i \ne j}^{n} |a_{ij}| \right\}, \tag{8}$$

$$\mu_\infty(A) = \max_{1 \le i \le n} \left\{ a_{ii} + \sum_{j \ne j}^{n} |a_{ij}| \right\}. \tag{9}$$

From Eq. (9), it is easy to see that, for all diagonal matrices $P = diag(d_1, d_2, \dots, d_n)$ $(d_i \ne 0, i = 1, 2, \dots, n)$, the following inequality holds

$$\mu_\infty(A) \le (\min_{1\le i\le n}(d_i))^{-1}\cdot\mu_\infty(PA). \tag{10}$$

For details about the matrix norm and matrix measure, see, for example, Horn and Johnson (1991).

The matrix measure plays an extremely important role in characterizing stability of the linear system with time delays. More specifically, if $\mu(A) + \|B\| < 0$, then it is known that the linear system

$$\frac{d}{dt}x(t) = Ax(t) + Bx(t-\tau), t \ge 0$$

is exponentially stable (Mori et al., 1982). Therefore, we can reasonably expect that, for some special nonlinear systems, there may be a quantity similar to the matrix measure for its stability. In the following, we present such a quantity.

For this purpose, let us first observe that when R^n is endowed with 1-norm $\|\cdot\|$ the corresponding matrix measure of A, $\mu_1(A)$, can be equivalently defined by

$$\mu_1(A) = \sup_{x\ne 0}\frac{\langle A(x), \mathrm{sgn}(x)\rangle}{\|x\|_1}, \tag{11}$$

where $\langle u, v\rangle$ represents the inner product of vectors u, $v \in R^n$, and sgn(z) is the vector whose i-th component is the sign of the i-th component of z, i.e. $\mathrm{sgn}(z)=(\mathrm{sgn}(z_1),\mathrm{sgn}(z_2),\ldots,\mathrm{sgn}(z_n))^T$ and for any real t, sgn(t) is defined by

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0; \\ 0 & t + 0; \\ -1 & t < 0. \end{cases}$$

We also notice that, in functional analysis terms, a function f from an open subset Ω of R^n into R^n is said to be a Lipschitz operator on Ω whenever there exists a nonnegative constant M such that, for any x, $y \in \Omega$,

$$\|f(x) - f(y)\| \le M\|x - y\|,$$

where M is called the Lipschitz constant of f. Furthermore, the *minimal Lipschitz constant* (MLC in brief) of f is defined by Soderlind (1984)

$$L(f) = \sup_{x,y\in\Omega, x\ne y}\frac{\|f(x) - f(y)\|}{\|x - y\|}. \tag{12}$$

It is clear that a matrix A is a Lipschitz operator on R^n, and its MLC is nothing other than the matrix norm, i.e. $L(A) = \|A\|$. For the general case, the MLC, $L(\cdot)$, is known to be a semi-norm of the space of Lipschitz operators (Soderlind, 1984), that is, $L(\cdot)$ satisfies the following:

1. $L(f) \ge 0$, and, $L(f) = 0$ if and only if f is a constant operator;
2. $L(f + g) \le L(f) + L(g)$ for any Lipschitz operators f and g;
3. $L(\lambda f) = |\lambda| L(f)$ for any real number λ;
4. $L(f\cdot g) \le L(f)L(g)$ for any pair of Lipschitz operators f and g;

where $f\cdot g$ represents the compound operator of f and g, i.e. $f\cdot g(x) = f(g(x))$ for all vector x in Ω.

Notice that the MLC of a Lipschitz operator varies with the norm adopted. Throughout this paper, we denote $L_1(f)$ and $L_\infty(f)$ the MLCs of the Lipschitz operator f corresponding to 1-norm and ∞-norm, respectively.

With these preparations, we now introduce a new concept.

Definition 1. Let R^n be equipped with 1-norm $\|\cdot\|_1$, and f be a Lipschitz operator on a subset Ω of R^n. The constant

$$m(f) = \sup_{x,y\in\Omega, x\ne y}\frac{\langle f(x) - f(y), \mathrm{sgn}(x-y)\rangle}{\|x - y\|_1}$$

is called the nonlinear Lipschitz measure (NLM) of f on Ω.

We observe that, for any vector z in R^n,

$$\begin{cases} \|z\|_1 = \langle z, \mathrm{sgn}(z)\rangle \text{ and} \\ \|z\|_1 \ge \langle z, \mathrm{sgn}(y)\rangle \text{ for all } y \in R^n. \end{cases} \tag{13}$$

From Definition 1, this immediately implies that $m(f) \le L(f)$ for any Lipschitz operator f. Moreover, if f is a matrix, it is seen from Eqs. (11) and (12) that the NLM of f coincides exactly with its matrix measure. So, the NLM notion of a nonlinear Lipschitz operator is a direct generalization of matrix measure in linear operator (matrix). The NLM of a Lipschitz operator has many useful properties similar to matrix measure, some of which can be listed as follows.

Lemma 1. If $L(f)$ and $m(f)$ are the MLC and NLM of a Lipschitz operator f on a subset Ω of R^n, then

1. $m(f + g) \le m(f) + m(g)$;
2. $-L_1(f) \le -m(-f) \le m(f) \le L_1(f)$;
3. $m(\lambda f) = \lambda m(f)$ for all $\lambda \ge 0$; and
4. $m(\alpha I + f) = \alpha + m(f)$ for any real number α;

where $\alpha I + f$ is the operator mapping each vector $x \in \Omega$ onto $\alpha x + f(x)$.

Proof. It is immediate from the definitions of MLC and NLM.□

In addition, it should be noticed that when the Lipschitz operator f is continuously differentiable on Ω, one can easily show that

$$L_1(f) = \sup_{x\in\Omega}\|Df(x)\|_1 \text{ and } m(f) = \sup_{x\in\Omega}\mu_1(Df(x)),$$

where $Df(x)$ is the Jacobian matrix of f at x (cf. Peng & Xu, 1997).

Lemma 2. Assume R^n is endowed with 1-norm $\|\cdot\|$, and f is a Lipschitz operator on a subset Ω of R^n. If $m(f) < 0$, then

1. f is one-to-one, that is, $f(x) \neq f(y)$ whenever $x \neq y$. Moreover, if Ω of R^n, the range of f (denoted by $R(f)$) is the whole space Ω of R^n, therefore, f is a homeomorphism of Ω of R^n; and
2. the inverse function f^{-1} is a Lipschitz operator on $R(f)$ with

$$L(f^{-1}) \leq \frac{1}{-m(f)}. \tag{14}$$

Proof.

1. Since, for any $x, y \in R^n$,

$$\langle f(x) - f(y), \mathrm{sgn}(x-y) \rangle \leq m(f)\|x-y\|_1.$$

Then, by Eq. (13), we have

$$\|f(x) - f(y)\|_1 \geq -\langle f(x) - f(y), \mathrm{sgn}(x-y) \rangle \geq -m(f)\|x-y\|_1 \tag{15}$$

which shows that f is one-to-one because $-m(f) > 0$. Furthermore, if $\Omega = R^n$, let y be fixed and $\|x\|_1 \to \infty$, then the above inequality shows $\|f(x)\|_1 \to \infty$, that is, f is norm-coercive. By the norm-coercive theorem (e.g. Ortega & Rheinbold, 1970), f, therefore, is a homeomorphism of R^n.

2. Since f is one-to-one, the inverse f^{-1} is then well defined on $R(f)$. The estimation (14) is immediate from Eq. (15). This completes the proof.□

We also need a mathematical analysis result:

Lemma 3. If $a > c > 0$, then, for each nonnegative real number b, the equation

$$0 = \lambda - a + c\mathrm{e}^{\lambda b}$$

has a unique positive solution.

Proof. Denote $h(\lambda) - a + c\mathrm{e}^{\lambda b}$. Then it is seen that $h(\lambda > 0$ for all $\lambda > a$, and $h(0) < 0$ because $a > c > 0$. Hence, there exists a λ in (0, a) such that $h(\lambda) = 0$. Such λ is unique because $h(\lambda)$ is strictly monotone in the interval (0, ∞). The proof is completed.□

Definition 2. The time-delay system (5) is said to be exponentially stable on a neighborhood Ω of an equilibrium point x^* if there are two positive constants a and M such that

$$\|x - x^*\| \geq M \cdot \mathrm{e}^{\alpha(t-t_o)} \sup_{t_0 - b \leq s \leq t} \|x_0(s) - x^*\|, t \geq t_0,$$

where $b = \sup\{\tau_{ij}(t) : 1 \leq i,j \leq n, t \in R\}$, and $x(t)$ is the unique trajectory of the system initiated from $x_0(t) \in \Omega$ with $s \in (t_0 - b, t_0]$.

Moreover, if the system (5) has a unique equilibrium point, and it is exponentially stable on the whole space R^n, then Eq. (5) is said to be exponentially global stable.

With the above Lemmas 1–3, we now state and prove our main theorem in this section.

Theorem 1. Suppose that Ω is a neighborhood of an equilibrium x^* of the system (5), F and G both are Lipschitz operators on Ω, and the time-varying transmission delays $b = \sup\{\tau_{ij}(t) : 1 \leq i,j \leq n, t \in R\} < \infty$.

1. In the case when R^n is endowed with 1-norm, if, for some $A-diag(a_1, a_2, \ldots, a_n)$, with $a_i > 0$, $m(FA) + L_1(GA) < 0$ here, $m(FA)$ and $L_l(GA)$ denote the NLM of FA and the MLC of GA on $A^{-1}(\Omega)$, respectively, then the time-delay system (5) is exponentially stable on Ω. Moreover, the exponential decay estimate is given by

$$\|x(t) - y(t)\|_1 \leq \mathrm{e}^{-\lambda(t-t_0)} \cdot \sup_{t_0 - b \leq s \leq t_0} \|x_0(s)\|_1 \text{ for all } t \geq t_0, \tag{16}$$

where λ is the unique positive solution of the equation

$$0 = \lambda \cdot \min_{1 \leq i \leq n} a_i + m(FA) + L_1(GA)\mathrm{e}^{b\lambda}. \tag{17}$$

2. In the case when R^n is endowed with ∞-norm, if F is a matrix, and, for some $A-diag(a_1, a_2, \ldots, a_n)$ with $a_i > 0$, $\mu_\infty(Af) + L_\infty(AG) < 0$, then the time-delay system (5) is exponentially stable on Ω. Moreover, the exponential decay estimate is governed by

$$\|x(t) - y(t)\|_\infty \leq \mathrm{e}^{-\mu(t-t_0)} \cdot \sup_{t_0 - b \leq s \leq t_0} \|x_0(s) - y_0(s)\|_\infty, \tag{18}$$

where μ is the unique positive solution of the equation

$$0 = \mu \cdot \min_{1 \leq i \leq n} a_i + \mu_\infty(AF) + L_\infty(AG)\mathrm{e}^{\mu b.} \tag{19}$$

In either case, $x(t)$ and $y(t)$ are the trajectories of Eq. (5) initiated, respectively, from $x_0(s)$ and $y_0(s)$, where $x_0(s), y_0(s) \in \Omega$ for all $s \in (t_0 - b, t_0]$.

Proof. Let $\mu(t) = x(t) - y(t)$ for all $t \geq t_0$.

1. From Eq. (13), we find that, for any $s > 0$,

$$\frac{\|u(t)\|_1 - \|u(t-s)\|_1}{s} \leq \frac{1}{s}\langle u(t) - u(t-s), \mathrm{sgn}(u(t)) \rangle.$$

Hence, from Eq. (5), the derivatives of $\|u(t)\|_1$, which exist almost everywhere in (t_0, $+\infty$) because the function $t \mapsto \|u(t)\|_1$ is absolutely continuous in (t_0, $+\infty$) (e.g. Rudin, 1974, Theorem 8.19), satisfy almost everywhere in (t_0, $+\infty$) that

$$\frac{\mathrm{d}\|u(t)\|_1}{\mathrm{d}t} \le \langle \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \mathrm{sgn}(u(t))\rangle$$

$$= \langle F(x(t)) - F(y(t)), \mathrm{sgn}(A^{-1}u(t))\rangle + \langle G(x_\tau(t)) - G(y_\tau(t)), \mathrm{sgn}(u(t))\rangle$$

$$\le m(FA\|A^{-1}u(t)\|_1 + L_1(GA)\|A^{-1}x_\tau(t) - A^{-1}y_\tau(t)\|_1$$

$$\le \{m(FA)\|u(t)\|_1 + L_1(GA)u_t\}\cdot\left(\min_{1\le t\le n} a_i\right)^{-1}$$

where $u_t = \sup_{t-b\le s\le t}\|u(s)\|_1$. Since $m(FA) + L_1(GA) < 0$ and $L_1(GA) \ge 0$, then, by Halanay's inequality (Driver, 1977, pp. 389–391),

$$\|u(t)\|_1 \le \mathrm{e}^{-\lambda(t-t_0)}\cdot \sup_{t_0-b\le s\le t_0} \|u(s)\|_1,$$

where λ is the unique positive solution of the equation

$$\lambda = -m(FA)\cdot\left(\min_{1\le t\le n} a_i\right)^{-1} - L_1(GA)\cdot\left(\min_{1\le t\le n}\right)^{-1}\mathrm{e}^{b\lambda}.$$

Consequently, Eq. (16) directly follows.

2. By assumption, F is a matrix in this case, so from Eq. (5), we obtain

$$u(t) = \mathrm{e}^{F(t-s)}u(s) + \int_s^t \mathrm{e}^{F(t-r)}[G(x_\tau(r)) - G(y_\tau(r))]\mathrm{d}r$$

for all $t > s \ge t_0$. It is known that, for matrix measure $\mu_\infty(F)$, $\|\mathrm{e}^{Ft}\|_\infty \le \mathrm{e}^{\mu_\infty(F)t}$ holds for all real number t (Horn & Johnson, 1991). So, we have

$$\frac{\|u(t)\|_\infty - \|u(s)\|_\infty}{t-s} \le \frac{1}{t-s}\{(\|\mathrm{e}^{F(t-s)}\| - 1)\cdot\|u(s)\|_\infty + \int_s^t \|\mathrm{e}^{F(t-r)}\|\cdot\|G(x_\tau(r)) - G(y_\tau(r))\|_\infty \mathrm{d}r\}$$

$$\le \frac{1}{t-s}\{(\mathrm{e}^{\mu_\infty(F)(t-s)} - 1)\cdot\|u(s)\|_\infty + \left(\min_{1\le i\le n}\right)^{-1}\cdot L_\infty(AG) \times \int_s^t \mathrm{e}^{\mu_\infty(F)(t-r)}\|u_\tau(r)\|_\infty \mathrm{d}r\}.$$

Let $s \to t$ in the above inequality and use the estimation (10), then we deduce that the derivatives of $\|u(t)\|_\infty$ satisfy almost everywhere in (t_0, $+\infty$) (Rudin, 1974, Theorem 8.2) that

$$\frac{\mathrm{d}\|u(t)\|_\infty}{\mathrm{d}t} \le \mu_\infty(F)\|u(t)\|_\infty + L_\infty(AG)\|Au_\tau(t)\|_\infty\left(\min_{1\le i\le n}\right)^{-1}$$

$$\le \{\mu_\infty(AF)\|u(t)\|_\infty + L_\infty(AG)u_t\}\cdot\left(\min_{1\le t\le n}\right)^{-1}.$$

This, combined with Halanay's inequality (Driver, 1977, pp. 389–391), implies Eq. (17). With this, the proof is completed.□

Theorem 1 reveals that the MLC and NLM of a Lipschitz operator are tightly related to the stability of a nonlinear system in a similar way to the matrix norm and matrix measure in linear system stability analysis. Thus, the NLM of a Lipschitz operator is exactly the quantity we want to find in characterizing the stability of a nonlinear system. In the subsequent sections, we shall apply this quantity to derive some more general sufficient conditions of the stability of neural network (1) than those published recently. However, before that, the existence and uniqueness of the equilibrium point of the neural network system are studied in the next section by means of the so introduced quantity.

3. Existence and uniqueness of equilibrium

As mentioned previously, the neural networks model (1) is applied either as associative memories or as optimization solvers. When applied as optimization solvers, the existence (uniqueness) of an equilibrium point ensures in most cases existence (uniqueness) of optimal solution of the optimization problem. While when used as associative memories, the existence of an equilibrium point means the possibility of storing information in the networks, and the uniqueness in a specific region then implies that those stored memories can be distinguished (since the uniqueness implies that the stored memories can be separated from each other). Accordingly, it is necessary to discuss the existence and local uniqueness of an equilibrium point of Eq. (1). In this section, we apply the NLM of a Lipschitz operator to prove some sufficient conditions for the existence and uniqueness of equilibrium of network (1).

Theorem 2. Suppose Ω is a subset of R^n, and there exists a set of positive real numbers $r_i (i = 1, 2, \ldots, n)$ such that

$$w_{jj} + \sum_{i\ne j}^{n} \frac{r_j}{r_i}|w_{ij}| < (m_j R_j)^{-1},\ j = 1, 2, \ldots, n \tag{20}$$

where m_i is the MLC of f_i on Ω_i, the projection of Ω to the i-th axis. Then, corresponding to each group of external input I_i, the equilibrium point of Eq. (1) is unique in Ω. Furthermore, if $\Omega = R^n$, then there exists an equilibrium point.

Proof. Let $P = diag(r_1, r_2, \ldots, r_n)$, and define the function $F : R^n \to R^n$ by:

$$(F(u))_i = -\frac{u_i}{R_i} + \sum_{j=1}^{n} w_{ij} f_j(u_j) + I_i, i = 1, 2, \ldots, n,$$

where $u = (u_1, u_2, \ldots, u_n)^T$, $F(u) = (F_1(u), F_2(u), \ldots,$

$F_n(u))^T$, and let

$$\mu_j = w_{jj} + \sum_{i\neq j}^{n} \frac{r_j}{r_i}|w_{ij}|.$$

Since, for each $i = 1, 2, \ldots, n$, the transfer function f_i is increasing, or equivalently,

$$(f_i(t) - f_i(s))\mathrm{sgn}(t - s) = |f_i(t) - f_i(s)| \text{ for all } t, s \in R$$

it then follows that, for all $x, y \in P^{-1}(\Omega)$,

$$\langle P^{-1}F(Px) - P^{-1}F(Py), \mathrm{sgn}(x - y)\rangle$$

$$= \sum_{i=1}^{n}\left\{-\frac{x_i - y_i}{R_i} + \sum_{j=1}^{n} r_i^{-1} w_{ij}(f_j(r_jx_j) - f_j(r_jy_j)\right\}\mathrm{sgn}(x_i - y_i)$$

$$= -\sum_{i=1}^{n}\frac{|x_i - y_i|}{R_i} + \sum_{j=1}^{n}\sum_{i=1}^{n} r_i^{-1} w_{ij}(f_j(r_jx_j) - f_j(r_jy_j))\mathrm{sgn}(x_i - y_i)$$

$$= -\sum_{j=1}^{n}\frac{|x_j - y_j|}{R_j} + \sum_{j=1}^{n}\{r_j^{-1} w_{jj}[f_j(r_jx_j) - f_j(r_jy_j)]\mathrm{sgn}(x_j - y_j) + \sum_{i\neq j}^{n} r_i^{-1} w_{ij}[f_j(r_jx_j) - f_j(r_jy_j)]\mathrm{sgn}(x_i - y_i)\}$$

$$\leq -\sum_{j=1}^{n}\frac{|x_j - y_j|}{R_j} + \sum_{j=1}^{n}\{r_j^{-1} w_{jj}|f_j(r_jx_j) - f_j(r_jy_j)| + \sum_{i\neq j}^{n} r_i^{-1}|w_{ij}|\cdot|f_j(r_jx_j) - f_j(r_jy_j)|\}$$

$$\leq -\sum_{j=1}^{n}\frac{|x_j - y_j|}{R_j} + \sum_{j=1}^{n}|f_j(r_jx_j) - f_j(r_jy_j)|\cdot r_j^{-1}\mu_j$$

$$\leq -\sum_{j=1}^{n} R_j^{-1}(1 - \mu_j^{+}\cdot m_jR_j)\cdot|x_j - y_j|$$

$$\leq -\frac{\min_{1\leq j\leq n}\{1 - \mu_j^{+} m_jR_j)\}}{\max_{1\leq j\leq n} R_j}\|x - y\|_1,$$

where $\mu_j^{+} = \max\{0, \mu_j\}$. From the condition (20), this then implies that the NLM of the operator $P^{-1}FP$ is less than zero, i.e. $m(P^{-1}FP) < 0$. Therefore, by Lemma 2, $P^{-1}FP$ is one-to-one, thus, there is no more than one $x^* \in \Omega$ such that $P^{-1}FP(x^*) = 0$. Since P is non-singular, the equilibrium point of Eq. (1) is unique in Ω.

Furthermore, if $\Omega = R^n$, then, by Lemma 2, $P^{-1}FP$ is a homeomorphism of R^n, thus, F is a homeomorphism of R^n because P is non-singular. Therefore, there is a unique u^* in R^n such that $F(u^*) = 0$, that is, u^* is the unique equilibrium point of Eq. (1). The proof is completed.□

Remark 1. Gopalsamy and He (1994b) discussed the global existence and uniqueness of the equilibrium point of the bi-directional associative memory networks (2) in certain assumptions. They stated that if

$$\left.\begin{array}{l}\lambda_i \sum_{j=1}^{k}|a_{ji}| \leq c < 1 \\ \mu_i \sum_{j=1}^{k}|b_{ji}| \leq c < 1\end{array}\right\}, i = 1, 2, \ldots, n \tag{21}$$

for some $c > 0$, then the network (2) has a unique equilibrium point corresponding to each set of pair of input vectors $I = (I_1, I_2, \ldots, I_k)$ and $J = (J_1, J_2, \ldots, J_k)$.

We observe that, with $n = 2k$ and w_{ij} specified as in Eq. (3), Eq. (21) clearly implies Eq. (20) for $r_i = 1$, ($i = 1, 2, \ldots, n$). So, the theorem of Gopalsamy and He (1994b, Theorem 1) is a special case of Theorem 2.

The following example further shows that even if Eq. (21) is not satisfied we can actually choose properly r_i such that Eq. (20) holds. Therefore, the uniqueness of the equilibrium point of corresponding memory networks can be deduced.

Example 1. Consider the following bi-directional associative memory networks:

$$\begin{cases}\dfrac{du(t)}{dt} = -u(t) + 2S(v(t - \sigma(t))) + I \\ \dfrac{dv(t)}{dt} = -v(t) + 0.4S(u(t - \tau(t))) + J\end{cases}$$

where the delays $\sigma(t)$ and $\tau(t)$ are bounded. Taking $n = 2$ and $W = (w_{ij})_{2\times 2}$ such that $w_{11} = w_{22} = 0$, $w_{12} = 2$, and $w_{21} = 0.4$, then it is easy to see that Eq. (20) holds with $r_1 = 0.9$ and $r_2 = 0.4$ (therefore, the above networks has uniquely an equilibrium point), but Eq. (21) is not satisfied.

4. Stability analysis of neural networks

In this section, we apply Theorem 1 to derive some generic sufficient conditions for the stability of neural network (1). The presented conditions are either the generalization of those existing or new.

Theorem 3. Suppose u^* is an equilibrium point of Eq. (1), Ω is a neighborhood of u^*. If there exists a set of $r_i > 0, (I = 1, 2, \ldots, n)$ such that

$$m_jR_j \sum_{i=1}^{n} \frac{r_j}{r_i}|w_{ij}| < 1, j = 1, 2, \ldots, n \tag{22}$$

or

$$R_j \sum_{i=1}^{n} \frac{r_i}{r_j} |w_{ji}| m_i < 1, j = 1, 2, \ldots, n \tag{23}$$

where m_i is the MLC of f_i on Ω_i (the projection of Ω to the i-th axis), then, corresponding to each set of external inputs I_i, Eq. (1) is exponentially stable on Ω. Specifically, if $u(t)$ is the trajectory of Eq. (1) initiated from $u_0(s) \in \Omega$ with $s \in (t_0 - b, t_0]$, then

$$\|u(t) - u^*\| \leq e^{-\lambda(t-t_0)} \cdot \frac{\max_{1\leq i\leq n} r_i}{\min_{1\leq i\leq n} r_i} \cdot \sup_{t_0 - b \leq s \leq t_0} \|u_0(s) - u^*\| \tag{24}$$

where the vector norm $\|\cdot\|$ is respectively 1-norm and ∞-norm corresponding to Eqs. (22) and (23), and λ is the unique positive solution of the equation

$$\lambda \cdot \min_{1\leq i\leq n} R_i - 1 + c e^{\lambda b} = 0, \tag{25}$$

with

$$c = \max_{1\leq j\leq n} \left\{ m_j R_j r_j \sum_{i=1}^{n} r_i^{-1} |w_{ij}| \right\}$$

when Eq. (22) holds, or

$$c = \max_{1\leq j\leq n} \left\{ m_j R_j r_j^{-1} \sum_{i=1}^{n} r_i^{-1} |w_{ji}| \right\}$$

when Eq. (23) holds.

Proof. Let $P = diag(r_1, r_2, \ldots, r_n)$, $R = diag(R_1, R_2, \ldots, R_n)$ and $F = -R^{-1}$. Define $G : R^n \to R^n$ by

$$(G(u))_i = \sum_{j=1}^{n} w_{ij} f_j(u_j) + I_i, i = 1, 2, \ldots, n.$$

Then the network (1) can be equivalently cast as the time-delay system of type (5).

Let $A = R$. We then have that $m(FA) = -1$, and since, for all $x, y \in A^{-1}P^{-1}(\Omega)$,

$$\|P^{-1}G(PAx) - P^{-1}G(PAy)\|_1$$
$$= \sum_{i=1}^{n} r_i^{-1} \left| \sum_{j=1}^{n} w_{ij}(f_j(r_j R_j x_j) - f_j(r_j R_j y_j) \right|$$
$$\leq \sum_{i=1}^{n} r_i^{-1} \sum_{j=1}^{n} |w_{ij}| \cdot m_j r_j R_j |x_j - y_j|$$
$$\leq \sum_{j=1}^{n} |x_j - y_j| \cdot m_j R_j \sum_{i=1}^{n} |w_{ij}| \cdot \frac{r_j}{r_i} \leq c \cdot \|x - y\|_1,$$

we get that $L_1(P^{-1}GPA) \leq c$. Hence, when Eq. (22) is satisfied, it is seen that $m(FA) + L_1(P^{-1}GPA) < 0$ since $L_1(P^{-1}GPA) \leq c < 1$. Consequently, by Theorem 1, the trajectory $x(t)$ of the time-delay system

$$\frac{dx(t)}{dt} = F(x(t)) + P^{-1}GP(x_\tau(t)) \tag{26}$$

satisfies that, for all $t \geq t_0$,

$$\|x(t) - P^{-1}u^*\|_1 \leq e^{-\lambda(t-t_0)} \cdot \sup_{t_0 - b \leq a \leq t_0} \|x(s) - P^{-1}u^*\|_1$$

in which λ is the unique positive solution of Eq. (25). Thus, the decay estimate (24) directly follows for the solution $u(t)$ of system (1) if we notice that $x(t) = P^{-1}u(t)$ in this case is the solution of Eq. (26).

Now, assume Eq. (23) holds. Since $\mu_\infty(AF) = -1$, and for all $x, y \in P^{-1}(\Omega)$,

$$|(P^{-1}AGP(x))_i - (P^{-1}AGP(y))_i|$$
$$\leq r_i^{-1} R_i \left| \sum_{j=1}^{n} w_{ij}[f_j(r_j x_j) - f_j(r_j y_j) \right|$$
$$\leq r_i^{-1} R_i \sum_{j=1}^{n} |w_{ij}| \cdot m_j r_j \cdot |x_j - y_j|,$$

we find that $\mu_\infty(AF) + L_\infty(AP^{-1}GP) < 0$. Hence, by Theorem 1, the solution $x(t)$ of the system

$$\frac{dx(t)}{dt} = F(x(t)) + P^{-1}GP(x_\tau(t)) \tag{27}$$

satisfies

$$\|x(t) - P^{-1}u^*\|_\infty \leq e^{-\lambda(t-t_0)} \cdot \sup_{t_0 - b \leq s \leq t_0} \|x(s) - P^{-1}u^*\|_\infty,$$

where λ is the unique positive solution of Eq. (25) corresponding to Eq. (23). Therefore, Eq. (24) is derived from the relation between the solutions of Eqs. (1) and (27). With this, the proof of Theorem 3 is completed.□

Remark 2. It should be noted that if Ω is the whole space R^n in Theorem 3, then inequality (22) implies Eq. (20), thus, the unique equilibrium point of Eq. (1) exists by Theorem 2, and then Eq. (1) is exponentially global stable. Gopalsamy and He (1994a) and Hou and Qian (1998) have studied the global stability of Eq. (1) in the special case when the transfer function f_i is specified as in Eq. (4), and Gopalsamy and He (1994b) discussed the global stability of the bi-directional associative memory networks (2). All the results they obtained extended those previously known, but are special cases of Theorem 3 proved here. To make these clear, let us recall that a matrix $A = (a_{ij})_{n\times n}$ is a quasi-dominant matrix (Moylan, 1977) if there exists a set of real number $d_i > 0$ such that

$$d_i a_{ii} > \sum_{j=1}^{n} d_j |a_{ij}| \text{ for all } i = 1, 2, \ldots, n \tag{28}$$

Corollary 1. If the transfer function f_i is defined as in

Eq. (4) and the matrix

$$M = diag(R_1^{-1}, R_2^{-1}, \ldots, R_n^{-1}) - (\beta_j |w_{ij}|)_{n\times n}$$

is quasi-dominant, then Eq. (1) is exponentially global stable. Moreover, the exponential decay estimate is governed by

$$\|u(t) - u^*\|_\infty \le e^{-\lambda(t-t_0)} \cdot \frac{\max_{1\le i\le n} d_i}{\min_{1\le i\le n} d_i} \cdot \sup_{t_0 - b\le a\le t_0} \|u(s) - u^*\|_\infty$$

where u^* is the unique equilibrium point of Eq. (1), d_i as in Eq. (28), and λ is the unique positive solution of the equation

$$\lambda \cdot \min_{1\le i\le n} R_i - 1 + c e^{\lambda b} = 0,$$

with

$$c = \max_{1\le i\le n} \left\{ d_i^{-1} R_i \sum_{j=1}^{n} |w_{ij}| \beta_j d_j \right\}$$

and $b = \sup\{\tau_{ij}(t) : 1 \le i,j \le n, t \in R\}$.

Proof. Since M is quasi-dominant, then, by Eq. (28), there is a set of positive numbers d_i ($i = 1, 2, \ldots n$) such that

$$d_i(R_i^{-1} - \beta_i)|w_{ii}|) > \sum_{j\ne i}^{n} |w_{ij}| \beta_j d_j$$

for every $i = 1, 2, \ldots, n$. This, combined with $m_i \le \beta_i$, implies Eq. (23). Thus, Theorem 3 implies Corollary 1.□

Corollary 2. If Eq. (21) is satisfied, then the bi-directional associative memory network (2) is exponentially global stable. Moreover, the exponential decay estimate is governed by

$$\sum_{i=1}^{k} (|w_i(t) = w_i^*| + |v_i(t) - v_i^*|) \le e^{-\lambda(t-t_0)} \sup_{t_0 - b\le s\le t_0} \sum_{i=1}^{k} (|w_i(s) - w_i^*| + |v_i(s) - v_i^*|),$$

where (w^*, v^*) is the unique equilibrium point of Eq. (2), and λ is the unique positive solution of the equation

$$\lambda - 1 + \max_{1\le i\le n} \left\{ \lambda_i \sum_{j=1}^{k} |a_{ji}|, \mu_i \sum_{j=1}^{k} |b_{ij}| \right\} \cdot e^{\lambda b} = 0,$$

with $b = \max\{\sigma_{ij}, \tau_{ij} : 1 \le i,j \le n\}$.

Proof. It directly follows from Theorem 3 with $n = 2k, u(t) = (w_1, \ldots, w_k, v_1, \ldots, v_k)$ and w_{ij} defined as in Eq. (3).□

We remark that Corollary 1 here is basically the same with that in Hou and Qian (1998). However, the exponential decay rate in Corollary 1 is different from that in Hou and Qian (1998).

The exponentially global stability of Eq. (2) stated in Corollary 2 is due to Gopalsamy and He (1994b). Nevertheless, they applied Liapunov direct method and have not established the exact exponential decay estimation as that in Corollary 2. This reveals the advantage of NLM approach developed in this paper over the Liapunov direct method in conducting stability of neural network systems. It should be noted also that the techniques employed by Hou and Qian (1998) (also Liapunov direct method) cannot be applied to deduce the results of Gopalsamy and He (1994b).

Remark 3. When networks are applied as associative memories, the equilibrium point of networks corresponds to the stored pattern, and the stability of each equilibrium point characterizes the error-correction capability of the corresponding stored pattern. The larger the attraction basin of each equilibrium point, the stronger the error correction of the corresponding stored pattern. So, it is extremely important to test whether a given region belongs to the contraction basin of a specific equilibrium point of the system. As compared with other results (e.g. Gopalsamy & He, 1994a,b; Hou & Qian, 1998), the main benefit of Theorem 3 is that it can exactly provide us with such a test (i.e. test whether a given neighborhood of each equilibrium point of Eq. (1) can be an attraction basin).

Let us present a concrete example to make this benefit clear.

Example 2. Consider the following Hopfield-type neural network:

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + f_2(u_2(t - \tau_{12}(t))) \\ \dfrac{du_2(t)}{dt} = -u_2(t) + f_1(u_1(t - \tau_{21}(t))) \end{cases} \tag{29}$$

where the transfer functions f_1 and f_2 are defined as in Eq. (4), but with $\beta_1 = \beta_2 > 1$.

This network is not globally stable because we easily check that there exist at least three equilibrium points $(0, 0)^T$, $(a, a)^T$ and $(-a, -a)^T$ for some $0 < a < 1$ and that the equilibrium point $(0, 0)^T$ is not stable. This implies that all the results in Gopalsamy and He (1994a,b) and Hou and Qian (1998) cannot be applied. We will show that our Theorem 3, however, works. Let us consider the stability of the equilibrium point $(a, a)^T$ as an example.

Suppose given the specific neighborhood $\Omega = \{(x_1, x_2)^T : |x_1 - a| + |x_2 - a| < \delta\}$ of the equilibrium point $(a, a)^T$, we hope to test if Ω is a part of attraction basin of $(a, a)^T$. Then, to apply Theorem 3, we first find that the projection Ω_i of Ω corresponding to the i-th axis is the same interval $(a - \delta, a + \delta)$, and then calculate that the NLM, $m(f_i)$, of f_i on Ω is given by

$$m(f_i) = \sup_{u\in\Omega_i} \left| \frac{df_i(u)}{du} \right| = \frac{4\beta}{(e^{2\beta(a-\delta)} + 1)^2}.$$

By Theorem 3, we, thus, can deduce that whenever δ is chosen such that

$$4\beta < (e^{2\beta(a-\delta)} + 1)^2 \tag{30}$$

then, for any bounded time-varying delays $\tau_{12}(t)$ and $\tau_{21}(t)$, the equilibrium point $(a, a)^T$ is exponentially stable on the neighborhood Ω (indeed, if Eq. (30) is satisfied, then Eq. (22) holds with $n = 2$, $w_{11} = w_{22} = 0$, $w_{12} = w_{21} = 1$, $R_1 = R_2 = 1$, and $r_1 = r_2 = 1$).

Example 3. Consider the bi-directional associative memory networks in Example 1. It has been shown in Example 1 that Eq. (21) does not hold in this case. Therefore, the results in Gopalsamy and He (1994b) cannot be applied to show the stability of the model. However, through choosing $r_1 = 0.9$ and $r_2 = 0.4$, we can test that Eq. (22) holds with w_{ij} specified as in Example 1. Accordingly, our Theorem 3 can be successfully applied in this time to conclude that the model is exactly exponentially global stable.

Remark 4. When a Hopfield-type neural network is applied as optimization solver, the exponentially global stability of the network not only implies the convergence of the solver to the global optimum of the corresponding optimization problem, but also provides an exponential decay estimation (a convergence speed estimation) of the convergence. It is noted, however, that, given an optimization problem, it is not always possible and straightforward to map the optimization problem into a Hopfield-type network so that the network takes a global optimum of the optimization problem as its unique exponentially global stable equilibrium point. A thorough discussion on this mapping problem is clearly beyond the scope of the present research. However, we refer the reader to Xu and Kwong (1995) and our forthcoming paper (Xu & Peng, 2002).

5. Conclusions

Through generalizing the matrix measure concept, we have introduced an important quantity called the nonlinear Lipschitz measure (NLM) of a nonlinear operator. The introduced quantity is applied to develop a new approach to the stability analysis of Hopfield-type neural networks with time-varying delays. By using the new approach, we have proved some generic sufficient conditions of the exponential stability. The obtained results are either the generalization of those existing or new.

The significance and advantage of the new approach include the following: (i) it can characterize the stability of Hopfield-type neural networks with any bounded time-varying delays even if the transfer functions are not differentiable; (ii) with the new approach, one can quantify the error-correction capability of each stored pattern (or, mathematically, the attraction basin of the stability of each equilibrium point) by directly computing the MLC of transfer functions on a given region when the networks are used as associative memories; (iii) the approach can yield stability conditions with a set of adjustable parameters (cf. Theorem 3) which then often offer much more generic stability criteria than other approach; and (iv) it can be applied to the stability analysis of more general systems with time delays, not only Hopfield-type neural networks.

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References

Baldi, P., & Attiya, A. F. (1994). How delays affect neural dynamics and learning. *IEEE Trans. Neural Networks*, *5* (4), 612–621.

Belair, J. (1993). Stability in a model of a delayed neural network. *J. Dynamics Differential Equations*, *5*, 607–628.

Belair, J., Campbell, S. A., & Driessche, P. V. D. (1996). Frustration, stability, and delay-induced oscillations in a neural network model. *SIAM J. Appl. Math.*, *56* (1), 245–255.

Driver, R. D. (1977). *Ordinary and delay differential equations*. New York: Springer-Verlag.

Fang, Y. G., & Kincaid, T. G. (1996). Stability analysis of dynamical neural networks. *IEEE Trans. Neural Networks*, *7* (4), 996–1006.

Gopalsamy, K., & He, X. Z. (1994a). Stability in asymmetric Hopfield nets with transmission delays. *Physica D*, *76*, 344–358.

Gopalsamy, K., & He, X. Z. (1994b). Delay-independent stability in bi-directional associative memory networks. *IEEE Trans. Neural Networks*, *5*, 998–1002.

Grujic, L. T., & Michel, A. N. (1991). Exponential stability and trajectory bounds of neural networks under structural variations. *IEEE Trans. Circuits Syst.*, *38* (10), 1182–1192.

Hopfield, J. J. (1984). Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Natl. Acad. Sci. USA*, *81*, 3088–3092.

Hou, C. H., & Qian, J. X. (1998). Stability analysis for neural dynamics with time-varying delays. *IEEE Trans. Neural Networks*, *9* (1), 221–223.

Marcus, C. M., & Westervelt, R. M. (1989). Stability of analog networks with time delay. *Physics Rev. A*, *39*, 347–359.

Matsuoka, K. (1992). Stability conditions for nonlinear continuous neural networks with asymmetric connection weights. *Neural Networks*, *5*, 495–500.

Mori, T., et al. (1982). On an estimate of the decay rate for stable linear delay systems. *Int. J. Contr.*, *36* (1), 95–97.

Ortega, J. M., & Rheinboldt, W. C. (1970). *Iterative solution of nonlinear equations in several variables*. New York: Academic Press.

Pakdaman, K., & Malta, C. P. (1998). A note on convergence under dynamical thresholds with delays. *IEEE Trans. Neural Networks*, *9* (1), 231–233.

Peng, J. G., & Xu, Z. B. (1997). On Soderlind's conjectures on nonlinear Lipschitz operators (in Chinese). *Acta Math. Sinica*, *40* (5), 701–708.

Soderlind, G. (1984). On nonlinear difference and differential equations. *BIT*, *24*, 667–680.

Xu, Z. B., & Kwong, C. P. (1995). Global convergence and asymptotic stability of asymmetric Hopfield neural networks. *J. Math. Anal. Appl.*, *191*, 405–427.

Xu, Z. B. & Peng, J. G. (2002). The generalized gradient system: a unified neural optimization model. (in preparation).