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A note on Söderlind's conjecture ☆

Jigen Peng*, Zong-Ben Xu

Research Center for Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, China

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Abstract

Let f be a Lipschitz operator from a path-connected set $D \subseteq \mathbb{C}^m$ into $\mathbb{C}^m$, with the lub-Lipschitz constant $L(f)$ and the so-called "Gerschgorim range radius" $r(f)$ subordinate to a given vector norm $\|\cdot\|$ of $\mathbb{C}^m$. In 1986 [Numer. Math. 50 (1986) 27], Söderlind's conjectured that if $r(f) < L(f)$, then there exists a new vector norm $\|\cdot\|^*$ of $\mathbb{C}^m$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(f)$. In this paper, we affirmatively prove Söderlind's conjecture for several class of Lipschitz operators f, whilst we construct a counterexample to disprove Söderlind's conjecture.

Keywords: Lub-Lipschitz constant; Gerschgorin range radius; Numerical radius; Logarithmic norm; Söderlind's conjecture; Strict stability

1. Introduction

Let $f : D \subseteq \mathbb{C}^m$ be a Lipschitz operator on a subset D, and $\|\cdot\|$ be a strictly homogeneous vector norm of $\mathbb{C}^m$ (i.e., $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in \mathbb{C}^m$ and all complex numbers λ). Subordinate to the given vector norm, the *lub-Lipschitz constant* $L(f)$ and the *logarithmic Lipschitz constant* $M(f)$ of f on D are respectively defined by (cf. [7])

$$L(f) = \sup_{x,y\in D,\, x\neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \tag{1.1}$$

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* Corresponding author. Tel.: +86-29-266-8005; fax: +86-29-266-9559.

E-mail addresses: jgpeng@xjtu.edu.cn (J. Peng), zbxu@xjtu.edu.cn (Z.-B. Xu).

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and

$$M(f) = \lim_{r\to+\infty}(L(f+rI)-r). \tag{1.2}$$

Clearly, $L(\cdot)$ and $M(\cdot)$ are just the nonlinear versions of operator norm (or matrix norm) and logarithmic norm of matrix, respectively. That is, if f is a matrix, then $L(f)$ and $M(f)$ are nothing but the operator norm and logarithmic norm of f, respectively.

To generalize numerical range of matrix to the nonlinear case, Söderlind [6] introduced the so-called *Gerschgorin domain* $G(f)$ of f:

$$G(f) = \bigcap_{\phi\in[0,2\pi)} C_\phi(f), \tag{1.3}$$

where $C_\phi(f) = \{z \in \mathbb{C} : \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\phi}z) \leqslant M(\mathrm{e}^{-\mathrm{i}\phi}f)\}$. From the definition, it directly follows that $G(f)$ is a compact and convex set contained in the circle $\{z \in \mathbb{C} : |z| \leqslant L(f)\}$. It is also easy to verify that, in the special case when f is linear, $G(f)$ is just the convex hull of the numerical range of f and further coincides with the numerical range of f provided the endowed vector norm of $\mathbb{C}^m$ is induced by an inner product. So, Gerschgorin domain is a proper generalization of numerical range of matrix.

It should be noted that these generalizations are not trivial. In fact, in his papers [6–8], Söderlind had shown that these introduced qualities $L(\cdot)$, $M(\cdot)$ and $G(\cdot)$ play important roles in characterizing the invertibility of f, the convergence of difference equation

$$x(n+1) = f(x(n)), \quad n = 1, 2, \ldots \tag{1.4}$$

and the stability of differential equation

$$x'(t) = f(x(t)), \quad t \geqslant 0. \tag{1.5}$$

Let $r(f) = \sup\{|z| : z \in G(f)\}$, which will be from now on refereed to as *Gerschgorin range radius*. Then, there always holds $r(f) \leqslant L(f)$, and $r(f)$ is nothing but the numerical radius when f is a matrix. It is known that, when f is linear, its operator norm can arrive at its fixed numerical radius as the endowed vector norm of $\mathbb{C}^m$ varies. That is, given a numerical radius $r(A)$ of matrix A (i.e., $r(A)$ is induced by a fixed vector norm on $\mathbb{C}^m$), one can always find a new vector norm on $\mathbb{C}^m$ such that the induced operator norm of A equals to $r(A)$ (see, e.g., [3]). Now that $r(\cdot)$ is a generalization of numerical radius of matrix, it is natural to expect that this result is likewise valid for a general Lipschitz operator. As a conjecture, this was specifically formulated by Söderlind [6].

Conjecture 1. *Assume that f is a Lipschitz operator from an open path-connected subset D into $\mathbb{C}^m$, with lub-Lipschitz constant $L(f)$ and Gerschgorin domain $G(f)$ subordinate to a given vector norm of $\mathbb{C}^m$. Let $r(f) = \sup\{|z| : z \in G(f)\}$. If $r(f) < L(f)$, then there exists a new vector norm $\|\cdot\|^*$ of $\mathbb{C}^m$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(f)$.*

A clear significance of Söderlind's conjecture exists in, for example, that one can apply $r(f)$ but the commonly used Lipschitz constant to the convergence analysis of difference equation of the type (4) as long as this conjecture is proved for f. However, as far as we know, Söderlind's conjecture is still open.

In this paper, our main aim is to settle well the conjecture. In the next section, we investigate Söderlind's conjecture towards positive direction. As a main result, we establish several equivalent assertions to Söderlind's conjecture, and hence deduce some positive results. On the other hand, based on these equivalent assertions, a counterexample is constructed to disprove Söderlind's conjecture in Section 3.

2. Positive results for Söderlind's conjecture

Let $f : D \subset \mathbb{C}^m \to \mathbb{C}^m$ be a Lipschitz operator. By Rademacher theorem, we know that f is differentiable almost everywhere in D. Set $\Omega = \{x \in D : f$ is differentiable at $x\}$ and $\partial(f) = \{f'(x) : x \in \Omega\}$, it is then easy to show that

$$\sup\{\|A\|_m : A \in \overline{\partial(f)}\} \leqslant L(f), \tag{2.1}$$

and the equality holds whenever D is convex (cf. [6]), where $\|\cdot\|_m$ denotes the operator norm of the space of $m \times m$ matrices. In what follows, we frequently adopt these notations.

To present the main theorem below, we recall that a subset $U \subseteq \mathbb{C}^m$ is said to be balanced if $\lambda U = U$ for all complex numbers λ with $|\lambda| = 1$, and to be absorbing if, for all $x \in \mathbb{C}^m$, there corresponds a $\lambda > 0$ such that $\lambda x \in U$.

Theorem 1. *Assume that $\mathbb{C}^m$ is endowed with a vector norm $\|\cdot\|$, and $D \subset \mathbb{C}^m$ is an open convex subset. Let f be a Lipschitz operator from D into $\mathbb{C}^m$, with the lub-Lipschitz constant $L(f)$ and the Gerschgorin range radius $r(f)$ subordinate to the given vector norm $\|\cdot\|$. If $r(f) < L(f)$, then the following assertions are equivalent:*

(i) *there exists a new vector norm $\|\cdot\|^*$ of $\mathbb{C}^m$ such that $\|f(x) - f(y)\|^* \leqslant r(f)\|x - y\|^*$ for all $x, y \in D$.*
(ii) *there exists a compact, balanced and convex neighborhood U of the zero vector $x = 0$ such that $A(U) \subset r(f)U$ for all $A \in \partial(f)$.*
(iii) *there is a positive real constant M such that $r(A_1A_2\cdots A_n) \leqslant Mr(f)^n$ for any set of $A_1, A_2, \ldots, A_n \in \partial(f), n = 1, 2, \ldots$*
(iv) *the zero vector $x = 0$ is one interior point of the set defined by*

$$V = \left\{x : \|A_1A_2\cdots A_nx\| \leqslant (r(f))^n,\ A_i \in \partial(f) \cup \{r(f)I\},\ n = 1, 2, \ldots\right\}.$$

Proof. (i) ⇒ (ii). Let $U = \{x : \|x\|^* \leqslant 1\}$. Then, by the equivalence between these two norms $\|\cdot\|^*$ and $\|\cdot\|$, it is easy to show that U is a compact, balanced and convex neighborhood of $x = 0$. Since the lub-Lipschitz constant subordinate to $\|\cdot\|^*$ is not larger than $r(f)$, we have by (2.1) that $\|A\|_m^* \leqslant r(f)$ for all $A \in \partial(f)$. From this, we deduce that $A(U) \subset r(f)U$ for all $A \in \partial(f)$.

(ii) $\Rightarrow$ (iii). Since U is a compact, balanced and convex neighbourhood of the zero vector, there corresponds a vector norm of $\mathbb{C}^m$ defined by

$$\|x\|^* = \inf\{\lambda > 0 : x \in \lambda U\}. \tag{2.2}$$

Clearly, when $\mathbb{C}^m$ equipped with this norm, U is just the unit ball of $\mathbb{C}^m$. Given any set of $A_1, A_2, \ldots, A_n \in \partial(f) \cup \{r(f)I\}$. By assumption that $A(U) \in r(f)U$ for all $A \in \partial(f)$, we have $A_1A_2\cdots A_n(U) \subset (r(f))^n U$, that is, $\|A_1A_2\cdots A_n x\|^* \leqslant (r(f))^n\|x\|^*$ for all $x \in U$. So, subordinate to the defined vector norm $\|\cdot\|^*$, the operator norm $\|A_1A_2\cdots A_n\|_m^* \leqslant r(f)^n$. Now, noticing that the numerical radius, as a vector norm on the space of $m \times m$ matrices, is equivalent to the operator norm $\|\cdot\|_m^*$ induced by $\|\cdot\|^*$, we further can find a positive real constant M such that

$$r(A_1A_2\cdots A_n) \leqslant M\|A_1A_2\cdots A_n\|_m^* \leqslant Mr(f)^n,$$

as expected.

(iii) $\Rightarrow$ (iv). By the equivalence between the numerical radius and any operator norm (both as vector norms of the space of $m \times m$ matrices), there exists a positive constant K such that $\|B\|_m \leqslant Kr(B)$ for all $m \times m$ matrix B, where $\|B\|_m$ is the operator norm induced by the given vector norm of $\mathbb{C}^m$. So, with such constant K, there holds

$$\|A_1A_2\cdots A_n\|_m \leqslant Kr(A_1A_2\cdots A_n) \leqslant KMr(f)^n$$

for any set of $A_1, A_2, \ldots, A_n \in \partial(f)$, $n = 1, 2, \ldots$ With which, it is easy to verify that the ball $\{x \in \mathbb{C}^m : \|x\| \leqslant K^{-1}M^{-1}\}$ is contained in V, and thus the zero vector is one interior point of V.

(iv) $\Rightarrow$ (i). For each $x \in \mathbb{C}^m$, let

$$\|x\|^* = \inf\{\lambda \geqslant 0 : x \in \lambda V\}. \tag{2.3}$$

Noticing that V is a absorbed set since the zero vector is one interior point of V, one can see that $\|x\|^*$ is defined well for every $x \in \mathbb{C}^m$. We prove below that $\|\cdot\|^*$ is a vector norm of $\mathbb{C}^m$.

(a) Homogeneity: It is easy to check that V is balanced. So, by (2.3), we have that, for all $x \in \mathbb{C}^m$ and $0 \neq \mu \in \mathbb{C}$,

$$\begin{aligned}\|\mu x\|^* &= \inf\{\lambda \geqslant 0 : x \in \mu^{-1}\lambda V\} \\ &= \inf\{\lambda \geqslant 0 : x \in |\mu|^{-1}\lambda V\} \\ &= |\mu| \cdot \|x\|^*,\end{aligned}$$

which justifies the homogeneity of $\|\cdot\|^*$.

(b) Subadditivity: Noticing that V is convex, we can show that $x + y \in (\lambda_1 + \lambda_2)V$ whenever $x \in \lambda_1 V$ and $y \in \lambda_2 V$. And then, we get the subadditivity of $\|\cdot\|^*$.

(c) $\|x\| = 0 \Leftrightarrow x = 0$: Clearly $\|0\|^* = 0$. Let $\|x\|^* = 0$. Then there exists a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive real numbers such that $x \in \lambda_k V$ and $\lambda_k \to 0$. Obviously, $x = 0$ if V is bounded. Actually, it is easy to check that V is contained in the unit ball $\{x : \|x\| \leqslant K\}$ and thus bounded. Therefore, $x = 0$.

We now show that, subordinate to $\|\cdot\|^*$, the lub-Lipschitz constant $L^*(f) \leqslant r(f)$, and then close the proof. Given any $A \in \partial(f)$. It is clear that $A(V) \subset r(f)V$. So, by (2.3), there holds $\|Ax\|^* \leqslant r(f)\|x\|^*$ for all $x \in \mathbb{C}^m$. That is, $\|A\|_m^* \leqslant r(f)$. Therefore, by (2.1) and the convexity of D, $L^*(f) \leqslant r(f)$ holds. □

Remark 1. Theorem 1 means that Söderlind's conjecture would be affirmative as long as any one of assertions (ii), (iii) and (iv) is justified. Actually, in either case (ii) or case (iv), the expected new vector norm of $\mathbb{C}^m$ in Söderlind's conjecture can be specifically constructed by

$$\|x\|^* = \inf\{\lambda \leqslant 0 : x \in \lambda U\}$$

in the case (ii), or by

$$\|x\|^* = \sup\left\{\frac{\|A_1A_2\cdots A_n x\|}{r(f)^n} : A_i \in \partial(f) \cup \{r(f)I\},\ n = 1, 2, \ldots\right\}$$

in the case (iv).

With those equivalent assertions in Theorem 1, we can positively prove Söderlind's conjecture for some class of Lipschitz operators f (even though Söderlind's conjecture is disproved in Section 3). As examples, we list several corollaries below.

Corollary 1 [3]. *Let A be an $m \times m$ complex matrix, and $r(A)$ be a numerical radius of A (subordinate to a given vector norm $\|\cdot\|$ of $\mathbb{C}^m$). Then there exists a new vector norm $\|\cdot\|^*$ of $\mathbb{C}^m$ such that the induced operator norm $\|A\|_m^* \leqslant r(A)$.*

Proof. Let $f = A$. Then, by Theorem 1, to complete the proof it suffices to verify that the zero vector is an interior point of the subset

$$V = \{x \in \mathrm{C}^m :\ \|A^n x\| \leqslant r(A)^n,\ n = 0, 1, 2, \ldots\}.$$

It is well known that, as a vector norm of the space M_m of $m \times m$ matrices, the numerical radius $r(\cdot)$ subordinate to $\|\cdot\|$ is spectral dominant (i.e., $\rho(B) \leqslant r(B)$ for all $B \in M_m$, where $\rho(B)$ is the spectral radius). So, by [1], $r(\cdot)$ is stable. That is, there is a positive constant K_1 such that $r(B^n) \leqslant K_1 r(B)^n$ for all $B \in M_m$. Moreover, by the equivalence between numerical radius and operator norm (both as vector norms of M_m), there exists a constant $K_2 > 0$ such that $\|B\|_m \leqslant K_2 r(B)$ for all $B \in M_m$. With these constants K_i, it is easy to check that the ball $\{x : \|x\| \leqslant K_1 K_2^{-1}\}$ is contained in V. Hence, the zero vector is an interior point of V. □

Recall that a vector norm $\|\cdot\|$ of $\mathbb{C}^m$ is said to be absolutely monotonic if, for all $x, y \in \mathbb{C}^m$, $\|x\| = \||x|\|$ and $|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|$, where $|x| = (|x_1|, |x_2|, \ldots,$

$|x_m|)^{\mathrm{T}} \in \mathbb{C}^m$ is the module vector of x. For example, all Hölder norms $\|x\|_p = (\sum_{i=1}^m |x_i|^p)^{1/p}$ (specially l^1-norm and l^∞-norm of $\mathbb{C}^m$) are absolutely monotonic.

Corollary 2. *Assume that $\mathbb{C}^m$ is endowed with an absolutely monotonic vector norm $\|\cdot\|$. Let f be a Lipschitz operator from an open convex set D into $\mathbb{C}^m$, and A be an $m \times m$ matrix such that $|f(x) - f(y)| \leqslant A(|x - y|)$ for all $x, y \in D$. Then, there exists a new vector norm $\|\cdot\|^*$ of $\mathbb{C}^m$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(A)$, where $r(A)$ is the numerical radius of A subordinate to $\|\cdot\|$.*

Proof. Let $V = \{x : \ \|A^n(|x|)\| \leqslant (r(A))^n, \ n = 0, 1, 2, \ldots\}$. Then, by Corollary 1, we can show that V is a compact, balanced and convex neighborhood of the zero vector $x = 0$. From the assumption that $|f(x) - f(y)| \leqslant A(|x - y|)$ for all $x, y \in \mathbb{C}^m$, it follows that A is real and positive (i.e., all elements are real and positive). Hence, $|Ax| \leqslant A(|x|)$ for all $x \in \mathbb{C}^m$. From which, we get $A(V) \subset r(A)V$. Therefore, by the equivalence between (i) and (ii) in Theorem 1 (where $f = A$) and by Remark 1, we find that, subordinate to the vector norm defined by

$$\|x\|^* = \inf\{\lambda \geqslant 0 : x \in \lambda V\}, \quad x \in \mathbb{C}^m,$$

the operator norm $\|A\|_m^* \leqslant r(A)$.

Furthermore, by the positivity of A and the absolute monotonicity of the vector norm $\|\cdot\|$, we can verify that $y \in V \Rightarrow x \in V$ and $\|x\|^* \leqslant \|y\|^*$ whenever $|x| \leqslant |y|$. It is also seen that $x \in V \Leftrightarrow |x| \in V$. Hence $\|x\|^* = \||x|\|^*$ for all $x \in \mathbb{C}^m$. That is, $\|\cdot\|^*$ is absolutely monotonic. Consequently, we have

$$\begin{aligned} \|f(x) - f(y)\|^* &= \||f(x) - f(y)|\|^* \\ &\leqslant \|A(|x - y|)\|^* \\ &\leqslant \|A\|^* \cdot \|x - y\|^* \\ &\leqslant r(A)\|x - y\|^*, \end{aligned}$$

that is, $L^*(f) \leqslant r(A)$. The proof is completed. □

Remark 2. Assume that $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^{\mathrm{T}}$ is continuously differentiable in its domain D. Let $A = (a_{ij})_{m\times m}$ of which $a_{ij} = \sup\{|\partial f_i/\partial x_j(x)| : x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \in D\}$. Then, there holds $|f(x) - f(y)| \leqslant A(|x - y|)$ for all $x, y \in \mathbb{C}^m$. So, by Corollary 2, any numerical radius $r(A)$ induced by an absolutely monotonic norm of $\mathbb{C}^m$ can serve as a Lipschitz constant of f. When applied to convergence analysis of the iteration $x_{n+1} = f(x_n)$, this indicates that if $r(A) < 1$, then the iteration sequence initiated from everywhere is always convergent to a unique fixed point.

We proceed with affirming Söderlind's conjecture for a class of simple nonlinear operators, namely, diagonal operators. A operator $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^{\mathrm{T}}$

is said to be diagonal if each component $f_i(x)$ is one-dimension function of the corresponding element x_i of variable x.

Corollary 3. *Assume f is a diagonal Lipschitz operator from an open convex set D into $\mathbb{C}^m$. Let $r(f)$ be the Gerschgorin range radius subordinate to any a given norm $\|\cdot\|$ on $\mathbb{C}^m$. Then, there exists a new vector norm $\|\cdot\|^*$ of $\mathbb{C}^m$ such that*

$$\|f(x) - f(y)\|^* \leqslant r(f)\|x - y\|^*, \quad x, y \in D.$$

Proof. For each $x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \in \mathbb{C}^m$, let $D_x = \mathrm{diag}(x_1, x_2, \ldots, x_m)$ (the diagonal matrix with principal elements x_i). Define a set U by $U = \{x \in \mathbb{C}^m : \rho(D_x) \leqslant 1\}$, where $\rho(\cdot)$ is the spectral radius. It can be seen that U is really the unit ball $\{x \in \mathbb{C}^m : \|x\|_\infty \leqslant 1\}$, where $\|\cdot\|_\infty$ is l^∞-norm of $\mathbb{C}^m$. To close the proof, by Theorem 1 it is sufficient to verify that $A(U) \subset r(f)U$ for all $A \in \partial(f)$.

Since f is diagonal, each matrix $A \in \partial(f)$ is diagonal. So, for all $x \in \mathrm{C}^m$ and all $A \in \partial(f)$, there holds $D_{Ax} = AD_x = D_x A$, and then $\rho(D_{Ax}) = \rho(D_x A) \leqslant \rho(A)\rho(D_x)$. Noticing that $\rho(A) \leqslant r(A) \leqslant r(f)$, we thus have $A(U) \subset r(f)U$ for all $A \in \partial(f)$, as expected. □

3. A counterexample

In this section, we use those equivalence established in Theorem 1 to find a way to disprove Söderlind's conjecture, and specifically give a counterexample. To this end, we first recall that a notion, namely stability of numerical radius, is mentioned in the proof of Corollary 1 when numerical radius is viewed as a vector norm of the space of $m \times m$ matrices.

In general, a vector norm $v(\cdot)$ of the space of $m \times m$ matrices is said to be *stable* if there is a positive constant K such that

$$v(A^n) \leqslant Kv(A)^n, \quad n = 1, 2, \ldots \tag{3.1}$$

holds for all $m \times m$ matrix A. Friedland and Zenger [1] proved that a vector norm $v(\cdot)$ is stable if and only if it is spectrally dominant (i.e., $v(A) \geqslant \rho(A)$ for all $m \times m$ matrix A). It is known that, as a vector norm of the space of $m \times m$ matrices, numerical radius induced by any vector norm of $\mathbb{C}^m$ is spectrally dominant. Thus, any numerical radius is stable. By the equivalence between assertion (i) and (iii) in Theorem 1, it is seen that Söderlind's conjecture seems to be closely related to a more "strict" stability of numerical radius. Motivated by this, we develop the following notion.

Definition 1. Let $v(\cdot)$ be a vector norm of the space M_m of $m \times m$ matrices, and Γ be a bounded subset of M_m. Then, $v(\cdot)$ is said to be strictly stable w.r.t. Γ if there exists a positive constant K such that

$$v(A_1A_2\cdots A_n) \leqslant K v_{\max}^n, \quad A_i \in \Gamma, \ \ n = 1, 2, \ldots, \tag{3.2}$$

where $v_{\max} = \sup\{v(A) : A \in \Gamma\}$.

In term of this notion, we can deduce the following proposition from the equivalence between assertions (i) and (iii) in Theorem 1.

Proposition 1. *Assume that $\|\cdot\|$ is a vector norm of $\mathbb{C}^m$ and $r(\cdot)$ is the numerical radius subordinate to $\|\cdot\|$. Let f be a Lipschitz operator from an open convex set D into $\mathbb{C}^m$, with $r(f) < L(f)$. If $r(\cdot)$ is not strictly stable w.r.t. $\partial(f)$, then, one cannot find a vector norm of $\mathbb{C}^m$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(f)$.*

Proof. By (2.1), the convexity of D gives $r(f) = \sup\{r(A) : A \in \partial(f)\} = r_{\max}$. Since $r(\cdot)$ is not strictly stable w.r.t. $\partial(f)$, then, by Definition 1, there cannot exist a positive constant K such that

$$r(A_1A_2\cdots A_n) \leqslant K \cdot r(f)^n, \quad A_i \in \partial(f), \ \ n = 1, 2, \ldots$$

That is, the assertion (iii) in Theorem 1 is not true. So, by Theorem 1, one cannot find a vector norm of $\mathbb{C}^m$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(f)$. The proof is completed. □

This proposition indicates that Söderlind's conjecture will be disproved as long as we find a Lipschitz operator f such that the given numerical radius $r(\cdot)$ is shown not to be strictly stable w.r.t. $\partial(f)$. To find such a operator f, let us list some useful properties of strict stability below.

Proposition 2. *Let $v(\cdot)$ be a vector norm of the space of $m \times m$ matrices and Γ be a set of $m \times m$ matrices. Then, one can prove the following properties* (a)–(e):

(a) *if $v(\cdot)$ is stable, then it is strictly stable w.r.t. any one-element set $\{A\}$;*
(b) *if $v(\cdot)$ is strictly stable w.r.t. the unit ball $\mathbf{B}_1 = \{A : v(A) \leqslant 1\}$ of the $m \times m$ matrix space, then it is stable;*
(c) *$v(\cdot)$ is strictly stable w.r.t. any bounded set of $m \times m$ matrices if and only if it is strictly stable w.r.t. the unit ball $\mathbf{B}_1$;*
(d) *if $v(\cdot)$ is sub-multiplicative (i.e., $v(AB) \leqslant v(A)v(B)$ for all $m \times m$ matrices A and B), then it is strictly stable w.r.t. any bounded set of $m \times m$ matrices;*
(e) *if $v(\cdot)$ is strictly stable w.r.t. Γ, then it is strictly stable w.r.t. $\overline{\Gamma}$, the closure of Γ under the topology induced by $v(\cdot)$.*

Proof. The properties (a), (c), (e) and (f) are immediate by definitions of stability and strict stability. In the following, we give the proof of (b). Let $v(\cdot)$ be strictly stable w.r.t. $\mathbf{B}_1$. Then, there is a positive constant K such that

$$v(A_1A_2\cdots A_n) \leqslant K, \quad \forall A_i \in \mathbf{B}_1, \ \ n = 1, 2, \ldots, \tag{3.3}$$

where we notice that $v_{\max} = \sup\{v(A) : A \in B_1\} = 1$. For all $m \times m$ matrix A, let $A_i = v(A)^{-1}A \in \mathbf{B}_1$ in (3.3), then we get $v(A^n) \leqslant Kv(A)^n$ for all $n = 1, 2, \ldots$ That is, $v(\cdot)$ is stable. The proof is completed. □

Now, let us consider the following example.

Example 1. Let D be an open convex subset of $\mathbb{C}^2$ defined by

$$D = \{x = (z_1, z_2)^{\mathrm{T}} : \ |z_1| + |z_2| < 1\} \tag{3.4}$$

and f a Lipschitz operator from D into $\mathbb{C}^2$ defined by

$$f((z_1, z_2)^{\mathrm{T}}) = (z_2^2, z_1^2)^{\mathrm{T}}, \quad (z_1, z_2)^{\mathrm{T}} \in D. \tag{3.5}$$

Denote by $r(f)$ and $L(f)$ the Gerschgorin range radius and the lub-Lipschitz constant of f subordinate l^2-norm of $\mathbb{C}^2$, respectively. Then, $r(f) < L(f)$ holds, but, there cannot exist a vector norm $\|\cdot\|^*$ of $\mathbb{C}^2$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(f)$. That is, the conclusion of Söderlind's conjecture is not true for the present f with $r(f)$ being subordinate to l^2-norm of $\mathbb{C}^2$.

Proof. It is easy to show that, for each $A \in \partial(f)$, there corresponds a certain $z = (z_1, z_2)^{\mathrm{T}} \in D$ such that

$$A = f'(z) = \begin{bmatrix} 0 & 2z_2 \\ 2z_1 & 0 \end{bmatrix}. \tag{3.6}$$

Let $r(\cdot)$ be the numerical radius subordinate to l^2-norm of $\mathbb{C}^2$. Then, by (2.1) and the convexity of D, we get $r(f) = \sup\{r(A) : A \in \partial(f)\}$. By Propositions 1 and 2, to close the proof it suffices to show that $r(\cdot)$, as a vector norm of the space of 2×2 matrices, cannot be strictly stable w.r.t. $\overline{\partial(f)}$.

A direct estimation yields

$$\begin{aligned} r(f) &= \sup_{A\in\partial(f)} r(A) = \sup_{z=(z_1,z_2)^{\mathrm{T}}\in D} r(f'(z)) \\ &= \sup_{(z_1,z_2)^{\mathrm{T}}\in D} \sup\{|\langle f'(z)x, x\rangle| : x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{C}^2, \|x\|_2 = 1\} \\ &= \sup_{(z_1,z_2)^{\mathrm{T}}\in D} \sup\{|2(z_1 + z_2)x_1x_2| : \ |x_1|^2 + |x_2|^2 = 1\} \\ &= 1. \end{aligned}$$

For each $i = 1, 2, \ldots$, let A_{2i} and $A_{2i-1} \in \overline{\partial(f)}$ be respectively of the following special forms:

$$A_{2i-1} = f'((0, 1)^{\mathrm{T}}) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \tag{3.7}$$

and

$$A_{2i} = f'((1,0)^{\mathrm{T}}) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}. \tag{3.8}$$

Then, for any $n = 2, 4, \ldots,$ we have

$$A_1 A_2 \cdots A_n = \begin{bmatrix} 2^n & 0 \\ 0 & 0 \end{bmatrix}.$$

From which, it follows that $r(A_1 A_2 \cdots A_{n-1} A_n) \geqslant 2^n$. Thus, letting $n \to \infty$ leads

$$\lim_{n\to\infty} \frac{r(A_1 A_2 \cdots A_{n-1} A_n)}{(r(f))^n} = \infty.$$

That is, $r(\cdot)$ cannot be strictly stable w.r.t. $\overline{\partial(f)}$, as expected. □

Remark 3. We can also prove the conclusion made in Example 1 directly by reduction to absurdity. In fact, suppose there is a new vector norm $\|\cdot\|^*$ such that the induced lub-Lipschitz constant $L^*(f) \leqslant r(f)$, that is, for any pair of $x, y \in D$, there holds

$$\|f(x) - f(y)\|^* \leqslant r(f)\|x - y\|^*.$$

Denote by $\|\cdot\|_m^*$ the operator norm induced by the new vector norm $\|\cdot\|^*$. Then, $\|A\|_m^* \leqslant r(f) = 1$ for all $A \in \overline{\partial(f)}$, and thus $\|A_1 A_2 \cdots A_n\|_m^* \leqslant r(f)^n = 1$ for any set of $A_1, A_2, \ldots, A_n \in \overline{\partial(f)}$. But, for those A_{2i} and A_{2i-1} defined by (3.7) and (3.8), respectively, we have

$$A_1 A_2 \cdots A_{n-1} A_n = \begin{bmatrix} 2^n & 0 \\ 0 & 0 \end{bmatrix}.$$

From which, it follows that $\|A_1 A_2 \cdots A_n\|_m^* \geqslant 2^n$ because of the clear fact that the spectral radius of $A_1 A_2 \cdots A_n$ is 2^n. So, letting $n \to \infty$ leads $\|A_1 A_2 \cdots A_n\|_m^* \to +\infty$, a contradiction.

Remark 4. For such defined f, we can further prove that, subordinate to any vector norm of $\mathbb{C}^2$, the lub-Lipschitz constant $L(f) \geqslant 2$. In fact, if there is a vector norm $\|\cdot\|^*$ such that $L^*(f) < 2$, then, by (2.1), there holds $\|A\|_2^* \leqslant L^*(f) < 2$ for all $A \in \overline{\partial(f)}$, where $\|A\|_m^*$ denotes the operator norm of A subordinate to $\|\cdot\|^*$. So, by the sub-multiplicativity of the operator norm $\|\cdot\|_2^*$, the inequality $\|AB\|_2^* \leqslant L^*(f)^2 < 4$ holds for all $A, B \in \overline{\partial(f)}$, and particularly it holds for A and B being defined by

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

However, for such given matrices A and B, a routine computing gives

$$AB = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, by the spectral domination of $\|\cdot\|_2^*$, we get $\|AB\|_2^* \geqslant 4$. Therefore, there is a contradiction. This contradiction tells us that Example 1 really gives a negative answer to the problem "if there is a new vector norm of $\mathbb{C}^m$ such that the induced lub-Lipschitz constant $L^*(f)$ of f satisfies $r(f) < L^*(f) < L(f)$, provided $r(f) < L(f)$", which clearly is a weaker version of Söderlind's conjecture.

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