A Reference Model Approach to Stability Analysis of Neural Networks

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***Abstract*—In this paper, a novel methodology called a reference model approach to stability analysis of neural networks is proposed. The core of the new approach is to study a neural network model with reference to other related models, so that different modeling approaches can be combinatively used and powerfully cross-fertilized. Focused on two representative neural network modeling approaches (the neuron state modeling approach and the local field modeling approach), we establish a rigorous theoretical basis on the feasibility and efficiency of the reference model approach. The new approach has been used to develop a series of new, generic stability theories for various neural network models. These results have been applied to several typical neural network systems including the Hopfield-type neural networks, the recurrent back-propagation neural networks, the BSB-type neural networks, the bound-constraints optimization neural networks, and the cellular neural networks. The results obtained unify, sharpen or generalize most of the existing stability assertions, and illustrate the feasibility and power of the new method.**

***Index Terms*—Local field neural network model, reference model approach, stability analysis, static neural network model.**

I. Introduction

Depending upon whether neuron states (the external states of neurons) or local field states (the internal states of neurons) are taken as basic variables, a dynamical neural network can frequently be cast either as *a static neural network model* or as *a local field neural network model* [15], [16], [40]. The recurrent back-propagation networks (ReBP-type NNs) [1], [31], [33], for example, are given in a static network model form by

$$\tau \frac{dv_i}{dt} = -v_i + g_i\left(\sum_{j=1}^{N} w_{ij}v_j + \theta_i\right), \; i = 1, 2, \ldots, N \quad (1)$$

where v_i is the state of neuron i with $u_i = \sum_{j=1}^{N} w_{ij}v_j + \theta_i$ being its local field state, g_i the activation function of neuron i, θ_i the external input imposed on neuron i, w_{ij} the synaptic connectivity value between neuron i and neuron j, and N the number of neurons in the network. In contrast, the Hopfield neural networks (Hopfield-Type NNs) [18], [19] are modeled in a local field model form:

$$C_i \frac{du_i}{dt} = -\frac{u_i}{R_i} + \sum_{j=1}^{N} w_{ij}g_j(u_j) + I_i, \qquad i = 1, 2, \ldots, N \quad (2)$$

where u_i is the local field state, $v_i = g_i(u_i)$ is the state of neuron i, and C_i, R_i and I_i are fixed physical parameters. More generally, the brain-state-in-a-box/domain type networks (BSB-type NNs) [25], [35] and the optimization-type neural networks (Op-type NNs) studied recently in [5], [12], [13], [27], [37] are all in the static neural network model forms, which can be written in the matrix form:

$$\tau \frac{dx}{dt} = -x + G(Wx + q), \qquad x(0) = x_0; \quad (3)$$

whereas, putting parameters aside, the Hopfield-Type NNs, the bidirectional associative memory networks (BAM-type NNs) [23] and the cellular neural networks (CNNs) [8], [30], [34] all are local field neural network models, which can be written as

$$\tau \frac{dy}{dt} = -y + WG(y) + q, \qquad y(0) = y_0. \quad (4)$$

Here $x = (x_1, x, \ldots, x_N)$ is the neural network state, $y = (y_1, y_2, \ldots, y_N)$ is the local field vector, $W = (w_{ij})_{N\times N}$ is the synaptic weight matrix and G: $\mathbf{R}^N \to \mathbf{R}^N$ is the nonlinear mapping associated with the network's activation functions.

The static neural network model (3) and the local field neural network model (4) typically represent two fundamental modeling approaches in the current neural network research [15], [16], [40]. However, they have been applied somehow in a separate manner and hardly been cross-fertilized. As a result, some types of networks such as the Hopfield-type NNs have been attracting considerable interests, and many deep theoretical results have been obtained for the models (see, e.g., [3], [6], [7], [9], [14], [17], [20], [26], [32], [43]). In contrast, other types of neural networks such as the ReBP-type NNs have not received so much attention and have been fallen short of a systematic and in-depth theoretical analysis [1], [15], [16], [31], [33].

On the other hand, the models (3) and (4) can also be explained as the modeling approaches from an external and internal state point of view respectively. So there must be certain similarity and connections between them. Motivated by this observation, a precise theoretical comparison on the dynamics of the models (3) and (4) has been made in [40], which established a series of equivalence results on the equilibria sets, stability, asymptotic stability, exponential stability as well as global convergence in some sense for both models. This comparison study

Manuscript received April 18, 2002; revised August 15, 2002. This work was supported by the Natural Science Foundation of China (Grant 10101019) and the Hi-Tech R&D (863) Program (Grant 2001AA113182). This paper was recommended by Associate Editor C. W. Tae.

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Digital Object Identifier 10.1109/TSMCB.2002.804368

not only revealed certain consistency and inconsistency properties of the static and local field neural network models, but also, more importantly, paved a way to the development of new, more effective, cross-fertilization type methodologies for neural network research.

The purpose of the present paper is to develop such a cross-fertilization type approach for stability analysis of neural networks, which will be called a reference model approach in the present paper. Through carefully capturing the subtle differences and intrinsic virtues of each model, we show how the intrinsic advantages originally exhibited in each individual model can be combined, or borrowed from each other, to yield a unified, deeper understanding on different types of neural networks deduced from different modelings. In particular, We demonstrate by presenting a series of examples that the new approach can be used to derive various new results on stability and convergence of the ReBP-type NNs, BSB-type NNs, Op-type NNs, and CNNs.

The study in [40] answered the following question: does there exist any difference and connection between (3) and (4)? The aim of the present paper is to answer two other related questions: which model is more beneficial when a specific analysis purpose is concerned? and how can these two modelings be cross-fertilized?

The remainder of the paper is organized as follows. In Section II, we briefly review the relationships between the dynamics of models (3) and (4) in terms of their trajectory transformation property, equilibria correspondence property, convergence and stability in many different senses. The theory introduced will serve as a theoretical basis of subsequent investigation. In Section III a set of unified definitions, followed by a series of examples, is introduced to characterize some common features of the nonlinear activation mappings G, which serves to illuminate how the abstract results obtained and to be obtained can be directly applied. In Section IV, some fundamental lemmas are proved to highlight the intrinsic properties and merits of the models (3) and (4), which further underly the feasibility and efficiency of the reference model approach. Section V is devoted to applications of the new methodology. The static model (3) and the local field model (4) are alternatively used as a reference model to investigate their counterparts. A series of new, unified stability results on both models as well as their specifications will be derived. The obtained results will demonstrate the power and efficiency of the proposed methodology. Concluding remarks are presented in Section VI.

We conclude this section by introducing some notations. For any set $V \subset \mathbf{R}^N$, $\overline{V}$ denotes the closure of V and $\overline{Co}V$ the convex hull of V. Given $q \in \mathbf{R}^N$, W_q stands for the affine transformation on $\mathbf{R}^N$ defined by $W_q x = Wx + q$. For any matrix A, A^t stands for the transpose of A. Finally, given an operator $F\colon V \to \mathbf{R}^N$, we denote by $\Re(F)$ the range of F, $F^{-1}(0)$ the null set, i.e., $F^{-1}(0) = \{x \in V | F(x) = 0\}$, and F^{-1} the inverse operator from $\mathbf{R}(F)$ to V (if it exists).

II. Review of Relationships Between Dynamics of Two Models

In this section the relationships between the dynamics of models (3) and (4) will be briefly reviewed, detailed proofs of which can be found in [40]. They will constitute a basis of the reference model approach proposed.

Assume that there exist a unique solution $x(t, x_0)$ to (3) and a unique solution $y(t, y_0)$ to (4) for any given x_0, y_0 in $\mathbf{R}^N$. (This is the case, e.g., when G is locally Lipschitz continuous.) As usual, the solution $x(t, x_0)$ is also called a *trajectory* of (3) through x_0, denoted henceforth by $\Gamma_1(x_0)$ [correspondingly, the trajectory $y(t, y_0)$ of (4) is denoted by $\Gamma_2(y_0)$]. Let $D \subset \mathbf{R}^N$ be a subset. D is said to be invariant under the system (3) if $x_0 \in D$ implies $\Gamma_1(x_0) \subseteq D$. A point x^* is called a ω-limit point of $\Gamma_1(x_0)$ if there is a subsequence $\{t_i\}$ such that $x^* = \lim_{i\to\infty} x(t_i, x_0)$. All the ω-limit points constitute the ω-limit set $\omega(\Gamma_1(x_0))$ of $\Gamma_1(x_0)$. The ω-limit set is invariant under the dynamics. Recall that a constant vector x^* is said to be an equilibrium state of the system (3) if x^* is a zero point of operator F_1, defined by

$$F_1(x) = -x + G(Wx + q), \qquad \forall x \in \mathbf{R}^N \tag{5}$$

that is, $F_1(x^*) = 0$. All equilibrium states of (3) is denoted by $F_1^{-1}(0)$. Similarly, $F_2^{-1}(0)$ denotes the set of equilibrium states of system (4), where F_2 is defined by

$$F_2(y) = -y + WG(y) + q, \qquad \forall y \in \mathbf{R}^N. \tag{6}$$

The equilibrium state x^* is said to be *stable* if any trajectory of (3) can stay within a small neighborhood of x^* whenever the initial x_0 is close to x^*, and is said to be *attractive* if there is a neighborhood $\Xi(x^*)$, called the *attraction basin* of x^*, such that any trajectory of (3) initialized from a state in $\Xi(x^*)$ will approach to x^* as time goes to infinity. An equilibrium state is said to be *asymptotically stable* if it is both stable and attractive, whilst the equilibrium state x^* is said to be *exponentially stable* if there exist a constant $\alpha > 0$ and a strictly increasing function $M\colon \mathbf{R} \to \mathbf{R}^+$ with $M(0) = 0$ such that

$$\|x(t, x_0) - x^*\| \leq M(\|x_0 - x^*\|)e^{-\alpha t}. \tag{7}$$

Further, x^* is said to be *globally asymptotic stable* if it is asymptotically stable, and $\Xi(x^*) = \mathbf{R}^N$. A system [say, (3)] is said to be *globally convergent* if $x(t, x_0)$ converges to an equilibrium state of (3) for every initial point $x_0 \in \mathbf{R}^N$ [the limit of $x(t, x_0)$ may not be the same for different x_0], whilst it is said to be *exponentially convergent* if it is globally convergent with $x(t, x_0)$ and its limit x^* satisfying (7).

To clarify the relationships between (3) and (4), we introduce the following four coherent systems:

$$\tau \frac{dx}{dt} = -x + G(Wx + q), \quad x(0) = x_0 \in \overline{Co}\Re(G) \tag{8}$$

$$\tau \frac{dx}{dt} = -x + G(Wx + q), \quad x(0) = x_0 \in \overline{Co}\Re(GW_q) \tag{9}$$

$$\tau \frac{dy}{dt} = -y + WG(y) + q, \quad y(0) = y_0 \in \Re(W_q) \tag{10}$$

$$\tau \frac{dy}{dt} = -y + WG(y) + q, \quad y(0) = y_0 \in \overline{\Re(WG)} + q. \tag{11}$$

Then the relationships between the dynamics of models (3) and (4) can be summarized in Theorems 1 and 2.

Theorem 1: i) If $\Gamma_1(x_0)$ is the trajectory of (3) through x_0, then $Wx(t, x_0)+q$ is the trajectory of (4) through Wx_0+q, i.e., $\Gamma_2(Wx_0+q) = W\Gamma_1(x_0)+q$. Conversely, if $\Gamma_2(Wx_0+q) := y(t)$ is the trajectory of (4) through Wx_0+q, then $x(t, x_0)$, defined by

$$x(t, x_0) = e^{-(t/\tau)}\, x_0 + \frac{1}{\tau}\int_0^t e^{s/\tau}\, G(y(s))\, ds, \qquad (12)$$

is the trajectory of (3) through x_0, i.e., $\Gamma_1(x_0) = x(t, x_0)$.

ii) The numbers of equilibrium states of systems (3) and (4) are identical, and there is a one-to-one and onto correspondence between $F_1^{-1}(0)$ and $F_2^{-1}(0)$. Precisely, $x^* \in F_1^{-1}(0)$ iff $y^* = Wx^*+q \in F_2^{-1}(0)$; conversely $y^* \in F_2^{-1}(0)$ iff $x^* = G(y^*) \in F_1^{-1}(0)$.

iii) The coherent systems (8)–(11) all are well-defined, i.e., $\overline{Co}\Re(G)$, $\overline{Co}\Re(GW_q)$, $\Re(W_q)$, and $\overline{\Re(WG)}+q$ [if $\Re(G)$ is bounded and convex] are invariant sets or manifolds of systems (8)–(11), respectively.

Theorem 1 i) implies that the trajectories between systems (3) and (10) can be transferable from one to the other. However, the trajectories of (3) and (4) can not be transposed if one does not presume the nonsingularity of W. [This is because, in this case, $\Re(W_q)$ is just a manifold strictly included in R^N, so any trajectory $\Gamma_2(y_0)$ of (4) may not be expressed in terms of solutions of (3).] Theorem 1 ii) means that there is exactly one one-to-one mapping between the two equilibrium state sets of systems (3) and (4), and this fact is actually independent of the regularity of G and W. From this property it follows that the activation mapping G, no matter whether it is invertible, is invertible when restricted to the equilibrium state set $F_1^{-1}(0)$ and the inverse G^{-1} is given by

$$G^{-1}(x^*) = Wx^* + q, \qquad \forall\, x^* \in F_1^{-1}(0). \qquad (13)$$

Similarly, we have that W, when restricted to $F_2^{-1}(0) - q$, is regular even though it may not be so itself. As a result, given an equilibrium state x^* of (3), there is a unique equilibrium state $y^* = G^{-1}(x^*)$ of (4) corresponding to x^*, and vise versa. Such a pair of equilibria x^* and y^* is called as *a pair of mutually-mapped equilibria* of (3) and (4), denoted henceforth by (x^*, y^*). With this notion the following stability invariance property between systems (3) and (10) as well as between (8) and (11), can be justified (see [40]).

Theorem 2: Let $(x^*, y^*) \in F_1^{-1}(0) \times F_2^{-1}(0)$ be any pair of mutually-mapped equilibria of systems (3) and (4). Assume that G is a Lipschitzian, i.e., there is a positive constant $L(G)$ such that

$$\|G(y_1) - G(y_2)\| \le L(G)\, \|y_1 - y_2\|, \qquad \forall\, y_1, y_2 \in R^N.$$

Then the stability of systems (3) and (10), as well as (8) and (11), is invariant in the following sense.

i) x^* is stable (asymptotically stable/exponentially stable) in system (3) or (8) iff y^* is stable (asymptotic stable/exponential stable) in system (10) or (11).

ii) x^* is globally asymptotically stable (globally exponentially stable) in system (3) or (8) iff y^* is globally asymptotically stable (globally exponentially stable) in system (10) or (11).

iii) System (3) is globally convergent (globally exponentially convergent) iff system (10) is globally convergent (globally exponentially convergent).

iv) System (8) is globally convergent (globally exponential convergent) iff system (11) is globally convergent (globally exponential convergent).

The stability-invariance property stated in Theorem 2 should be precisely understood. It says that x^* is stable in a sense (say, globally or asymptotically) iff y^* is so in the same sense (namely, globally or asymptotically, too). Nevertheless, it should be carefully discriminated that "x^* is stable in system (3)" makes sense in the topology of R^N, because (3) is a dynamical system defined on the whole space R^N, whereas "x^* is stable in system (8)" then makes sense in the topology of $\overline{Co}\Re(G)$, which is, of course, a relative topology of R^N, since (8) is a dynamical system defined on the manifold $\overline{Co}\Re(G)$. Similarly, the stability of y^* in systems (10) or (11) should be understood respectively in the topologies of $\Re(W_q)$ and $\overline{\Re(WG)}+q$. Particularly, that system (8) is globally convergent means that trajectories $\Gamma_1(x_0)$ starting from any x_0 in $\overline{Co}\Re(G)$ are convergent, while that (11) is globally convergent only implies the convergence of trajectories $\Gamma_2(y_0)$ for every y_0 in $\overline{\Re(WG)}+q$.

From those aforementioned, a natural question arises: whether or not the stability-invariance property mentioned in Theorem 2 for systems (3) and (10), as well as (8) and (11), can be extended to the case between (3) and (4)? Unfortunately, this is still an open problem [40]. Nevertheless, since the stability of system (4) in any sense can sufficiently imply the stability of system (10) in the same sense, we have the following immediate consequences of Theorem 2.

Corollary 1: If $y^* \in F_2^{-1}(0)$, as an equilibrium state of (4), is stable in some sense, then $x^* \in F_1^{-1}(0)$, as an equilibrium state of (3), is stable in the same sense; if (4) is globally convergent, then so is (3).

Corollary 2: If $x^* \in F_1^{-1}(0)$, as an equilibrium state of (8), is stable in some sense, then $y^* \in F_2^{-1}(0)$, as an equilibrium state of (11), is stable in the same sense; if (8) is globally convergent, then so is (11).

Corollary 3: If $x^* \in F_1^{-1}(0)$, as an equilibrium state of (3) is stable in some sense, then $y^* \in F_2^{-1}(0)$, as an equilibrium state of (10), is stable in the same sense; if (3) is globally convergent, then so is (10).

Corollaries 1–3 underlie the feasibility of using system (4) to study system (3), using (8) to study (11), and using (3) to study (10). Such a methodology of using one model to study another model will be referred to as a *reference model approach*. Clearly, a reference model approach, even feasible, can be effective or beneficial only if the referenced model can gain by comparison. Thus, ascertaining the intrinsic properties and merits of models (3), (8) and (4) is a necessity for the effective use of the reference model approach. This will be elucidated in Section IV.

III. Characteristics of Activation Mappings

We introduce a set of abstract operator definitions to unify and formulate some common features and properties of the activation mappings G, as a basis of our further study. We present a series of examples to show how neural networks like the Hopfield-type NNs, ReBP-type NNs, BSB-type NNs, BAM-type NNs, Op-type NNs and CNNs naturally possess those properties, so that the abstract results established in this paper can directly apply.

We begin with the following definition.

Definition 1: Let G: $\mathbf{R}^N \rightarrow \mathbf{R}^N$ be a nonlinear mapping and let $F_2^{-1}(0)$ be the set of equilibrium states of system (4).

i) G is said to be diagonally nonlinear (or, is of diagonal nonlinearity) if G is defined componentwisely by

$$G(x) = (g_1(x_1),\, g_2(x_2),\, \ldots,\, g_N(x_N))^t$$

where each g_i is a one-dimensional nonlinear function.

ii) G is said to be a nearest point projection if there is a bounded, closed, convex subset $\Omega \subset \mathbf{R}^N$ such that

$$G(x) = \arg\min_{z\in\Omega} \|x - z\|.$$

Such a mapping is denoted by P_Ω, i.e., $G(x) = P_\Omega(x)$.

iii) G is said to be a diagonal projection if it is diagonally nonlinear, and each component function g_i is a one-dimensional nearest point projection.

iv) G is said to be uniformly anti-monotonic if there is a constant $\alpha_0 > 0$ such that, for any $y^*(1), y^*(2) \in F_2^{-1}(0)$,

$$\begin{aligned}&\langle G(y^*(1)) - G(y^*(2))\,,\, y^*(1) - y^*(2)\rangle\\ &\qquad \geq \alpha_0 \|G(y^*(1)) - G(y^*(2))\|^2. \end{aligned}\tag{14}$$

v) G is said to be an inversely pseudo-Lipschitzian if there is a constant $\beta_0 > 0$ such that, for any $y^*(1), y^*(2) \in F_2^{-1}(0)$,

$$\begin{aligned}&\langle G(y^*(1)) - G(y^*(2))\,,\, y^*(1) - y^*(2)\rangle\\ &\qquad \leq \beta_0 \|G(y^*(1)) - G(y^*(2))\|^2. \end{aligned}\tag{15}$$

vi) G is said to be diagonally uniform anti-monotonic if it is diagonally nonlinear, and each g_i is uniformly anti-monotonic: $\forall\, i = 1, 2, \ldots, N$, and $\forall\, y^*(1), y^*(2) \in F_2^{-1}(0)$,

$$\begin{aligned}&[g_i(y_i^*(1)) - g_i(y_i^*(2))] \cdot (y_i^*(1) - y_i^*(2))\\ &\qquad \geq \alpha_i [g_i(y_i^*(1)) - g_i(y_i^*(2))]^2. \end{aligned}\tag{16}$$

vii) G is said to be a diagonally inverse pseudo-Lipschitzian if it is diagonally nonlinear, and each g_i is inversely pseudo-Lipschitzian: $\forall\, i = 1, 2, \ldots, N$, and $\forall\, y^*(1), y^*(2) \in F_2^{-1}(0)$

$$\begin{aligned}&[g_i(y_i^*(1)) - g_i(y_i^*(2))]\, [y_i^*(1) - y_i^*(2)]\\ &\qquad \leq \beta_i (g_i(y_i^*(1)) - g_i(y_i^*(2)))^2. \end{aligned}\tag{17}$$

Examples 1–3 below show that the activation mappings appeared in most of the currently-known neural networks naturally possess the properties defined as above.

Example 1: $G = P_\Omega$ is a nearest point projection [25], [35]. In this case, G is uniformly anti-monotonic with $\alpha_0 = 1$.

Actually, it is known (see, e.g., [22]) that P_Ω is the nearest point projection of $\mathbf{R}^N$ onto Ω if and only if it satisfies

$$\langle x - P_\Omega(x),\, P_\Omega(x) - z\rangle \geq 0, \qquad \forall\, z \in \Omega,\ x \in R^N.$$

Taking $z := P_\Omega(y)$ with any $y \in \mathbf{R}^N$ then gives $\langle x - P_\Omega(x), P_\Omega(x) - P_\Omega(y)\rangle \geq 0$. Similarly we have $\langle y - P_\Omega(y), P_\Omega(y) - P_\Omega(x)\rangle \geq 0$. Adding these two inequalities leads to the result

$$\begin{aligned}\langle P_\Omega(x) - P_\Omega(y),\, x - y\rangle &\geq \|P_\Omega(x) - P_\Omega(y)\|^2,\\ &\qquad \forall\, x,\, y \in R^N. \end{aligned}\tag{18}$$

That is, G is uniformly anti-monotonic (with $\alpha_0 = 1$) in a more broad sense.

Such a projection nonlinearity has been extensively used in the BSP-type NNs (see, e.g., [25], [35]) and the Op-type NNs (see, e.g., [12], [13], [38], [37]).

Example 2: $G(x) = (g_1(x_1),\, g_2(x_2),\, \ldots,\, g_N(x_N))^t$ is a diagonal projection [8], [25], [30], [34]. Then G is diagonally uniform anti-monotonic with $\alpha_o = 1$.

This is because, in this case, each g_i is a one-dimensional nearest point projection.

In the BSP-type NNs (see, e.g., [25], [35]), CNNs (see, e.g., [8], [30], [34]) and the bound constrain optimization neural networks (BCOp-type NNs) (see, e.g., [5], [12], [27]), the following specific one-dimensional nearest point projection g_i: $\mathbf{R} \rightarrow [a_i, b_i]$ is used: either

$$g_i(s) = \begin{cases} a_i, & s \leq a_i\\ s, & s \in [a_i, b_i]\\ b_i, & s \geq b_i \end{cases}\tag{19}$$

or

$$g_i(s) = \tfrac{1}{2}\,(|s+1| - |s-1|).\tag{20}$$

Example 3: Let g_i be a continuously differentiable, strictly increasing function with range in $[a_i,\, b_i]$. Define $G(x) = (g_1(x_1),\, g_2(x_2),\, \ldots,\, g_N(x_N))^t$. Then G is both diagonally uniform anti-monotonic and a diagonally inverse pseudo-Lipschitzian, with $\alpha_i = 1/L_i$ and $\beta_i = 1/l_i$, where the constants l_i and L_i are defined, respectively, by

$$l_i = \min\{\|g_i'((W\zeta)_i)\| : \zeta \in \Omega\} > 0,$$
$$L_i = \max\{\|g_i'((W\zeta)_i)\| : \zeta \in \Omega\} < +\infty$$

with $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$.

In fact, it is easy to see that, in this case, there holds the following estimate:

$$\begin{aligned}\frac{1}{L_i}\,|g_i(x_i) - g_i(y_i)|^2 &\leq [g_i(x_i) - g_i(y_i)]\,(x_i - y_i)\\ &\leq \frac{1}{l_i}\,|g_i(x_i) - g_i(y_i)|^2, \end{aligned}\tag{21}$$

from which the conclusion directly follows.

In many neural networks such as the Hopfield-type NNs [18], [19], BAM-type NNs [23] and ReBP-type NNs [1], [31], [33], the nonlinear mapping G is frequently defined via a set of so-called sigmoid functions g_i, which may, for example, be defined by

$$g_i(s) = \tanh\left(\frac{\delta_i s}{2}\right) = \frac{1 - e^{-\delta_i s}}{1 + e^{-\delta_i s}}\tag{22}$$

where $\delta_i > 0$ is a parameter controlling the steepness of the sigmoidal curve. Thus, $\Re(G) = [-1, 1]^N$ and

$$l_i = \frac{2\delta_i e^{\delta_i r_i}}{(1 + e^{\delta_i r_i})^2}, \qquad L_i = \delta_i/2$$

where $r_i = \sum_{j=0}^{N} |w_{ij}| + |q_i|$ $(i = 1, 2, \ldots, N)$. So G is both diagonally uniform anti-monotonic with $\alpha_i = 2/\delta_i$ and a diagonally inverse pseudo-Lipschitzian with $\beta_i = (1 + e^{\delta_i r_i})^2/(2\delta_i e^{\delta_i r_i})$.

IV. Establishment of Fundamental Lemmas on Two Models

With the above examples in mind, we establish in this section some fundamental lemmas for the static neural network model (3) and the local field neural network model (4). These lemmas will highlight certain intrinsic properties and merits of each model and underlie the efficiency of subsequent applications of the reference model approach in the next section.

A. The Static Model

The first result on the static neural network model (3) is the finiteness of its equilibrium states.

Lemma 1. (Finiteness of Equilibrium States): System (3) has at most a finite number of equilibrium states if one of the following conditions **C1)**–**C5)** is satisfied.

C1): $G = P_\Omega$ is a diagonal projection with Ω being a hyperrectangle, say,

$$\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N]$$

and 1 is not an eigenvalue of any principal submatrix of W.

C2): G is uniformly anti-monotonic with α_0 satisfying that

$$\alpha_0 > \lambda_{\max}\left(\frac{W + W^t}{2}\right),$$

where $\lambda_{\max}((W + W^t)/2)$ stands for the largest eigenvalue of matrix $(W + W^t)/2$.

C3): G is an inversely pseudo-Lipschitzian with β_0 satisfying that

$$\beta_0 < \lambda_{\min}\left(\frac{W + W^t}{2}\right)$$

where $\lambda_{\min}((W + W^t)/2)$ denotes the smallest eigenvalue of matrix $(W + W^t)/2$.

C4): G is diagonally uniform anti-monotonic with coefficients α_i $(i = 1, 2, \ldots, N)$, and there is a positive parameter matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that

$$\Lambda\Gamma - \frac{\Gamma W + W^t\Gamma}{2} \text{ is positive definite}$$

where $\Lambda = diag\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$.

C5): G is a diagonally inverse pseudo-Lipschitzian with the coefficients β_i $(i = 1, 2, \ldots, N)$, and there is a positive parameter matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that

$$B\Gamma - \frac{\Gamma W + W^t\Gamma}{2} \text{ is negative definite}$$

where $B = diag\{\beta_1, \beta_2, \ldots, \beta_N\}$.

Proof: Assume first that condition **C1)** is satisfied. Then $\overline{Co}\Re(G) = \Omega$ is a N-dimensional hyperrectangle. It is known that a k-dimensional face Ω_k^0 of Ω is a subset of the interior of Ω, in which every element $x = (x_1, x_2, \ldots, x_N)$ has k fixed coordinates with each taking extreme value a_i or b_i when x_i is fixed. Thus, for any fixed k, there exist $\binom{k}{N} \cdot 2^k$ k-dimensional faces of Ω. So, totally, from 0 to $N-1$, Ω has K_0 faces, where

$$K_0 = \sum_{k=0}^{N-1} \binom{k}{N} 2^k = 3^N - 2^N.$$

If we can prove that system (3) has at most one equilibrium state on each face of Ω, then (3) has at most 3^N equilibrium states by taking into account the fact that the 2^N extreme points may all be equilibrium states of the system. This then proves the lemma under condition **C1)**.

For any k-dimensional face Ω_k^0, let $J(k)$ be the index set such that every x_i in $x = (x_1, x_2, \ldots, x_N) \in \Omega_k^0$ takes a fixed extreme value when $i \in J(k)$ and let $I(k) = \{1, 2, \ldots, N\}\backslash J(k)$. Note that the cardinal number of $I(k)$ is $N - k$ and that the projection mapping $G = P_\Omega$ can be written as $G = (g_1, g_2, \ldots, g_N)^t$ with $g_i\colon R \to [a_i, b_i]$ being defined by (19). We claim that equation $-x + G(Wx + q) = 0$ has at most one solution on Ω_k^0. Suppose this is not true. Then there would exist $x^*(1) = (x_1^*(1), x_2^*(1), \ldots, x_N^*(1))^t$ and $x^*(2) = (x_1^*(2), x_2^*(2), \ldots, x_N^*(2))^t$ on Ω_k^0 such that $x^*(1) \neq x^*(2)$ and

$$x^*(i) = G\left(Wx^*(i) + q\right), \qquad i = 1, 2. \tag{23}$$

Since $x^*(1), x^*(2) \in \Omega_k^0$, then $x_j^*(1) = x_j^*(2)$ if $j \in J(k)$, and $x_i^*(1)$ and $x_i^*(2)$ are in the interior of Ω_k^0 if $i \in I(k)$. Thus, by (23),

$$x_i^*(l) = g_i\left(\sum_{j=1}^{N} w_{ij}x_j^*(l) + q_i\right) = \sum_{j=1}^{N} w_{ij}x_j^*(l) + q_i$$

with $l = 1, 2$ and $i \in I(k)$, so that

$$\begin{aligned} x_i^*(1) - x_i^*(2) &= \sum_{j=1}^{N} w_{ij}\left(x_j^*(1) - x_j^*(2)\right) \\ &= \sum_{j \in I(k)} w_{ij}\left(x_j^*(1) - x_j^*(2)\right), \qquad i \in I(k). \end{aligned}$$

Denote by W_I the $(N-k)$th-order principal submatrix of W, that is, $W_I = (w_{ij}\colon i, j \in I(k))$ and let $\zeta_I = (x_i^*(1) - x_i^*(2), i \in I(k))^t$. Then the above equation can be rewritten as

$$\zeta_I = W_I\zeta_I$$

which, together with the fact that $\zeta_I \neq 0$, implies that 1 is an eigenvalue of W_I. This contradicts the assumption that 1 is not an eigenvalue of any principal submatrix of W, and the lemma is thus proved under the condition **C1)**, as remarked above.

Suppose now one of conditions **C2)**–**C5)** is satisfied. We want to prove that (3) has exactly one equilibrium state. If this is not the case, then there would be two equilibria $x^*(1)$ and $x^*(2)$ in $F_1^{-1}(0)$ such that $x^*(1) \neq x^*(2)$ and (23) holds. Since, when restricted to the equilibrium state set $F_1^{-1}(0)$, the inverse of G, G^{-1}, exists, it follows from (23) that

$$G^{-1}(x^*(i)) = Wx^*(i) + q, \qquad i = 1, 2.$$

Thus, for any positive parameter matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ we have

$$\begin{aligned}&\left\langle G^{-1}(x^*(1)) - G^{-1}(x^*(2)), \Gamma(x^*(1) - x^*(2))\right\rangle \\ &\qquad = \langle W(x^*(1) - x^*(2)), \Gamma(x^*(1) - x^*(2))\rangle. \qquad (24)\end{aligned}$$

If G is diagonally uniform anti-monotonic, then from (16) it follows that

$$\begin{aligned}&\left\langle G^{-1}(x^*(1)) - G^{-1}(x^*(2)), \Gamma(x^*(1) - x^*(2))\right\rangle \\ &= \sum_{i=1}^{N} \xi_i \left[g_i^{-1}(x_i^*(1)) - g_i^{-1}(x_i^*(2))\right] \cdot \left[x_i^*(1) - x_i^*(2)\right] \\ &= \sum_{i=1}^{N} \xi_i \left(g_i^{-1}(x_i^*(1)) - g_i^{-1}(x_i^*(2))\right) \\ &\quad \cdot \left[g_i\left(g_i^{-1}(x_i^*(1))\right) - g_i\left(g_i^{-1}(x_i^*(2))\right)\right] \\ &\geq \sum_{i=1}^{N} \xi_i \alpha_i \left[g_i\left(g_i^{-1}(x_i^*(1))\right) - g_i\left(g_i^{-1}(x_i^*(2))\right)\right]^2 \\ &= \sum_{i=1}^{N} \xi_i \alpha_i \left[x_i^*(1) - x_i^*(2)\right]^2 \\ &= \langle x^*(1) - x^*(2), \Lambda\Gamma[x^*(1) - x^*(2)]\rangle. \qquad (25)\end{aligned}$$

This, combined with (24), yields

$$\begin{aligned}0 &\leq \langle(\Gamma W - \Lambda\Gamma)(x^*(1) - x^*(2)), (x^*(1) - x^*(2))\rangle \\ &= \left\langle\left(\frac{\Gamma W + W^t\Gamma}{2} - \Lambda\Gamma\right)(x^*(1) - x^*(2)), (x^*(1) - x^*(2))\right\rangle.\end{aligned}$$

If **C4**) is satisfied, then, since $x^*(1) \neq x^*(2)$ and by the positive definiteness of $\Lambda\Gamma - (\Gamma W + W^t\Gamma)/2$, the right hand side of the above inequality is negative. This leads to a contradiction.

Now if condition **C5**) is satisfied, then (24), combined with (25) with Λ being replaced by B, gives

$$\left\langle\left[B\Gamma - \frac{\Gamma W + W^t\Gamma}{2}\right][x^*(1) - x^*(2)], (x^*(1) - x^*(2))\right\rangle \geq 0.$$

By the assumption, $B\Gamma - (\Gamma W + W^t\Gamma)/2$ is negative, so the left hand side of the above inequality is less than zero. We thus have a contradiction.

On the other hand, apply condition **C2**) to deduce

$$\begin{aligned}&\langle W(x^*(1) - x^*(2)), x^*(1) - x^*(2)\rangle \\ &\quad \leq \lambda_{\max}\left(\frac{W + W^t}{2}\right)\|x^*(1) - x^*(2)\|^2 \\ &\quad < \alpha_0 \|x^*(1) - x^*(2)\|^2 \\ &\quad \leq \left\langle G^{-1}(x^*(1)) - G^{-1}(x^*(2)), (x^*(1) - x^*(2))\right\rangle\end{aligned}$$

whilst use of condition **C3**) gives

$$\begin{aligned}&\left\langle G^{-1}\left(x^*(1)\right) - G^{-1}(x^*(2)), (x^*(1) - x^*(2))\right\rangle \\ &\quad \leq \beta_0 \|x^*(1) - x^*(2)\|^2 \\ &\quad < \lambda_{\min}\left(\frac{W + W^t}{2}\right)\|x^*(1) - x^*(2)\|^2 \\ &\quad \leq \langle W(x^*(1) - x^*(2)), x^*(1) - x^*(2)\rangle.\end{aligned}$$

Either of these two inequalities contradicts to the identity (24). The proof of Lemma 1 is thus complete. □

Remark 1: In the proof of Lemma 1 an essential role is played by both the invertibility of the mapping G when restricted to the equilibria set $F_1^{-1}(0)$ and the specific structure of $-x + G(Wx + q)$. In contrast, similar results [e.g., in the setting of **C1**)] may not be so easily deduced from the local field model (4). In fact, assuming two distinct equilibria $y^*(1)$ and $y^*(2)$ of (4) can only lead to $y^*(1) - y^*(2) = W[G(y^*(1)) - G(y^*(2))]$, which is in general intractable further since W is not invertible. This highlights the benefit of the static neural network model (3).

The importance of Lemma 1 lies in the fact that, when combined with the LaSalle invariance principle in dynamical system theory (say, [24]), it can naturally lead to the global convergent dynamics of system (3), as obtained in Lemma 2.

To state Lemma 2, we need to recall some basic facts from dynamical system theory [24]. Let D be a subset of R^N and let E: $D \to R$ be a continuously differentiable function. Then

i) E is said to be an energy function of system (3) if it decreases along the trajectory of (3), that is, $d(E(x(t, x_0)))/dt \leq 0$ for all $x_0 \in D$.

ii) The energy function E is said to be *strict* if its derivative along the trajectories vanishes only at the equilibria of (3), i.e.,

$$\frac{dE(x(t, x))}{dt} = 0, \text{ iff } x \in F_1^{-1}(0).$$

iii) (*LaSalle Invariance Principle*): If system (3) has an energy function E on D and $\Gamma_1(x_0) \subseteq D$ is bounded, then there is a constant c such that

$$\omega(\Gamma_1(x_0)) \subseteq \Sigma \cap E^{-1}(c),$$

where $E^{-1}(c) = \{x | E(x) = c\}$ and Σ is the largest invariant set contained in

$$E_0 = \left\{x \,\middle|\, \frac{dE(x)}{dt} = 0, x \in \overline{D}\right\}.$$

It is worth noting that the LaSalle invariance principle provides us only with enclosure information on the ω-limit set $\omega(\Gamma_1(x_0))$ in the sense that $\omega(\Gamma_1(x_0)) \subset M \cap E^{-1}(c)$, or, it only implies the quasiconvergence by Hirsch [17]. It says, however, nothing on convergence of the trajectory $\Gamma_1(x_0)$ itself (since $\Gamma_1(x_0) = x(t, x_0)$ may still be oscillatory even though it is quasiconvergent; see, e.g., [41] for an example).

We now apply Lemma 1 to establish a basic result on global convergence of system (3).

Lemma 2. (Global Convergence): i) System (3) is of globally convergent dynamics if it has an exact energy and one of the conditions **C1**)–**C5**) in Lemma 1 is satisfied.

ii) System (8) is globally convergent if either

a) $G = P_\Omega$ is a nearest point projection, W is symmetric and satisfies one of conditions **C2**)–**C3**) in Lemma 1, or,

b) $G = P_\Omega$ is a diagonal projection, there is a positive definite diagonal matrix Γ such that ΓW is symmetric, and W satisfies condition **H**) or one of the conditions **H1**)–**H4**) below.

H): 1 is not an eigenvalue of any principal submatrix of W.

H1): There is a norm $\|\cdot\|$ such that $\|W\| < 1$.

H2): There are constants $\alpha_i > 0$ $(i = 1, \ldots, N)$ such that

$$|w_{ii} - 1| > \max\left\{\sum_{j\neq i}^{N} \frac{\alpha_j}{\alpha_i}|w_{ij}|, \sum_{j\neq i}^{N} \frac{\alpha_i}{\alpha_j}|w_{ji}|\right\} \quad \forall i = 1, 2, \ldots, N. \tag{26}$$

H3): $I - W$ or $W - I$ is a nonsingular M-matrix.

H4): $I - W$ or $W - I$ is a positive definite matrix.

Proof: By the LaSalle invariance principle, the existence of an exact energy function to system (3) ensures that the ω-limit set, $\omega(\Gamma_1(x_0))$, of system (3) is included in the set $F_1^{-1}(0)$ of equilibrium states and that the trajectory $x(t, x_0)$ approaches to the largest invariant subset of $F_1^{-1}(0)$ as time goes to infinity.

On the other hand, the ω-limit set is invariant and connected (see, e.g., [36, p. 323]). So $x(t, x_0)$ either converges to a single equilibrium state or approaches to an invariant subset in $F_1^{-1}(0)$ with the cardinal number being infinite. This latter case, however, does not occur since, by Lemma 1, the equilibrium set $F_1^{-1}(0)$ is finite. Thus, (3) must have globally convergent dynamics, which implies the assertion i) of Lemma 2.

To prove ii), let us first verify that, under either of conditions a) and b), system (8) does have an exact energy function E: $\Omega \to R$ satisfying that

$$\frac{dE(x)}{dt} = \left\langle \nabla E(x), \frac{dx}{dt}\right\rangle \le -C\|F_1(x)\|^2 \le 0 \tag{27}$$

with a positive constant C, where F_1 is defined as in (5). In fact, if a) is satisfied, then $G = P_\Omega$ is a nearest point projection so $\overline{Co}\Re(G) = \Omega$. Theorem 1 iii) thus assures that (8) is a well-defined dynamical system on set Ω and therefore $x(t, x_0) \in \Omega$ for all $t \ge 0$ as long as $x_0 \in \Omega$. When W is symmetric, we can take $E(x) = (1/2)\, x^t(I - W)x - x^t q$ so

$$\begin{aligned}\left\langle \nabla E(x), \frac{dx}{dt}\right\rangle &= \langle x - Wx - q, -x + P_\Omega(Wx + q)\rangle\\ &= -\langle Wx + q - x, P_\Omega(Wx + q) - P_\Omega(x)\rangle\\ &\le -\|P_\Omega(Wx + q) - P_\Omega(x)\|^2\\ &= -\|P_\Omega(Wx + q) - x\|^2\\ &= -\|F_1(x)\|^2 \le 0,\end{aligned}$$

where property (18) has been used to derive the last inequality. Thus, (27) follows with $C = 1$.

If condition b) is satisfied, we can define $E(x) = (1/2)\, x^t(\Gamma - \Gamma W)x - x^t\Gamma q$ and, using the symmetry assumption of ΓW and the diagonal property of P_Ω, obtain

$$\begin{aligned}\left\langle \nabla E(x), \frac{dx}{dt}\right\rangle &= -\langle \Gamma(Wx + q - x), P_\Omega(Wx + q) - P_\Omega(x)\rangle\\ &\le -\sum_{i=1}^{N} \alpha_i \left|g_i\left((Wx + q)_i\right) - g_i(x_i)\right|^2\\ &\le -\min_{1\le i\le N}\{\alpha_i\}\|F_1(x)\|^2 \le 0.\end{aligned}$$

So again (27) holds with $C = \min_{1\le i\le N}\{\alpha_i\}$.

We now prove that (8) has a finite number of equilibrium states. First by Lemma 1 this is true if one of **C2**)–**C3**) and **H**) is satisfied. We now proceed by showing that any of conditions **H1**)–**H4**) actually implies **H**). That **H2**) implies **H**) follows from the well-known Gerschgorin theorem (say, e.g., [4]). From the facts that the norm of any principal submatrix of W is not larger than the norm of W and that any principal submatrix of W inherits the nonsingularity, the M-matrix property and the positive definiteness property of W, the implications **H1**) $\Longrightarrow$ **H**), **H3**) $\Longrightarrow$ **H**) and **H4**) $\Longrightarrow$ **H**) follow immediately. Statement ii) follows by arguing similarly as in the proof of i). This completes the proof of Lemma 2. □

Lemma 2 provides us with a very fundamental global convergence result on systems (3) and (8). We will make use of such a generic result to study the local field model (4) in the next section.

B. The Local Field Model

In contrast to the benefit of easily identifying finiteness of equilibria with the static model (3), which then leads to an exposition of generic convergent dynamics property of the system, the local field model (4) has an exclusive advantage apt to conduct stability analysis. This is supported by the currently existing various in-depth results on Hopfield-type neural networks. See, e.g., [2], [3], [6], [7], [10], [11], [14], [20], [21], [26], [28], [29], [32], [42], [43] and the references therein. In the following lemmas we summarize the basic stability results of system (4).

Lemma 3. (Globally Exponential Stability): Assume that $G = (g_1, g_2, \ldots, g_N)^t$ is diagonally nonlinear with each g_i being Lipschitz continuous [that is, $|g_i(s) - g_i(t)| \le L_i|s - t|$ for any $s, t \in R$]. For any diagonal matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$, we write

$$M = L^{-1} - W$$

$$M(\Gamma) = L^{-1}\Gamma - \frac{\Gamma W + W^t\Gamma}{2}$$

where $L = diag\{L_1, L_2, \ldots, L_N\}$. Then, system (4) has a unique equilibrium state y^*. Further, y^* is globally exponentially stable if there is a positive definite diagonal matrix Γ such that, for any $i \in \{1, 2, \ldots, N\}$, one of the following conditions **D**) and **D1**)–**D8**) is satisfied.

D) $M(\Gamma)$ is positive definite.
D1): $L_i^{-1}\xi_i - \xi_i w_{ii} > \sum_{j\neq i}^N \xi_j |w_{ij}|$.
D2): $L_i^{-1}\xi_i - \xi_i w_{ii} > \sum_{j\neq i}^N \xi_j |w_{ji}|$.
D3): $L_i^{-1}\xi_i - \xi_i w_{ii} > (1/2) \sum_{j\neq i}^N |\xi_j w_{ji} + \xi_i w_{ij}|$
D4): $\lambda_{\min}(M(\Gamma)) > 0$ [i.e., the matrix measure $\mu(-\Gamma M) < 0$].
D5): $\Gamma W + W^t \Gamma$ is nonnegative.
D6): $L_i^{-1}\xi_i > \{\xi_i w_{ii} + \sum_{j\neq i}^N \xi_j |w_{ij}|\}^+$.
D7): $\xi_i > L_i \xi_i (w_{ii})^+ + \sum_{j\neq i}^N L_j \xi_j |w_{ij}|$.
D8): $\xi_i > L_i \xi_i (w_{ii})^+ + (1/2) \sum_{j\neq i}^N \{L_j \xi_j |w_{ij}| + \xi_i L_i |w_{ji}|\}$
where $(a)^+ = \max\{0, a\}$.

Proof: Under the assumption of positive definiteness of matrix $M(\Gamma)$ [namely, condition **D**)], it is easy to see that there is a unique equilibrium state y^* to system (4) (cf. Lemma 1). Let

$$E(y) = \sum_{i=1}^N \xi_i \int_{y_i^*}^{y_i} (g_i(s) - g_i(y_i*))\, ds. \tag{28}$$

It can be verified similarly as in [43] that E gives a strict energy function of system (4). Thus the global convergence (or attractivity) of the trajectory $\Gamma_2(y_0)$ follows immediately from the LaSalle invariance principle (cf. the proof of Lemma 2). Further, an exponential estimate as (7) on the decay of solution $y(t, y_0)$ can be established similarly as in [26], [43].

Clearly, either of conditions **D4**) and **D5**) implies **D**). Note that any of conditions **D1**)–**D3**) and **D6**)–**D8**) can imply that the comparison matrix of W is a nonsingular M-matrix, so there is a positive definite diagonal matrix Γ such that $M(\Gamma)$ is positive definite [4]. Thus, any of conditions **D1**)–**D3**) and **D6**)–**D8**) sufficiently implies **D**). The lemma is thus proved. □

Remark 2: Lemma 3 has been proved by many authors under certain specific conditions as above. In particular, the same or similar results have been established in [12], [26], [43] under condition **D**), in [14], [11], [29], [32] under **D1**), in [10], [14] under **D2**), in [42] under **D3**), in [21] under **D5**), and in [7], [11] under **D6**)–**D8**).

The following lemma contains the latest stability results due to Chen and Amari [6].

Lemma 4. (Global Attractivity): Assume $G = (g_1, g_2, \ldots, g_N)^t$ with $g_i(s) = \tanh(L_i s)$, $L_i > 0$ $(i = 1, 2, \ldots, N)$. Then (4) has a unique equilibrium state y^*, and y^* is globally attractive if there is a positive definite diagonal matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that one of the following conditions **E1**)–**E3**) holds: for $i = 1, 2, \ldots N$.
E1): $\xi_i \geq L_i \{\xi_i w_{ii} + \sum_{j\neq i}^N \xi_j |w_{ji}|\}^+$.
E2): $\xi_i \geq L_i \xi_i (w_{ii}^+ + \sum_{j\neq i}^N L_j \xi_j w_{ij}|$.
E3): $\xi_i \geq L_i \xi_i (w_{ii}^+ + (1/2) \sum_{j\neq i}^N \{L_i \xi_j w_{ij}| + \xi_i L_j |w_{ji}|\}$.

It is interesting to note that, as contrasted to so abundant and in-depth stability results of the local field model (4), there is few generic results on stability of the static model (3) [1], [17], [25], [27], [33], [37]. We will apply the reference model approach in the next section to transfer all the stability results on model (4) to model (3).

V. Application of Reference Model Approach

As an application of the reference model approach, in this section the static and local field neural network models (3) and (4) are alternatively used as a reference to investigate their counterparts, and a set of new, unified stability results on both models will be derived.

A. Local Field Model Used as a Reference

As explained in Corollary 1 of Section II, the stability properties of model (3) can be totally mirrored from those of model (4). This provides a rationale of the application in this subsection. Furthermore, we have seen in the last section that there have been abundant in-depth stability results on model (4). By taking advantage of such a benefit of model (4), the following series of generic results on model (3) can be obtained.

Theorem 3. (Globally Exponential Stability): Assume that $G = (g_1, g_2, \ldots, g_N)^t$ is diagonally nonlinear with each g_i being Lipschitz continuous [that is, $|g_i(s) - g_i(t)| \leq L_i |s - t|$ for any $s, t \in R$]. Let $L = diag\{L_1, L_2, \ldots, L_N\}$. Then, under any of the following conditions **P1**)–**P4**), system (3) has a unique equilibrium point x^*, and x^* is globally exponential stable.

P1): There is a positive definite diagonal matrix Γ such that $M(\Gamma) = L^{-1}\Gamma - (\Gamma W + W^t \Gamma)/2$ is positive definite.

P2): Any of conditions **D1**)–**D8**) in Lemma 3 is satisfied.

P3): G is a diagonal projection and there is a positive definite diagonal matrix Γ such that $\Gamma - (\Gamma W + W^t \Gamma)/2$ is positive definite.

P4): G is a diagonal projection and any of conditions **D1**)–**D8**) in Lemma 3 is satisfied with $L = I$.

Proof: If either **P1**) or **P2**) is satisfied, then by Lemma 3 there exists a unique equilibrium state y^* of system (4) and y^* is globally exponentially stable. Thus, as a subsystem of (4), system (10) has a unique equilibrium state y^* that is globally exponentially stable. Consequently, by Theorem 1 and Corollary 1, there is a unique equilibrium state x^* $(=G(y^*))$ of (3) such that x^* is globally exponentially stable.

Now if $G = (g_1, g_2, \ldots, g_N)^t$ is a diagonal projection, then each g_i is a one-dimensional nearest point projection, so, from (18), g_i satisfies the Lipschitz condition with constant $\alpha_i = 1$. Thus Theorem 3 follows from Lemma 3, Theorem 1 and Corollary 1 in the case when either **P3**) or **P4**) is satisfied. This proves the theorem. □

Theorem 4. (Global Attractivity): Assume that $G = (g_1, g_2, \ldots, g_N)^t$ is a diagonal nonlinear mapping with $g_i(s) = \tanh(L_i s)$, $L_i > 0$ $(i = 1, 2, \ldots, N)$. Then (3) has a unique equilibrium state x^*, and x^* is globally attractive if there is a positive definite diagonal matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that any of conditions **E1**)–**E3**) in Lemma 4 is satisfied.

Proof: This directly follows from Lemma 4, Theorem 1 and Corollary 1. □

Remark 3: The generic globally exponential stability and convergence in Theorems 3 and 4 are taken in the sense of the topology of the whole space $\mathbf{R}^N$. These results are new for model (3). In particular, they unify, sharpen or generalize all the relevant results previously reported in [3], [5], [12], [17], [25], [27], [33], [37].

Theorems 3–4 can be directly applied to various specific types of neural networks such as the ReBP-type NNs [1], [31], [33], BCOp-type NNs [5], [12], [13], [27], [38], [39] and

BSB-type NNs [25], [35]. These networks, as in their most standard forms, are, respectively, modeled by

ReBP-typeNNs:

$$\tau\,\frac{dx_i}{dt} = -x_i + g_i\left(\sum_{j=1}^{N} w_{ij}x_j + \theta_i\right), \quad i = 1, 2, \ldots, N \tag{29}$$

or

$$\tau\,\frac{dx}{dt} = -x + G(Wx + \theta) \tag{30}$$

BCOp-typeNNs:

$$\tau\,\frac{dx_i}{dt} = -x_i(t) + g_i\left(x_i - \alpha_i\sum_{j=1}^{N} Q_{ij}x_j + q_i\right) \quad i = 1, 2, \ldots, N \tag{31}$$

or

$$\tau\,\frac{dx}{dt} = -x + P_\Omega(x - \Lambda Qx + q) \tag{32}$$

BSB-typeNNs:

$$\frac{dx_i}{dt} = -x_i + g_i\left(x_i + \alpha\sum_{j=1}^{N} w_{ij}x_j + \alpha b_i\right) \quad i = 1, 2, \ldots, N \tag{33}$$

or

$$\frac{dx}{dt} = -x + P_\Omega(x + \alpha Wx + \alpha b). \tag{34}$$

In (29)–(34), $W = (w_{ij})$, $Q = (Q_{ij})$, $\Lambda = diag\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ are $N \times N$ matrices, q, b and θ are constant N-dimensional vectors, and α, α_i, τ are all positive parameters. The nonlinear activation mapping $G = (g_1, g_2, \ldots, g_N)^t$ in ReBP-type NNs (29)–(30) is of diagonal nonlinearity with each g_i being sigmoid defined as in (22) and $\Re(G) = [-1, 1]^N$. The function g_i in BC-Op-type NNs (31) is the one-dimensional nearest point projection defined by (19) with $\Omega_i = [a_i, b_i]$, and the g_i in BSB-type NNs (33) is defined by (20) with $\Omega_i = [-1, +1]$. Finally, the nonlinear mapping P_Ω in (32) and (34) are generic nearest point projections of R^N onto some closed, bounded convex subset Ω, which, different from those in (31) and (33), are not of diagonal nonlinearity in general.

Application of Theorems 1, 3–4 and Lemma 2 to the above networks are straightforward, and we have, for example, the following corollaries.

Corollary 4. (ReBP-Type NNs): Assume $g_i(s) = \tanh(\delta_i s/2)$ ($\delta_i > 0$) is used in (29). Then the ReBP-type networks (29) defines a unique trajectory $x(t, x_0)$, starting from any $x_0 \in R^N$, and has a nonempty equilibrium state set $F_1^{-1}(0)$. Moreover, the trajectory $x(t, x_0)$ has the following properties:

i) $x(t, x_0) \in [-1, 1]^N$ if $x_0 \in [-1, 1]^N$ [Theorem 1 iii)].

ii) $F_1^{-1}(0)$ contains only one state, say, x^*; x^* is globally asymptotic attractive if there is a set of positive parameters $\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that one of the conditions **E1**)–**E3**) in Lemma 4 is satisfied with $L_i = \delta_i/2$ (Theorem 4).

iii) The unique equilibrium state x^* is globally exponentially stable if there is a positive definite diagonal parameter matrix $\Gamma = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$ such that the matrix $L^{-1}\Gamma - (W\Gamma + \Gamma W^t)/2$ is positive definite, where

$$L = \tfrac{1}{2}\, diag\{\delta_1, \delta_2, \ldots, \delta_N\}.$$

This is particularly true if any of conditions **D1**)–**D8**) in Lemma 3 is fulfilled (Theorem 3).

Corollary 5. (BCOp-Type NNs): For any $x_0 \in \Omega$, there is a unique solution $x(t, x_0)$ to the BCOp-type NNs (32) [(31)]. The solution converges to the unique equilibrium state of (32) if Q is symmetric and positive definite [Lemma 2 i)]. Furthermore, if $x(t, x_0)$ is defined through model (31), then

- i) it exponentially converges to the unique equilibrium state, starting from any $x_0 \in R^N$, if there is a positive definite diagonal matrix Γ such that $(\Gamma Q + Q^t\Gamma)/2$ is positive definite. This is true, particularly when one of conditions **D1**)–**D8**) in Lemma 3 is satisfied with $L_i = 1$ and $W = I - Q$ [Theorem 3 P2)];
- ii) it converges to an equilibrium state of (31), starting from any $x_0 \in \Omega$, if there is a positive definite diagonal matrix Γ such that ΓQ is symmetric and one of the following conditions is satisfied:
  - a) ΓQ is positive definite [Lemma 2 i)];
  - b) all the principal submatrices of Q are nonsingular [Lemma 2 ii)];
  - c) any of the conditions H1)–H4) in Lemma 2 is satisfied with $W = (I - Q)$ [Lemma 2 (ii)].

Corollary 6. (BSB-Type NNs): There is a unique solution $x(t, x_0)$ to the BSB-type model (34) [(33)] for any $x_0 \in R^N$. If W is symmetric and negative definite, then the equilibrium state of (34) is unique, and the unique equilibrium state is globally attractive [Lemma 2 i)]. Furthermore, if the solution $x(t, x_0)$ is defined through model (33), then

- i) there is a unique equilibrium state x^*, and x^* *is* globally exponentially stable if there is a positive definite diagonal matrix Γ such that $(\Gamma W + W^t\Gamma)/2$ is negative definite [Theorem 3-P3)]. This is true, particularly when any of conditions **D1**)–**D8**) in Lemma 3 is satisfied with $L_i = 1$ and $W := I + \alpha W$ [Theorem 3-P4)];
- ii) it converges to an equilibrium state of (33), starting from any $x_0 \in \Omega$, if there is a positive definite diagonal matrix Γ such that ΓW is symmetric and one of the following conditions is satisfied:
  - a) ΓW is positive definite [Theorem 3-P3)];
  - b) all the principal submatrices of ΓW are nonsingular [Lemma 2 ii)];
  - c) any of conditions H1)–H4) in Theorem 8 with $W := I + \alpha W$ [Lemma 2 ii)].

Remark 4: i) It is known that the globally convergent dynamics of ReBP-type NNs has been a prerequisite for their application in learning and recognition [16], [31]. Owing to its difficulty, there has been lack of a systematic and in-depth analysis on such dynamical property (cf. [1], [33]). Corollary 4 provides us with such a general theory, which, in particular, generalizes the analysis in [1], [33].

ii) The BCOp-type NNs (31) and (32) are extensively studied in recent years by many authors (see, e.g., [3], [5], [12], [13], [27], [37], [38] and the references quoted there). The aim of such networks is to solve bound-constraint [or, more generally, domain (Ω)-constraint] quadratic optimization problems:

$$\min E(x) = x^t Qx + x^t q + c$$
$$s.t.\ a_i \le x_i \le b_i, \qquad i = 1, 2, \ldots, N \tag{35}$$

[or, $\min_{x\in\Omega} E(x)$ with a bounded closed convex set Ω in R^N]. Recently obtained is the result on convergence of the network that the trajectory of (4) globally and exponentially converge to the unique solution of (35) if Q is positive definite and $(I-\Lambda Q)$ is invertible [26], [28]. Corollary 5 sharpens and generalizes this result in the sense that it not only totally dismisses the invertibility assumption on $(I-\Lambda Q)$, but also gives various nonpositive definiteness types of convergence conditions such as condition that all the principal submatrices of Q are nonsingular. Corollary 6 also unifies and generalizes the corresponding stability results in [25], [35].

B. The Static Model Used as a Reference

We now take the static model (3) as a reference to study properties of the local field model (4), presenting only one particular example.

Consider the CNNs with the case when G is a nearest point projection [8], [29], [33]. In this case, G is uniformly anti-monotonic with the constant $\alpha_0 = 1$, and we have the following result.

Theorem 5: Assume that G is a nearest point projection, not necessarily diagonal, and that W is symmetric with $I - W$ positive definite. Then there is a unique equilibrium state y^* to systems (11) and (4). As an equilibrium state of (11), y^* is globally attractive [or, system (11) is globally convergent].

Proof: Let $Q = I - W$. Then $W = I - Q$ is symmetric and $\lambda_{\max}((W + W^t)/2) < 1$. So, by Lemmas 1 and 2, (3) [and therefore (8)] has a unique equilibrium state x^*, and as an equilibrium state of (8) x^* is globally attractive [or, (8) is globally convergent]. Thus, by Theorem 1 and Corollary 2, (11) has a unique equilibrium state y^* which is globally attractive for system (11). This means that system (11) is globally convergent with $y(t, y_0)$ converging to y^* for any $y_0 \in \overline{\Re(WG)} + q$. □

Theorem 5 can be further strengthened if G is of diagonal nonlinearity.

Theorem 6: Assume that G is a diagonal projection and that $y_1(t, y_0)$ and $y_2(t, y_0)$ are the unique solutions of systems (10) and (11), respectively. Then

i) there is a unique equilibrium state y^* of (4) [and therefore, of (10) and (11)]. As an equilibrium state of (4), (10) and (11), y^* is globally exponentially stable if there is a positive definite diagonal matrix Γ such that either $M(\Gamma) = \Gamma - (\Gamma W + W^t\Gamma)/2$ is positive definite or any of conditions **D1**)–**D8**) is satisfied with $L_i = 1$;

ii) system (11) is globally convergent [i.e., $y_2(t, y_0)$ converges to an equilibrium state for any $y_0 \in \overline{\Re(WG) + q}$)] if there is a positive definite diagonal matrix Γ such that ΓW is symmetric and either 1 is not an eigenvalue of any principal submatrix of W [e.g., in the case when any of conditions **H1**)–**H4**) in Lemma 2 is satisfied] or $\Gamma - \Gamma W$ is positive definite.

Proof: Statement i) follows directly from Lemma 3, whilst ii) follows from Lemma 2 ii) and Corollary 3. □

Remark 5: i) The globally exponential stability and convergence in Theorems 5 and 6 are taken in the sense of the topology of the whole space $\mathbf{R}^N$.

ii) Theorems 5 and 6 can be directly applied to the CNNs discussed in the literature (see, e.g., [8], [30], [34]). Such type of networks typically are of 2-D neuron structures (say, $M \times N$) and modeled by

CNNs:
$$\frac{dv_{ij}}{dt} = -v_{ij} + \sum_{(k,\, l)\in N_r(i,\, j)} w_{ij,\, kl} sat(v_{kl}) + d_{ij},$$
$$i = 1, \ldots, M,\ j = 1, \ldots, N,$$

or

$$\frac{dV}{dt} = -V + W sat(V) + D,$$

where $V = (v_{ij})_{N\times M}$ is the local field state of neuron (i, j) and the nonlinear activation function $sat(t) = g_i(t)$ is the one-dimensional nearest point projection (called the linear saturating function in CNNs paradigm), defined as in (20). Theorems 5 and 6 unify and generalize the existing stability results on such networks [30], [34].

VI. Conclusion

We have proposed a new methodology, called a reference model approach, for stability analysis of neural networks. The main point of the approach consists in studying a neural network model with reference to other related models, so that different modeling approaches can be combinatively used and powerfully cross-fertilized. The feasibility of such an approach lies in the equivalence or stability-invariance intrinsically existed in two cross-referenced models in some sense, but its efficiency relies on the merits or benefit the reference model may gain by comparison. Focused on two types of representative neural network models, the static and local field neural network models (3) and (4), a theoretical foundation on feasibility and efficiency of the new approach has been established, through exploration of a stability invariance principle existed between models (3) and (4) in the sense of Theorems 1 and 2, as well as establishment of some fundamental stability-related lemmas for each model (Lemmas 1–4). By virtue of the fundamental lemmas, we have proved a set of very generic stability/convergence results for models (3) and (4), which have in turn, by using the reference model approach, been transferred from one model to the other, resulting in various unified stability results for different neural network models deduced from different modelings. We have also applied the abstract results to several typical neural network systems including the Hopfield-type NNs, ReBP-type NNs, BSB-type NNs, BCOp-type NNs and CNNs. The results have unified and generalized most of the existing stability assertions and demonstrated the feasibility and efficiency of the new methodology.

The significance of the proposed approach consists not only in the findings of many new stability assertions for different individual neural network models, but also in the effect on formalization of a unified stability theory for many different neural network models deduced from different modelings. The results presented clearly illustrate just a few of many possible consequences the new methodology may bring. A more systematic, broad and in-depth application of the reference model approach is needed. The potential advantage of the new approach is also needed to be further exploited. On this line, many research opportunities exist, for example, to extend or modify the stability invariance property found between (3) and (10), as well as (8) and (11), to the case between (3) and (4) or between (10) and (11), to ascertain if the stability conditions for the subsystem (8) [(10), (11)] can be intrinsically weaker than that of general system (3) [(4)], to study instability properties of systems (3) and (4) by using the reference model approach. All of these problems deserve further investigation.

It is also worth noting that we have introduced in Section III a set of abstract operator definitions to capture some common and useful properties of the neural network activation mappings. Such series of concepts and in particular as the one of *uniformly anti-monotonic operators*, have proven to be very useful. It was such notion that had made it possible to develop the generic stability theories in the present paper. Furthermore, such abstract concepts have also greatly simplified and conciliated the proofs of all our theorems. This benefit can also be made use of in other related research.

ACKNOWLEDGMENT

The authors wish to thank the Associate Editor and the anonymous reviewers for their helpful comments and suggestions.

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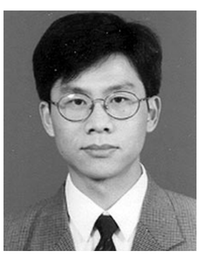

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