Harmless Feedback Control for Permanence and Global Asymptotic Stability in Nonlinear Delay Population Equation

By Linfei Nie, Jigen Peng, and Zhidong Teng

In this paper, we consider whether or not the feedback control has influence on a nonlinear population equation with several time delays. The general criteria of integrable form on the permanence are established, and we note that the feedback control has no influence to the permanence of system. By constructing suitable Lyapunov functionals, a set of easily verifiable sufficient conditions are derived for global asymptotic stability of any positive solutions to the system.

1. Introduction

In the last decades, delay differential equations, and among them delay equations of logistic type, have been extensively used as models in biology and other sciences, with particular emphasis in population dynamics.

Lisena [18] considered the nonautonomous logistic equation with delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)[r(t) - a(t)x(t) - b(t)x(t-\tau)]. \tag{1}$$

He establish the sufficient conditions for the existence, uniqueness and global asymptotic stability of positive periodic solutions of system (1) (see theorem 4.2 and 4.3 in [18]).

Address for correspondence: Linfei Nie, Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, China; e-mail: nielinfei@xju.edu.cn

Published by Blackwell Publishing, 350 Main Street, Malden, MA 02148, USA, and 9600 Garsington Road, Oxford, OX4 2DQ, UK.

However, we note that ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. In ecology, just we know that the practical interest question is whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. Whereas, the control variables discussed in most literatures are constants or time dependent [5, 13, 15].

Recently, we see that the dynamic behaviors for the differential system with feedback control are studied in [1–4, 6, 14, 20] and the references cited therein. In particular, Chen and Chen [4] investigated existence of almost-periodic solution for the following delay population equation with feedback control

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)[r(t) - a(t)x(t) - x(t-\tau) - c(t)u(t)] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + g(t)x(t-\tau). \end{aligned} \tag{2}$$

Chen [6] studied existence of almost-periodic solution for the following delay nonlinear population equation with feedback control

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)\left[r(t) - a(t)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\tau_i) - c(t)u(t)\right] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + \sum_{i=1}^{n} g_i(t)x^{\beta_i}(t-\tau_i), \end{aligned} \tag{3}$$

where $\alpha > 0,\ \beta_i > 0,\ \tau_i \geq 0$ and $a(t),\ c(t),\ \eta(t),\ b_i(t),\ g_i(t)(i = 1, 2, \ldots, n)$ are continuous functions. However, we see that for system (2) and (3), up until now, there is not any study work of the permanence and global asymptotic stability of positive solutions.

In this paper, motivated by the above works, we study whether or not the feedback control has influence on the ultimate boundedness, permanence and global asymptotic stability of positive solutions for system (3), and obtain that the feedback control has no influence on the permanence of system (3). Further, by numerical simulation, we put forward a conjecture: the feedback control has no influence on the global asymptotic stability of system (3).

This paper is organized as follows. In Section 2, we present some basic assumptions for system (3) and two important lemmas. In Section 3, we state and prove a general criterion for the permanence of system (3), which is independent of the feedback control. We also by means of suitable Lyapunov functionals, a set of easily verifiable sufficient conditions are derived for global asymptotic stability of any positive solutions of system (3). In additional, we also give the sufficient conditions for the permanence and global stability of any positive solution for system (2). Some specific examples are given to illustrate our results in the last section.

2. Preliminaries

Let $R_{+0} = [0, \infty)$ and $R_+ = (0, \infty)$. For a bounded continuous function $g(t)$ on R, we use the notations: $g^u = \sup_{t\in R} g(t),\ g^l = \inf_{t\in R} g(t)$.

For system (3), we introduce the following assumptions.

($\mathbf{H_1}$) Functions $r(t),\ a(t),\ c(t),\ \eta(t),\ b_i(t),\ g_i(t)$ are continuous and bounded on R_{+0}, and $a(t) \geq 0,\ c(t) \geq 0,\ \eta(t) \geq 0,\ b_i(t) \geq 0$ and $g_i(t) \geq 0$ for all $t \geq 0,\ i = 1, 2, \ldots, n$.

($\mathbf{H_2}$) There are positive constants μ_1 and μ_2, such that

$$\liminf_{t\to\infty} \int_t^{t+\mu_1} r(s)\,\mathrm{d}s > 0, \qquad \liminf_{t\to\infty} \int_t^{t+\mu_2} a(s)\,\mathrm{d}s > 0.$$

($\mathbf{H_3}$) There is a constant μ_3 such that

$$\liminf_{t\to\infty} \int_t^{t+\mu_3} \eta(s)\,\mathrm{d}s > 0.$$

Let $\tau = \max_{1\leq i\leq n}\{\tau_i\}$. We define $C[-\tau, 0]$ be the Banach space of bounded continuous functions $\phi : [-\tau, 0] \to R$ with the supremum norm defined by

$$\|\phi\|_c = \sup_{-\tau\leq s\leq 0} |\phi(s)|.$$

By the fundamental theory of functional differential equations [12, 15, 17] we know that for any $(\phi,\ \psi) \in C[-\tau, 0] \times C[-\tau, 0]$ system (3) has a unique solution $X(t,\ \phi,\ \psi) = (x(t,\ \phi),\ u(t,\ \psi))$ satisfying the initial condition $X_{t_0}(\cdot, \phi, \psi) = (\phi, \psi)$.

Defined $C_+[-\tau, 0] = \{\phi \in C[-\tau, 0] : \phi(s) \geq 0$ and $\phi(0) > 0$ for all $s \in [-\tau, 0]\}$. Motivated by the biological background of system (3), in this paper we are concerned with only positive solutions of system (3). It is not difficult to prove that the solution $X(t,\ \phi,\ \psi)$ of system (3) is positive, if the initial function $(\phi,\ \psi) \in C_+[-\tau, 0] \times C_+[-\tau, 0]$.

First, we consider the single-species nonautonomous logistic system

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)[r(t) - a(t)y^{\alpha}(t)]. \tag{4}$$

By lemma 1 given in Teng [9], we have

LEMMA 1. *Suppose that assumptions* ($\mathrm{H_1}$) *and* ($\mathrm{H_2}$) *hold, then there is a constant* $M > 1$ *such that*

$$M^{-1} \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq M$$

for any positive solution $y(t)$ *of system (4).*

Now, we consider the single-specie nonautonomous system with a parameter

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\eta(t)v(t) + \sum_{i=1}^{n} g_i(t)(\beta)^{\beta_i}, \tag{5}$$

where $\beta \in (0,\ \beta_0]$, and $\beta_0 > 0$ is constant.

In system (5), when parameter $\beta = 0$ we obtain the following system

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\eta(t)v(t). \tag{6}$$

It is well known that system (6) has a trivial equilibria $E = 0$, and E is is globally asymptotically stable.

For any $\theta > 0$ and $t_0 \in R_{+0}$, let $\beta \in [0,\ \beta_0]$, and $v_\beta(t,\ t_0,\ v_0)$ be the positive solution of system (5) with initial value $v_\beta(t_0) = v_0 \in [-\theta,\ \theta]$. It is easy to prove that for all $t \geq t_0,\ v_\beta(t,\ t_0,\ v_0) \geq 0$ if the initial value $v_0 \geq 0$, and $v_\beta(t,\ t_0,\ v_0) > 0$ if the initial value $v_0 > 0$. By lemmas 1 and 2 given in Teng [9], we further have

Lemma 2. *Suppose that assumptions* (H_1) *and* (H_3) *hold, then*

(a) There is a constant $M > 0$ *such that* $\limsup_{t\to\infty} v_\beta(t) \leq M$ *for any positive solution* $v_\beta(t)$ *of system (5).*

(b) If there is a constant $\omega > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+\omega} \sum_{i=1}^{n} g_i(s)\,\mathrm{d}s > 0$, *then there is a constant* $M > 1$ *such that*

$$M^{-1} \leq \liminf_{t\to\infty} v_\beta(t) \leq \limsup_{t\to\infty} v_\beta(t) \leq M.$$

(c) Each fixed positive solution v_β *of system (5) is globally uniformly attractive on* R_{+0} .

(d) For any $\varepsilon > 0$ *and* $\theta > 0$, *there are positive constants* $T = T(\varepsilon, \theta)$ *and* $\delta = \delta(\varepsilon)$ *such that for any* $t_0 \in R_{+0}$ *and* $v_0 \in [0,\ \theta]$ *we have* $v_\beta(t, t_0, v_0) < \varepsilon$ *for all* $t \geq t_0 + T$ *and* $\beta < \delta$.

To the convenience of statement in the following of this paper, we introduce the definition on persistence.

Definition 1. *System (2) is said to be persistent, if there are positive constants m and M such that*

$$m \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M$$

for any positive solution $(x(t),\ u(t))$ *of system (2).*

Remark 1. In system (2), $u(t)$ is control variable, thus we do not consider the permanence of control variable.

3. Main results

In this section, we proceed to discussion on the ultimate boundedness, permanence and global asymptotically stable of any positive solution of system (3).

Firstly, on the ultimate boundedness of any positive solution for system (3), we can get

THEOREM 1. *Suppose that assumptions* (H_1)–(H_3) *hold, then system (3) is ultimately bounded, in the sense that there are positive constants M and T such that if* $t \geq T$, *then* $x(t) \leq M$ *and* $u(t) \leq M$ *for all positive solutions* $X(t) = (x(t), u(t))$ *of system (3).*

Proof: Let $X(t) = (x(t), u(t))$ be any positive solution of system (3). We first prove that the component x of system (3) is ultimately bounded. From the first equation of system (3) we have

$$\frac{dx(t)}{dt} \leq x(t)[r(t) - a(t)x^{\alpha}(t)].$$

By the Comparison Theorem and Lemma 1, we can obtain that there is a constant M_1 such that for any positive solution $(x(t), u(t))$ of system (3), there is a $T_1 > 0$ such that $x(t) \leq M_1$ for all $t \geq T_1$.

Further, from the second equation of system (3) we have

$$\frac{du(t)}{dt} \leq -\eta(t)u(t) + \sum_{i=1}^{n} g_i(t)M_1^{\beta_i}$$

for all $t \geq T_1 + \tau$. Hence, using the Comparison Theorem and the conclusion (a) of Lemma 2, we further can obtain that there is constant $M_2 > 0$ such that for any positive solution $(x(t), u(t))$ of system (3) there is a $T_2 \geq T_1 + \tau$ such that $u(t) \leq M_2$ for all $t \geq T_2$. Therefore, the solution $(x(t), u(t))$ is ultimately bounded. This completes the proof of this theorem. ■

Next, on the permanence of components for system (3), we have the result

THEOREM 2. *Suppose that assumptions* (H_1)– (H_3) *hold, then system (3) is permanent.*

Proof: Let $X(t) = (x(t), u(t))$ be any positive solution of system (3). From Theorem 1, there is a constant $M > 0$ such that for any positive solution $X(t)$ of system (3), there is a $T_1 \geq 0$ such that $x(t) \leq M$ and $u(t) \leq M$ for all $t \geq T_1$. Therefore, from the first equation of system (3) we have

$$\begin{aligned}\frac{dx(t)}{dt} &\geq x(t)\left[r(t) - a(t)M^{\alpha} - \sum_{i=1}^{n} b_i(t)M^{\beta_i} - c(t)M\right] \\ &\geq -\alpha_1 x(t)\end{aligned} \tag{7}$$

for all $t \geq T_1 + \tau$, where $\alpha_1 = \sup_{t \in R_{+0}}\{|r(t) - a(t)M^{\alpha} - \sum_{i=1}^{n} b_i(t)M^{\beta_i} - c(t)M|\}$. For any $t \geq T_1 + \tau$ and $s \in [-\tau, 0]$, integrating (7) from $t + s$ to t we obtain

$$x(t+s) \leq x(t)\exp(-\alpha_1 s) \leq x(t)\exp(\alpha_1 \tau). \tag{8}$$

For any $t_1,\ t_2$ and $t_2 \geq t_1 \geq 0$, integrating directly system (3) we have

$$x(t_2) = x(t_1)\exp\int_{t_1}^{t_2}\left[r(s) - a(s)x^{\alpha}(s) - \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s - \tau_i) - c(s)u(s)\right]\mathrm{d}s. \tag{9}$$

In the following, we will use two claims to complete the proof of Theorem 2.

CLAIM 1. *There is a constant* $\beta > 0$ *such that* $\limsup_{t\to\infty} x(t) > \beta$ *for any positive solution* $X(t)$ *of system (3).*

In fact, by assumptions (H_1) and (H_2), we can choose small enough positive constants ε and δ, and a large enough $T_2 \geq T_1$ such that for all $t \geq T_2 + \tau$

$$\int_{t}^{t+\mu_1}\left[r(s) - a(s)\varepsilon^{\alpha} - \sum_{i=1}^{n} b_i(s)[\varepsilon\exp(\alpha_1\tau)]^{\beta_i} - c(s)\varepsilon\right]\mathrm{d}s > \delta. \tag{10}$$

Consider the following system with a parameter

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\eta(t)u(t) + \sum_{i=1}^{n} g_i(t)[\beta\exp(\alpha_1\tau)]^{\beta_i}, \tag{11}$$

where $\beta \in [0,\ \beta_0]$ is a parameter. Let $u_\beta(t)$ be the solution of system (11), by the conclusions (c) and (d) of Lemma 2, $u_\beta(t)$ is globally asymptotically stable, and $u_\beta(t) \to 0$, as $\beta \to 0$ and $t \to \infty$. Thus, there are positive constants β, T_3, and $T_3 > T_2,\ \beta < \varepsilon$ such that

$$u_\beta(t) < \frac{\varepsilon}{2} \quad \text{for all} \quad t \geq T_3. \tag{12}$$

If Claim 1 is not true, then there is positive solution $(x(t),\ u(t))$ of system (3) such that $\limsup_{t\to\infty} x(t) < \beta$. Then, there is a constant $T_4 > T_3$ such that $x(t) < \beta$ for all $t \geq T_4$. From (8) and the second equation we obtain

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq -\eta(t) + \sum_{i=1}^{n} g_i(t)[\beta\exp(\alpha_1\tau)]^{\beta_i} \quad \text{for all} \quad t \geq T_4.$$

Using the Comparison Theorem and globally asymptotically stable of solution $u_\beta(t)$, we obtain that there is $T_5 \geq T_4$ such that

$$u(t) < u_\beta(t) + \frac{\varepsilon}{2} \quad \text{for all} \quad t \geq T_5. \tag{13}$$

Hence, from (12) and (13 it follows that

$$u(t) < \varepsilon \quad \text{for all} \quad t \geq T_5. \tag{14}$$

By (8), (9), and (14) we obtain

$$\begin{aligned}x(t) &= x(T_5)\exp\int_{T_5}^{t}\left[r(s) - a(s)x^{\alpha}(s) - \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\tau_i) - c(s)u(s)\right]\mathrm{d}s\\ &\geq x(T_5)\exp\int_{T_5}^{t}\left[r(s) - a(s)\varepsilon^{\alpha} - \sum_{i=1}^{n} b_i(s)[\varepsilon\exp(\alpha_1\tau)]^{\beta_i} - c(s)\varepsilon\right]\mathrm{d}s\end{aligned}$$

for all $t \geq T_5$. Thus from (10) we finally obtain $\lim_{t\to\infty}x(t) = \infty$ which leads to a contradiction. Therefore, Claim 1 is true.

CLAIM 2. *There is a constant* $\gamma > 0$ *such that* $\liminf_{t\to\infty}x(t) > \gamma$ *for any positive solution* $X(t)$ *of system (3).*

In fact, if Claim 2 is not true, then there is a sequence of initial value $\{X_n = (\phi_n, \psi_n)\} \subset C_+ \times C_+$ such that, for the solution $(x(t, X_n), u(t, X_n))$ of system (3),

$$\liminf_{t\to\infty} x(t, X_n) < \frac{\beta}{n^2}, \quad n = 1, 2, \ldots,$$

where constant β is given in Claim 1. By Claim 1, for every n there are two time sequences $\{s_q^{(n)}\}$ and $\{t_q^{(n)}\}$, satisfying $0 < s_1^{(n)} < t_1^{(n)} < s_2^{(n)} < t_2^{(n)} < \cdots < s_q^{(n)} < t_q^{(n)} < \cdots$ and $\lim_{q\to\infty}s_q^{(n)} = \infty$, such that

$$x\left(s_q^{(n)}, X_n\right) = \frac{\beta}{n}, \qquad x\left(t_q^{(n)}, X_n\right) = \frac{\beta}{n^2}, \tag{15}$$

and

$$\frac{\beta}{n^2} < x(t, X_n) < \frac{\beta}{n} \quad \text{for all} \quad t \in \left(s_q^{(n)},\ t_q^{(n)}\right). \tag{16}$$

From the ultimate boundedness of system (3), we can choose a positive constant $T^{(n)}$ for every n such that $x(t, X_n) < M$ and $u(t, X_n) < M$ for all $t > T^{(n)}$. Further, there is an integer $K_1^{(n)} > 0$ such that $s_q^{(n)} > T^{(n)} + \tau$ for all $q > K_1^{(n)}$. Let $q > K_1^{(n)}$, then for any $t \in [s_q^{(n)},\ t_q^{(n)}]$, we have

$$\begin{aligned}\frac{\mathrm{d}x(t, X_n)}{\mathrm{d}t} &\geq x(t, X_n)\left[r(t) - a(t)M^{\alpha} - \sum_{i=1}^{n} b_i(t)M^{\beta_i} - c(t)M\right]\\ &\geq -\gamma_0 x(t, X_n),\end{aligned}$$

where $\gamma_0 = \sup_{t\in R_+}\{r(t) + a(t)M^{\alpha} + \sum_{i=1}^{n} b_i(t)M^{\beta_i} + c(t)M\}$. Integrating the above inequality from $s_q^{(n)}$ to $t_q^{(n)}$, we can get

$$x\left(t_q^{(n)}, X_n\right) \geq x\left(s_q^{(n)}, X_n\right)\exp\left[-\gamma_0\left(t_q^{(n)} - s_q^{(n)}\right)\right].$$

Consequently, by (16)

$$\frac{\beta}{n^2} \geq \frac{\beta}{n} \exp\left[-\gamma_0\left(t_q^{(n)} - s_q^{(n)}\right)\right].$$

Thus,

$$t_q^{(n)} - s_q^{(n)} \geq \frac{\ln n}{\gamma_0} \quad \text{for all} \quad q > K_1^{(n)}.$$

By (10), there are positive constants P and ϱ such that

$$\int_t^{t+\kappa} \left[r(s) - a(s)\varepsilon^\alpha - \sum_{i=1}^n b_i(s)[\varepsilon \exp(\alpha_1 \tau)]^{\beta_i} - c(s)\varepsilon\right] \mathrm{d}s > \varrho \tag{17}$$

for all $t \geq 0$ and $\kappa \geq P$.

Let $\widetilde{u}_\beta(t)$ be the solution of system (11) with the initial value $\widetilde{u}_\beta(t) = u(s_q^{(n)}, X_n)$. By (8), (16) and condition of (H_1), we have

$$\frac{\mathrm{d}u(t, X_n)}{\mathrm{d}t} \leq -\eta(t)u(t, X_n) + \sum_{i=1}^n g_i(t)[\beta \exp(\alpha_1 \tau)]^{\beta_i}$$

for any $m,\ q$ and $t \in [s_q^{(m)},\ t_q^{(m)}]$. Using the Comparison Theorem, we have

$$u(t, X_n) \leq \widetilde{u}_\beta(t) \quad \text{for all} \quad t \in \left[s_q^{(n)},\ t_q^{(n)}\right].$$

By the conclusion (c) of Lemma 2, the solution $u_\beta(t)$ is globally uniformly attractive on R_{+0}, we obtain that there is a constant $T_0 \geq P$, and T_0 is independent of any n and $q \geq K^{(n)}$, such that

$$\widetilde{u}_\beta(t) < u_\beta(t) + \frac{\varepsilon}{2} \quad \text{for all} \quad t \geq s_q^{(n)} + T_0. \tag{18}$$

Choose an integer $N_0 > 0$ such that when $n \geq N_0$ and $q \geq K^{(n)}$,

$$t_q^{(n)} - s_q^{(n)} > T_0 + P.$$

Further, from (12) and (18) we obtain

$$u(t) < \varepsilon \quad \text{for all} \quad t \in \left[s_q^{(n)} + T_0,\ t_q^{(n)}\right]. \tag{19}$$

Hence, when $n \geq N_0$ and $q \geq K^{(n)}$, by (8), (9), (15), (16), (17), (19), and (H_1) it follows

$$\begin{aligned}
\frac{\beta}{n^2} &= x\left(t_q^{(n)}, X_n\right) \\
&= x\left(s_q^{(n)} + T_0, X_n\right) \exp \int_{s_q^{(n)}+T_0}^{t} \Bigg[r(s) - a(s)x^{\alpha}(s) \\
&\qquad - \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\tau_i) - c(s)u(s) \Bigg] \mathrm{d}s \\
&\geq \frac{\beta}{n^2} \exp \int_{s_q^{(n)}+T_0}^{t} \left[r(s) - a(t)\varepsilon^{\alpha} - \sum_{i=1}^{n} b_i(s)[\varepsilon \exp(\alpha_1 \tau)]^{\beta_i} - c(s)\varepsilon \right] \mathrm{d}s \\
&> \frac{\beta}{n^2}
\end{aligned}$$

which leads to a contradiction. Therefore, Claim 2 is true.

Finally, from Claims 1 and 2, we see that Theorem 2 is proved. ■

Remark 2. From the proof of Theorem 2, we note that the feedback control has no influence on the permanence of system (3).

Further, using theorem 1 given by Teng and Chen [19] on the existence of positive periodic solutions for the general n-species periodic Kolmogorov type systems with delays, we have the following theorem on the existence of positive periodic solutions for the periodic system (3).

THEOREM 3. *If system (3) is ω-periodic and assumptions (H_1)–(H_3) hold and there is a constant $\lambda > 0$ such that* $\liminf_{t\to\infty} \int_t^{t+\lambda} \sum_{i=1}^{n} g_i(s)\,\mathrm{d}s > 0$, *then system (3) has at least a positive ω-periodic solution.*

Proof: By assumptions (H_1)–(H_3) and Theorem 1 and 2, we obtain that the component x of system (3) is permanent, that is there are positive constants m_1, M_1 and T_1 such that for any positive solution $(x(t), u(t))$ of system (3) we have $m_1 \leq x(t) \leq M_1$ for all $t \geq T_1$. Further, from this and the second equation of system (3) we have

$$-\eta(t) + \sum_{i=1}^{n} g_i(t)(m_1)^{\beta_i} \leq \frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq -\eta(t) + \sum_{i=1}^{n} g_i(t)(M_1)^{\beta_i} \text{ for all } t \geq T_1 + \tau.$$

Therefore, using the Comparison Theorem and the conclusion (b) of Lemma 2, we can obtain that there are positive constants m_2, M_2 and $T_2 \geq T_1 + \tau$ such that for any positive solution $(x(t), u(t))$ of system (3) we have $m_2 \leq u(t) \leq M_2$ for all $t \geq T_2$. Now, we let $m = \min\{m_1, m_2\}$ and $M = \max\{M_1, M_2\}$, then for all $t \geq T_2$

$$m \leq x(t) \leq M, \qquad m \leq u(t) \leq M.$$

Finally, by theorem 1 in [19], it follows that the system has at least a positive ω-periodic solution, which completes the proof of Theorem 3. ■

Now, we discuss the global asymptotic stability of any positive solution of system (3). We first derive certain bounds estimates for positive solution of system (3).

THEOREM 4. *Let* $(x(t), u(t))$ *denote any positive solution of system (3). Suppose that assumption* (H_1) *hold, and* $\eta^l > 0$, $a^l > 0$, *assume further that the following hold,*

$$r^l > \sum_{i=1}^{n} \left(b_i^u + \frac{c^u}{\eta^l} g_i^u \right) \left(\frac{r^u}{a^l} \right)^{(\beta_i/\alpha)}, \tag{20}$$

then there is a constant $T > 0$ *such that if* $t \geq T$,

$$m < x(t) \leq M_1, \quad 0 < u(t) \leq M_2,$$

where

$$M_1 = \left(\frac{r^u}{a^l} \right)^{1/\alpha}, \quad M_2 = \frac{\sum_{i=1}^{n} g_i^u (M_1)^{\beta_i}}{\eta^l},$$
$$m = \left(\frac{r^l - \sum_{i=1}^{n} b_i^u (M_1)^{\beta_i} - c^u M_2}{a^u} \right)^{1/\alpha}. \tag{21}$$

The proof of Theorem 4 is similar to that of theorem 2.1 in [11], we therefore omit it here.

We now formulate global asymptotic stability of any positive solution of system (3).

THEOREM 5. *Let* $(x_1(t), u_1(t))$ *denote any positive solution of system (3). In addition to assumptions* $\eta^l > 0$, $a^l > 0$, (H_1) *and (20) hold, assume further that there are positive constants* c_1 *and* c_2 *such that*

$$(\mathbf{H}_4) \qquad \liminf_{t \to \infty} A_i(t) > 0,$$

where

$$A_1 = c_1 \alpha (m)^{\alpha} a(t) - \sum_{i=1}^{n} \beta_i (M_1)^{\beta_i} [c_1 b_i(t + \tau_i) + c_2 g_i(t + \tau_i)],$$
$$A_2(t) = c_2 \eta(t) - c_1 c(t), \tag{22}$$

then the solution $(x_1(t), u_1(t))$ *is globally asymptotically stable.*

Proof: Let $(x_2(t), u_2(t))$ be any positive solution of system (3). It follows from Theorem 3 that there exist positive constants T, m and M_i $(i = 1, 2.)$ (defined by (21)), such that for all $t \geq T$,

$$m < x_i(t) \leq M_1, \qquad 0 < u_i(t) \leq M_2, \quad i = 1, 2. \tag{23}$$

We define

$$V_1(t) = c_1 |\ln x_1(t) - \ln x_2(t)|.$$

Calculating the upper right derivative of $V_1(t)$ along solutions of system (3), it follows that

$$\begin{aligned} D^+ V_1(t) &= \operatorname{sgn}(x_1(t) - x_2(t)) c_1 \left[\frac{\dot{x}_1(t)}{x_1(t)} - \frac{\dot{x}_2(t)}{x_2(t)}\right] \\ &= \operatorname{sgn}(x_1(t) - x_2(t)) c_1 \left\{ - a(t)\big[x_1^{\alpha}(t) - x_2^{\alpha}(t)\big] - \sum_{i=1}^{n} b_i(t)\big[x_1^{\beta_i}(t - \tau_i) \right. \\ &\quad \left. - x_2^{\beta_i}(t - \tau_i)\big] - c(t)[u_1(t) - u_2(t)] \right\} \\ &\leq -c_1 a(t)\big|x_1^{\alpha}(t) - x_2^{\alpha}(t)\big| + c_1 \sum_{i=1}^{n} b_i(t)\big|x_1^{\beta_i}(t - \tau_i) - x_2^{\beta_i}(t - \tau_i)\big| \\ &\quad + c_1 c(t)|u_1(t) - u_2(t)| \end{aligned} \tag{24}$$

We note that

$$\big|x_1^{\alpha}(t) - x_2^{\alpha}\big| = \big|e^{\alpha \ln x_1(t)} - e^{\alpha \ln x_2(t)}\big|.$$

Using the Mean Value Theorem, we have

$$\big|x_1^{\alpha}(t) - x_2^{\alpha}\big| = \alpha e^{\xi_1(t)} |\ln x_1(t) - \ln x_2(t)|,$$

where $\xi_1(t)$ lies between $\ln x_1(t)$ and $\ln x_2(t)$. By (23) we have $\ln m < \xi_1(t) \leq \ln M_1$ for all $t \geq T$. Thus we obtain that for all $t \geq T$

$$\alpha(m)^{\alpha} |\ln x_1(t) - \ln x_2(t)| \leq \big|x_1^{\alpha}(t) - x_2^{\alpha}(t)| \leq \alpha(M_1)^{\alpha} |\ln x_1(t) - \ln x_2(t)|. \tag{25}$$

Similarly, we have

$$\begin{aligned} &\beta_i(m)^{\beta_i} |\ln x_1(t - \tau_i) - \ln x_2(t - \tau_i)| \\ &\quad \leq \big|x_1^{\beta_i}(t - \tau_i) - x_2^{\beta_i}(t - \tau_i)\big| \\ &\quad \leq \beta_i(M_1)^{\beta_i} |\ln x_1(t - \tau_i) - \ln x_2(t - \tau_i)| \end{aligned} \tag{26}$$

for all $t \geq T + \tau$. On substituting (25) and (26) into (24), we derive for all $t \geq T + \tau$ that

$$D^+V_1(t) \le -c_1\alpha(m)^{\alpha}a(t)|\ln x_1(t) - \ln x_2(t)| + c_1c(t)|u_1(t) - u_2(t)| + c_1\sum_{i=1}^{n}\beta_i(M_1)^{\beta_i}b_i(t)|\ln x_1(t-\tau_i) - \ln x_2(t-\tau_i)| \tag{27}$$

Define

$$V_2(t) = c_2|u_1(t) - u_2(t)|.$$

Calculating the upper right derivative of $V_2(t)$ along solutions of system (3), it follows that

$$\begin{aligned} D^+V_2(t) &= \operatorname{sgn}(u_1(t) - u_2(t))c_2[\dot{u}_1(t) - \dot{u}_2(t)] \\ &= \operatorname{sgn}(u_1(t) - u_2(t))c_2\Bigg\{ -\eta(t)[u_1(t) - u_2(t)] \\ &\quad + \sum_{i=1}^{n} g_i(t)\big[x_1^{\beta_i}(t-\tau_i) - x_2^{\beta_i}(t-\tau_i)\big]\Bigg\} \\ &\le -c_2\eta(t)|u_1(t) - u_2(t)| + c_2\sum_{i=1}^{n} g_i(t)\big|x_1^{\beta_i}(t-\tau_i) - x_2^{\beta_i}(t-\tau_i)\big| \end{aligned} \tag{28}$$

On substituting (26) into (28), we derive for all $t \ge T + \tau$ that

$$D^+V_2(t) \le -c_2\eta(t)|u_1(t) - u_2(t)| + c_2\sum_{i=1}^{n}\beta_i(M_1)^{\beta_i}g_i(t)|\ln x_1(t-\tau_i) - \ln x_2(t-\tau_i)| \tag{29}$$

We obtain from (27) and (29) that for all $t \ge T + \tau$

$$\begin{aligned} D^+V_1(t) + D^+V_2(t) &\le -c_1\alpha(m)^{\alpha}a(t)|\ln x_1(t) - \ln x_2(t)| \\ &\quad - \big[c_2\eta(t) - c_1c(t)\big]|u_1(t) - u_2(t)| \\ &\quad + \sum_{i=1}^{n}\beta_i(M_1)^{\beta_i}[c_1b_i(t) + c_2g_i(t)]|\ln x_1(t-\tau_i) \\ &\quad - \ln x_2(t-\tau_i)|. \end{aligned} \tag{30}$$

We now define

$$V(t) = V_1(t) + V_2(t) + \sum_{i=1}^{n} V_{1i}(t),$$

where

$$V_{1i}(t) = \beta_i(M_1)^{\beta_i}\int_{t-\tau_i}^{t}[c_1b_i(s+\tau_i) + c_2g_i(s+\tau_i)]|\ln x_1(s) - \ln x_2(s)|\,\mathrm{d}s. \tag{31}$$

It then follows from (22) and (30)–(31) that for $t \geq T + \tau$

$$D^+V(t) \leq -A_1(t)|\ln x_1(t) - \ln x_2(t)| - A_2(t)|u_1(t) - u_2(t)|, \tag{32}$$

where $A_1(t)$ and $A_2(t)$ are defined in (22).

By assumption (H_4), there exist positive constants α_1, α_2 and $T^* \geq T + \tau$ such that if $t \geq T^*$

$$A_i(t) \geq \alpha_i > 0, \quad i = 1, 2. \tag{33}$$

Integrating both sides of (32) on interval $[T^*, t]$,

$$V(t) + \int_{T^*}^{t} [A_1(s)|\ln x_1(s) - \ln x_2(s)| + A_2(s)|u_1(s) - u_2(s)|]\,\mathrm{d}s \leq V(T^*). \tag{34}$$

It follows from (33) and (34) that

$$V(t) + \int_{T^*}^{t} [\alpha_1|\ln x_1(s) - \ln x_2(s)| + \alpha_2|u_1(s) - u_2(s)|]\,\mathrm{d}s \leq V(T^*)$$

for all $t \geq T^*$. Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} |\ln x_1(s) - \ln x_2(s)|\,\mathrm{d}s < \infty, \qquad \int_{T^*}^{\infty} |u_1(s) - u_2(s)|\,\mathrm{d}s < \infty.$$

By Theorem 3, $|\ln x_1(t) - \ln x_2(t)|$ and $|u_1(t) - u_2(t)|$ are bounded on $[T^*, \infty)$.

On the other hand, it is easy to see that $\dot{\ln} x_i(t)$ and $\dot{u}_i(t)$ $(i = 1, 2)$ are bounded for $t \geq T^*$. Therefore, $|\ln x_1(t) - \ln x_2(t)|$ and $|u_1(t) - u_2(t)|$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's [12] Lemma (lemma 1.2.2 and 1.2.3), we conclude that

$$\lim_{t\to\infty} |\ln x_1(t) - \ln x_2(t)| = 0 \quad \left(\text{i.e. } \lim_{t\to\infty} |x_1(t) - x_2(t)| = 0\right)$$

and

$$\lim_{t\to\infty} |u_1(t) - u_2(t)| = 0.$$

This completes the proof of this theorem. ■

Remark 3. From the proof of Theorem 4, we note that the feedback control has influence to the permanence of system (3).

Finally, we observe that the system (3) degenerates into the system (2) when parameters $n = \alpha = \beta_1 = b_1(t) = 1$, $g_1(t) = g(t)$ and $\tau_1 = \tau$. On the permanence and global stability of any positive solution for system (2), directly applying Theorem 1–5 we have the following corollaries.

COROLLARY 1. *Suppose that assumptions* (H_1)–(H_3) *hold, then system (2) is ultimately bounded and permanent.*

Corollary 2. *If system (2) is ω-periodic and assumptions* (H_1)–(H_3) *hold and there is a constant* $\gamma > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+\gamma} g(s)\,\mathrm{d}s > 0$, *then system (2) has at least a positive ω-periodic solution.*

Corollary 3. *Let* $(x^*(t), u^*(t))$ *denote any positive solutions of system (2). In addition to assumptions* $\eta^l > 0$, $a^l > 0$ *and* (H_1) *hold, assume further that*

$$r^l > \left(1 + \frac{c^u}{\eta^l} g^u\right)\left(\frac{r^u}{a^l}\right)$$

and there are positive constants c_1 *and* c_2 *such that*

$$\liminf_{t\to\infty} A_i(t) > 0,$$

where

$$M_1 = \frac{r^u}{a^l}, \qquad M_2 = \frac{g^u M_1}{\eta^l}, \qquad m = \frac{r^l - M_1 - c^u M_2}{a^u},$$

$$A_1 = c_1 m a(t) - M_1[c_1 + c_2 g(t + \tau_i)], \qquad A_2(t) = c_2\eta(t) - c_1 c(t),$$

then the solution $(x^*(t), u^*(t))$ *is globally asymptotically stable.*

4. Example, numerical simulation and discussing

In this paper, we investigate a class of delay nonlinear population equation with feedback control. By using analytic method we give the criteria for the ultimate boundedness, permanence, the globally asymptotically stable of positive solution of system (3).

In the following, we present some numerical simulation of examples which validate these theoretical results obtained in this paper. To make the convenient, without loss of the generality, we will only consider the special case of system (3)

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)[r(t) - a(t)x(t) - b(t)x(t-\tau) - c(t)u(t)] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + g(t)x(t-\tau). \end{aligned} \tag{35}$$

In system (35), let $r(t) = 1.0 + 0.3\sin(2\pi t)$, $a(t) = 1.1 + 0.1\sin(2\pi t)$, $b(t) = 0.2 + 0.1\sin(2\pi t)$, $c(t) = 0.2 + 0.1\sin(2\pi t)$, $\eta(t) = 0.6 + 0.1\sin(2\pi t)$, $g(t) = 0.2 + 0.2\sin(2\pi t)$ and $\tau = 1$. We easily verify that assumption (H_1)–(H_3) hold. Therefore, from Theorems 1 and 2, system (35) is ultimate boundedness and permanent, and the control variable u has no influence to the ultimate bounded and permanence of system (35). Further, by Theorem 3 and Theorem 5, system (35) has a unique globally asymptotically stable positive periodic solution. Numerical simulation of the above results can be seen in Figure 1.

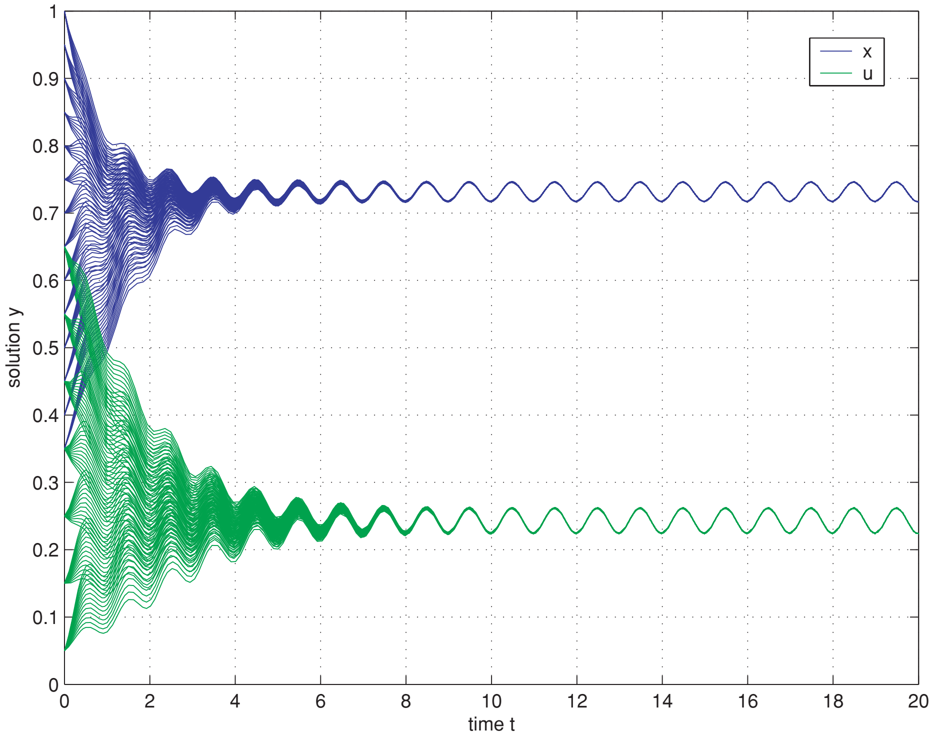

Figure 1. The ultimate boundedness, permanence, the globally asymptotically stable of positive solution of system (35) with $r(t) = 1.0 + 0.3\sin(2\pi t)$, $a(t) = 1.1 + 0.1\sin(2\pi t)$, $b(t) = 0.2 + 0.1\sin(2\pi t)$, $c(t) = 0.2 + 0.1\sin(2\pi t)$, $\eta(t) = 0.6 + 0.1\sin(2\pi t)$, $g(t) = 0.2 + 0.2\sin(2\pi t)$ and $\tau = 1$.

However, if in system (35), let $r(t) = 0.4 + 0.3\sin(2\pi t)$, $a(t) = 0.3 + 0.1\sin(2\pi t)$, $b(t) = 0.2 + 0.1\sin(2\pi t)$, $c(t) = 0.2 + 0.1\sin(2\pi t)$, $\eta(t) = 0.6 + 0.1\sin(2\pi t)$, $g(t) = 0.2 + 0.2\sin(2\pi t)$ and $\tau = 1$, it is easy to verify that assumption (H_1)–(H_3) hold. Therefore, from Theorem 1–3, system (35) is ultimate boundedness, permanent and has at least one positive periodic solution $(x^*(t), u^*(t))$, in which the control variable u has no influence to the ultimate bounded and permanence of system (35). It is also easy to verify that the condition (14) in Theorem 3 do not hold. Thus, we cannot guarantee the uniqueness and global stability of positive periodic solution $(x^*(t), u^*(t))$. On the other hand, we note that the without feedback controls which correspond to the system (35) is $\mathrm{d}\,x(t)/\mathrm{d}\,t = x(t)[0.4 + 0.3\sin(2\pi t) - (0.3 + 0.1\sin(2\pi t))x(t) - (0.2 + 0.1\sin(2\pi t))x(t - 1)]$. Similar to the discussion of Theorem 1–3 and 5, we get that the system is ultimate boundedness, permanent, and has a unique globally asymptotically stable positive periodic solution. By numerical simulation, we obtain that system (35) is ultimate boundedness, permanent

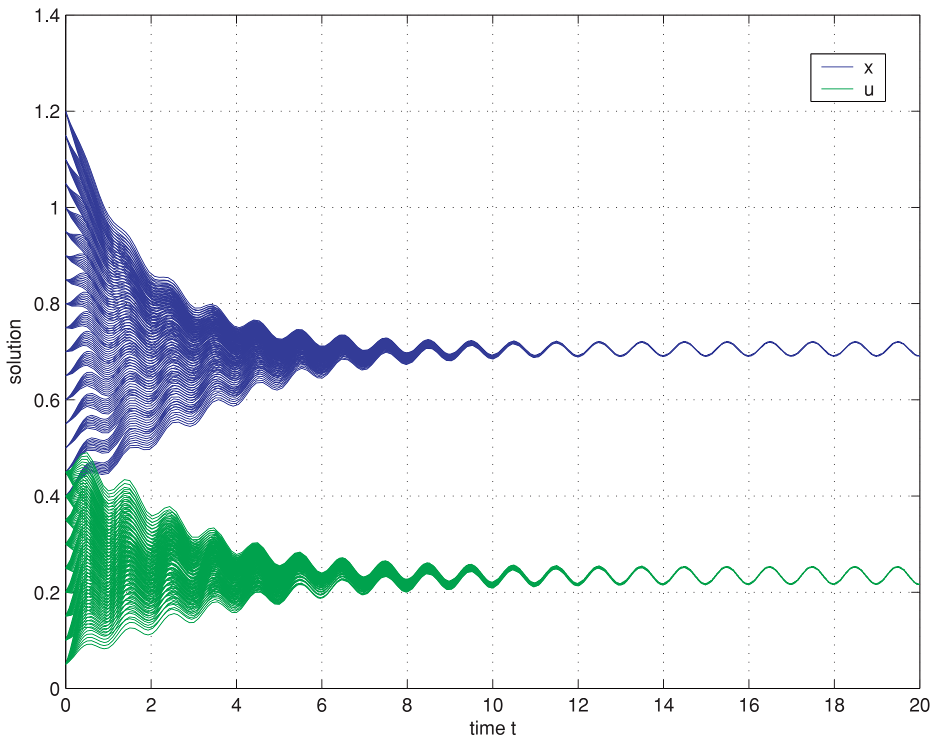

Figure 2. The ultimate boundedness, permanence, the globally asymptotically stable of positive solution of system (35) with $r(t) = 0.4 + 0.3\sin(2\pi t)$, $a(t) = 0.3 + 0.1\sin(2\pi t)$, $b(t) = 0.2 + 0.1\sin(2\pi t)$, $c(t) = 0.2 + 0.1\sin(2\pi t)$, $\eta(t) = 0.6 + 0.1\sin(2\pi t)$, $g(t) = 0.2 + 0.2\sin(2\pi t)$ and $\tau = 1$.

and has a unique globally asymptotically stable positive periodic solution, that is, Numerical simulation of the above results can be seen in Figure 2. Thus, we give an interesting open problem.

CONJECTURE 1. *The feedback control has no influence on the globally asymptotically stable of system (3).*

Acknowledgments

This work was partially supported by the Natural Science Foundation of China under the contact 10531030 and partially by NCET.

References

1. K. GOPALSAMY and P. WEN, Feedback regulation of logistic growth, *Int. J. Math Math. Sci.* 16:177–192 (1993).

2. Y. Li and P. Liu, Positive periodic solutions of a class of functional differential systems with feedback controls, *Nonlinear Anal*. 57:655–666 (2004).
3. H. Huo and W. Li, Positive periodic solutions of a class of delay differential system with fedback control, *Appl. Math. Comput*., 148:35–46 (2004).
4. C. Chen and F. Chen, Almost-periodic solutions of a delay population equation with feedback control, *Nonlinear Anal.: Real World Appl.* 7:559–571 (2006).
5. J. LaSalle and S. Lefschetz, *Stability by Lyapunov's Direct Method*, Academic Press, New York, 1961.
6. X. Chen, Almost periodic solutions of nonlinear delay population equation with feedback control, *Nonlinear Anal.: Real World Appl.* 8:62–72 (2007).
7. R. R. Vance and E. A. Coddington, A nonautonomous model of population growth, *J. Math. Biol.* 27:491–506 (1989).
8. Z. Teng, Permanence and stability in general nonautonomous single-species Kolmogorov systems with delays, *Nonlinear Anal.: Real World Appl.* 8:230–248 (2007).
9. Z. Teng and Z. Li, Pemanence criteria in non-autonomous predator-prey Kolmogorov systems and its applications, *Dyn. Syst.* 19:171–194 (2004).
10. G. Seifert, On a delay-differential equation for single specie population variation, *Nonlinear Anal.: TMA* 11:1051–1059 (1987).
11. W. Wang and Z. Ma. Harmless delays for uniform permanence, *J. Math. Anal. Appl.* 158:256–268 (1991).
12. K. Gopalsamy, *Stability and Oscillations in Delay Different Equations of Population Dynamics*, Kluwer Academic, Dordrecht/Norwell, MA, 1992.
13. S. Lefschetz, *Stability of Nonlinear Control System*, Academic Press, New York, 1965.
14. Y. Xiao and S. Tang, Permanence and periodic solution in competition system with feedback controls, *Math. Comput. Model.* 27:33–37 (1998).
15. Y. Kuang, *Delay Differential Equations, with Applications in Population Dynamics*, Academic Press, New York, 1993.
16. Z. Teng and M. Rehim, Permanence in nonautonomous predator-prey systems with infinite delays, *J. Comput. Appl. Math.* 197:302–321 (2006).
17. J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, Heidelberg, 1977.
18. B. Lisena, Global attractivity in nonautonomous logistic equations with delay, *Nonlinear Anal.: Real World Appl.* 9:53–63 (2008).
19. Z. Teng and L. Chen, The positive periodic solutions of periodic Kolmogorov type systems with delays, *Acta Math. Appl. Sin.* 22:456–464 (1999).
20. Q. Liu and R. Xu, Persistence and global stability for a delayed nonautonomous single-species model with dispersal and feedback control, *Diff. Equat. Dyn. Syst.* 11:353–367 (2003).

Xi'an Jiaotong University
Institute for Information and System Sciences
Xinjiang University

(Received August, 2007)