Nonlinear Analysis: Real World Applications 11 (2010) 1364–1373

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Qualitative analysis of a modified Leslie–Gower and Holling-type II predator–prey model with state dependent impulsive effects[☆]

Linfei Nie[a,b,*], Zhidong Teng[a], Lin Hu[a], Jigen Peng[c]

[a] College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China

[b] Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, China

[c] Institute for Information and System Sciences, Research Center for Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, China

ARTICLE INFO

Article history:
Received 4 January 2008
Accepted 25 February 2009

Keywords:
Impulsive differential equations
State-dependent
Leslie–Gower
Holling-type II
Predator–prey system
Periodic solution

ABSTRACT

In this paper, we present a two-dimensional autonomous dynamical system modeling a predator–prey food chain which is based on a modified version of the Leslie–Gower scheme and on the Holling-type II scheme with state dependent impulsive effects. By using the Poincaré map, some conditions for the existence and stability of semi-trivial solution and positive periodic solution are obtained. Numerical results are carried out to illustrate the feasibility of our main results.

1. Introduction

In the last few decades, impulsive differential equations (IDEs) have been extensively used as models in biology, physics, chemistry, engineering and other sciences, with particular emphasis on population dynamics. Many evolution processes in nature are submitted to short temporary perturbations that are negligible compared to the process duration. These short-time perturbations are often assumed to be in the form of impulses in the modeling process. Consequently, IDEs provide a natural description of such processes. In recent years, some IDEs have been introduced in population dynamics (see [1–14] and references therein), such as vaccination, chemotherapeutic treatment of disease, chemostat, birth pulse, control and optimization, etc. The majority of them just concern the systems with impulses at fixed times. However, impulsive state feedback control strategy is used widely in real life problems. In practical ecological systems, the control measures (by catching, poisoning or releasing the natural enemy, etc.) are taken only when the amount of species reaches a threshold value, rather than the usual impulsive fixed-time control strategy. Recently, a few studies on IDEs with state-dependent impulsive effects were made in [15,16,4,17–20]. In particular, Jiang and Lu [15,16] obtained the sufficient conditions of existence and stability of semi-trivial solution, and positive periodic solution for some systems by using the Poincaré map.

On the other hand, a two-dimensional system of autonomous differential equations modeling a predator–prey system, which incorporates a modified version of the Leslie–Gower functional response, that is the Holling-type II. The system

☆ This work was supported by the National Natural Science Foundation of PR China (60764003), the Major Project of The Ministry of Education of PR China (207130), the Scientific Research Programmes of Colleges in Xinjiang (XJEDU2007G01, XJEDU2006I05) and the Natural Science Foundation of Xinjiang University (XY080103,BS080105).

* Corresponding author at: College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China. Tel.:+86 991 8136698.
E-mail address: nielinfei@xju.edu.cn (L. Nie).

1468-1218/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.nonrwa.2009.02.026

describes a prey population x which serves as food for a predator y, and it can be written as follows:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \left[r_1 - b_1 x(t) - \dfrac{a_1 y(t)}{x(t)+k_1}\right] x(t) \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \left[r_2 - \dfrac{a_2 y(t)}{x(t)+k_2}\right] y(t), \end{cases} \tag{1.1}$$

where x and y represent the population densities at time t; b_1, r_i, a_i and k_i ($i = 1, 2$) are model parameters assuming only positive values. The dynamic behaviors for system (1.1) with impulsive effects at fixed times or not, which have been studied extensively in the literature. For example, Aziz-Alaoui [21] and Nindjin [22] studied system (1.1) and obtained the sufficient condition for boundedness of solutions, existence of an attracting set and global stability of the coexisting interior equilibrium, and Song [2] and Liu [10] considered system (1.1) with impulsive effects at fixed times and established the conditions for the linear stability of trivial periodic solution and semi-trivial periodic solutions, the permanence and existence of a stable pest-eradication periodic solution.

Motivated by the above works, in this paper, we consider the dynamic behaviors for system (1.1) with state dependent impulsive effects. The system is modeled by the following equations:

$$\begin{cases} \left.\begin{aligned} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} &= \left[r_1 - b_1 x(t) - \dfrac{a_1 y(t)}{x(t)+k_1}\right] x(t) \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} &= \left[r_2 - \dfrac{a_2 y(t)}{x(t)+k_2}\right] y(t) \end{aligned}\right\} & x \neq h, \\ \left.\begin{aligned} \Delta x(t) &= x(t^+) - x(t) = -px(t) \\ \Delta y(t) &= y(t^+) - y(t) = qy(t) + \alpha \end{aligned}\right\} & x = h, \end{cases} \tag{1.2}$$

where $h \in (0, \infty)$, $p \in (0, 1)$ and $q \in (-1, \infty)$. When the amount of the prey reaches the threshold h at time t_h, controlling measures are taken and the amount of prey and predator abruptly turn to $(1-p)h$ and $(1+q)y(t_h) + \alpha$, respectively.

This paper is organized as follows. In the next section, as preliminaries, we present some basic definitions, two Poincaré maps and an important lemma. In Section 3, we state and prove a general criterion for the semi-trivial periodic solution and positive periodic solution. Some specific examples are given to illustrate our results in the last section.

2. Preliminaries

The dynamic behaviors for system (1.1) clearly have an unstable focus (0, 0) and two saddle $(r_1/b_1, 0)$ and $(0, r_2k_2/a_2)$ and one locally stable focus (x^*, y^*) under the following condition

(H) $r_1 \le r_2, k_1 \ge k_2$ and $r_2k_2/a_2 < r_1k_1/a_1$,

where

$$x^* = \frac{1}{2a_2b_1}\left\{-(a_1r_2 - a_2r_1 + a_2b_1k_1) + [(a_1r_2 - a_2r_1 + a_2b_1k_1)^2 - 4a_2b_1(a_1r_2k_2 - a_2r_1k_1)]^{\frac{1}{2}}\right\}, \tag{2.1}$$
$$y^* = \frac{r_2(x^* + k_2)}{a_2}.$$

Throughout in this paper, we assume that (H) is held. By the biological background of system (1.2), we only consider system (1.2) in the biological meaning region $D = \{(x, y) : x \ge 0, y \ge 0\}$. Obviously, the global existence and uniqueness of solutions of system (1.2) are guaranteed by the smoothness properties of f, which denotes the mapping defined by right-side of system (1.2) — for details see Lakshmikantham et al. [23], Bainov and Simeonov [24].

Set $R = (-\infty, \infty)$. First, we give the notion of the distance between a point and a set. It is defined as follows. Let $S \in R^2 = \{(x, y) : x \in R, y \in R\}$ be an arbitrary set and $P \in R^2$ be an arbitrary point. Then the distance between the point P and the set S is denoted by

$$d(P, S) = \inf_{P_0 \in S} |P - P_0|.$$

Let $z(t) = (x(t), y(t))$ be any solution of (1.2). Next, we define the positive orbit through the point $z_0 \in R^2_+ = \{(x, y) : x \ge 0, y \ge 0\}$ for $t \ge t_0$ as:

$$O^+(z_0, t_0) = \{z \in R^2_+ : z = z(t), t \ge t_0, z(t_0) = z_0\}.$$

In order for the convenience of statement, in the rest of this paper, we introduce the definitions:

Definition 2.1 (*Orbital Stability*). $z^*(t)$ is said to be orbitally stable, if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any other solution, $z(t)$, of system (1.2) satisfying $|z^*(t_0) - z(t_0)| < \delta$, then $d(z(t), O^+(z_0, t_0)) < \varepsilon$ for $t > t_0$.

Definition 2.2 (*Asymptotic Orbital Stability*). $z^*(t)$ is said to be asymptotically orbitally stable if it is orbitally stable and for any other solution, $z(t)$, of system (1.2), there exists a constant $\eta > 0$ such that, if $|z^*(t_0) - z(t_0)| < \eta$, then $\lim_{t\to\infty} d(z(t), O^+(z_0, t_0)) = 0$.

To discuss the dynamics of system (1.2), we define two cross-sections to the vector field (1.2) by $\sum^p = \{(x, y) : x = (1-p)h, y > 0\}$ and $\sum^h = \{(x, y) : x = h, y > 0\}$. Suppose system (1.2) has a positive T periodic solution $z(t) = (\phi(t), \psi(t))$ with the initial condition $z_0 = z(0) = ((1-p)h, y_0)$, where $y_0 > 0$. Let the periodic trajectory $O^+(z_0, 0)$ intersect the sections $\sum^p$ and $\sum^h$ at the points $S^+((1-p)h, y_0)$ and $S(h, y_1)$, respectively. At the state S, the trajectory of (1.2) is subjected, by impulsive effects to jumps, to the point S^+ again. Thus

$$\phi(0) = (1-p)h, \qquad \psi(0) = y_0, \qquad \phi(T) = h \quad \text{and} \quad \psi(T) = y_1 = \frac{y_0 - \alpha}{1+q}.$$

Now, we consider another solution $\widetilde{z}(t) = (\widetilde{\phi}(t), \widetilde{\psi}(t))$ of small-amplitude perturbation of the periodic solution $z(t)$ with initial condition $\widetilde{z}_0 = \widetilde{z}(0) = ((1-p)h, \widetilde{y}_0)$. Suppose the trajectory $O^+(\widetilde{z}_0, 0)$ which starting from $A_1((1-p)h, \widetilde{y}_0)$ first intersects the section $\sum^h$ at the point $B_1(h, \widetilde{y}_1)$ when $t = T + \delta t$ and then jumps to the point $A_2((1-p)h, \widetilde{y}_2)$ on the section $\sum^p$. Then, we have

$$\widetilde{\phi}(0) = (1-p)h, \qquad \widetilde{\psi}(0) = \widetilde{y}_0, \qquad \widetilde{\phi}(T+\delta t) = h \quad \text{and} \quad \widetilde{\psi}(T+\delta t) = \widetilde{y}_1.$$

Let $u(t) = \widetilde{\phi}(t) - \phi(t)$ and $v(t) = \widetilde{\psi}(t) - \psi(t)$, then $u_0 = u(0) = \widetilde{\phi}(0) - \phi(0) = 0$ and $v_0 = v(0) = \widetilde{\psi}(0) - \psi(0)$. Let $v_1 = \widetilde{y}_2 - y_0$ and $v_0^* = \widetilde{y}_1 - y_1$. It is known that for $0 < t < T$, the variables $u(t)$ and $v(t)$ are described by the relation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + o(u_0^2 + v_0^2) = \Phi(t) \begin{pmatrix} 0 \\ v_0 \end{pmatrix} + o\begin{pmatrix} 0 \\ v_0^2 \end{pmatrix}, \tag{2.2}$$

where the fundamental solution matrix $\Phi(t)$ satisfies the matrix equation

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \begin{pmatrix} r_1 - 2b_1\phi(t) - \dfrac{a_1k_1\psi(t)}{(\phi(t)+k_1)^2} & -\dfrac{a_1\phi(t)}{\phi(t)+k_1} \\ \dfrac{a_2\psi^2(t)}{(\phi(t)+k_2)^2} & r_2 - \dfrac{2a_2(t)\psi(t)}{\phi(t)+k_2} \end{pmatrix} \Phi(t) \tag{2.3}$$

with $\Phi(0) = I$ (the unit matrix). Set $f_1(t) = \phi(t)[r_1 - b_1\phi(t) - a_1\psi(t)/(\phi(t)+k_1)]$ and $f_2(t) = \psi(t)[r_2 - a_2\psi(t)/(\phi(t)+k_2)]$. We can express the perturbed trajectory in a first-order Taylor expansion

$$\begin{cases} \widetilde{\phi}(T+\delta t) \approx \phi(T) + u(T) + f_1(T)\delta t, \\ \widetilde{\psi}(T+\delta t) \approx \psi(T) + v(T) + f_2(T)\delta t. \end{cases}$$

From $\widetilde{\phi}(T+\delta t) = \phi(T) = h$, we have

$$\delta t = -\frac{u(T)}{f_1(T)} \quad \text{and} \quad v_0^* = \widetilde{y}_1 - y_1 = v(T) - \frac{f_2(T)u(T)}{f_1(T)}.$$

In view of $\widetilde{y}_2 = (1+q)\widetilde{y}_1 + \alpha$ and $\widetilde{y}_2 - y_0 = (1+q)(\widetilde{y}_1 - y_1)$, thus $v_1 = (1+q)v_0^*$. So, we defined the Poincaré map of $\sum^p$ as follows:

$$v_2 = P_1(q, v_0) = (1+q)\left[v(T) - \frac{f_2(T)u(T)}{f_1(T)}\right], \tag{2.4}$$

where $u(T)$ and $v(T)$ are calculated according to (2.2).

Now, we consider the another Poincaré map. Suppose that the point $S_n(h, y_n)$ is on the section $\sum^h$. Then the trajectory $O^+(S_n, t_n)$ jumps to the point $S^+((1-p)h, (1+q)y_n + \alpha)$ on $\sum^p$ due to the impulsive effects, and then reaches the point $S_{n+1}(h, y_{n+1})$ on the section $\sum^h$ again, where y_{n+1} is decided by y_n and the parameters q and α. Therefore, we defined the Poincaré map of $\sum^h$ as follows:

$$y_{n+1} = P_2(q, \alpha, y_n). \tag{2.5}$$

Next, we consider the autonomous system with impulsive effects

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = P(x, y), & \dfrac{\mathrm{d}y}{\mathrm{d}t} = Q(x, y), & \varphi(x, y) \neq 0, \\ \Delta x = \xi(x, y), & \Delta y = \eta(x, y), & \varphi(x, y) = 0, \end{cases} \tag{2.6}$$

where $P(x, y)$ and $Q(x, y)$ are continuous differential functions defined on R^2, and $\varphi(x, y)$ is a sufficiently smooth function with $grad\varphi(x, y) \neq 0$. Let $(\mu(t), \nu(t))$ be a positive T-periodic solution of system (2.6). By Corollary 2 of Theorem 1 given in Simeonov and Bainov [3], there is the following lemma.

Lemma 2.1 (*Analogue of Poincaré's Criterion*). *If the Floquet multiplier μ satisfies the condition $|\mu| < 1$, where*

$$\mu = \prod_{j=1}^{n} \kappa_j \exp\left[\int_0^T \left(\frac{\partial P(\mu(t), \nu(t))}{\partial x} + \frac{\partial Q(\mu(t), \nu(t))}{\partial y}\right) \mathrm{d}t\right]$$

with

$$\kappa_j = \frac{(\frac{\partial\eta}{\partial y}\frac{\partial\varphi}{\partial x} - \frac{\partial\eta}{\partial x}\frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial x})P_+ + (\frac{\partial\xi}{\partial x}\frac{\partial\varphi}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y})Q_+}{\frac{\partial\varphi}{\partial x}P + \frac{\partial\varphi}{\partial y}Q}$$

and P, Q, $\partial\xi/\partial x$, $\partial\xi/\partial y$, $\partial\eta/\partial x$, $\partial\eta/\partial y$, $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ are calculated at the point $(\mu(\tau_j), \nu(\tau_j))$, $P_+ = P(\mu(\tau_j^+), \nu(\tau_j^+))$, $Q_+ = Q(\mu(\tau_j^+), \nu(\tau_j^+))$ and τ_j $(j \in N)$ is the time of the j-th jump. Then, $(\mu(t), \nu(t))$ is orbitally asymptotically stable.

Let $z(t) = (x(t), y(t))$ be a solution of system (1.2) with initial conditions $z_0 = z(t_0) = ((1-p)h, y_0) \in R_+^2$. This trajectory $O^+(z_0, t_0)$ starts from the point $E_0((1-p)h, y_0)$ first intersects the section $\sum^h$ at the point $F_0(h, \widetilde{y}_0)$, next jumps to the point $E_1((1-p)h, y_1)$ on the section $\sum^p$ due to the impulsive effects, and then reaches the point $F_1(h, \widetilde{y}_1)$ on the section $\sum^p$ again, etc. So, we have two impulsive points' sequences $\{E_m((1-p)h, y_k)\}$ and $\{F_k(h, \widetilde{y}_k)\}(k = 0, 1, 2, \ldots)$. We notice that the coordinates satisfy the relation $y_k = (1+q)\widetilde{y}_{k-1} + \alpha (k = 1, 2, \ldots)$.

Definition 2.3. A trajectory $O^+(z_0, t_0)$ of system (1.2) is said to be order-k periodic if there exist positive integer $k \geq 1$ such that k is the smallest integer for $y_0 = y_k$.

Definition 2.4. A solution $z(t) = (x(t), y(t))$ of system (1.2) is said a semi-trivial solution if its a component is zero and another is nonzero.

3. Main results

3.1. Existence and stability of semi-trivial periodic solution with $\alpha = 0$

Let $y(t) = 0$ for $t \in [0, \infty)$, then from system (1.2) we have

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = [r_1 - b_1 x(t)]x(t), & x \neq h, \\ \Delta x = x(t^+) - x(t) = -px, & x = h. \end{cases}$$

Set $x_0 = x(0) = (1-p)h$, then the solution of equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = [r_1 - b_1 x(t)]x(t)$$

is $x(t) = r_1 \exp(r_1 t)/[b_1 \exp(r_1 t) + c]$, where $c = [r_1 - (1-p)b_1 h]/(1-p)h$. Let $T = r_1^{-1} \ln\{[r_1 - (1-p)b_1 h]/[(1-p)(r_1 - b_1 h)]\}$, then $x(T) = h$ and $x(T^+) = x_0$. This means that system (1.2) with $\alpha = 0$ has the following semi-trivial periodic solution for $(k-1)T < t \leq kT$ $(k = 1, 2, \ldots)$,

$$\begin{cases} \phi(t) = \dfrac{(1-p)hr_1 \exp[r_1(t-(k-1)T)]}{(1-p)b_1 h \exp[r_1(t-(k-1)T)] + r_1 - (1-p)b_1 h}, \\ \psi(t) = 0. \end{cases} \tag{3.1}$$

On the stability of this semi-trivial periodic solution, we have the following result.

Theorem 3.1. *If the following condition*

$$-1 < q < \left[\frac{r_1 - (1-p)b_1 h}{(1-p)(r_1 - b_1 h)}\right]^{-\frac{r_2}{r_1}} - 1 \tag{3.2}$$

holds. Then, (3.1) is a stable semi-trivial periodic solution of system (1.2) with $\alpha = 0$.

Proof. In fact, from $\psi(t) = 0$ and (2.3), we have

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \begin{pmatrix} r_1 - 2b_1\phi(t) & -\dfrac{a_1\phi(t)}{\phi(t) + k_1} \\ 0 & r_2 \end{pmatrix} \Phi(t), \qquad M(0) = I. \tag{3.3}$$

Let

$$\Phi(t)=\begin{pmatrix}\phi_{11}(t) & \phi_{12}(t)\\ \phi_{21}(t) & \phi_{22}(t)\end{pmatrix}.$$

Then, by (3.3) we can obtain that

$$\begin{cases}\dfrac{\mathrm{d}\phi_{11}(t)}{\mathrm{d}t}=[r_1-2b_1\phi(t)]\phi_{11}(t)-\dfrac{a_1\phi(t)}{\phi(t)+k_1}\phi_{21}(t), & \phi_{11}(0)=1,\\ \dfrac{\mathrm{d}\phi_{12}(t)}{\mathrm{d}t}=[r_1-2b_1\phi(t)]\phi_{12}(t)-\dfrac{a_1\phi(t)}{\phi(t)+k_1}\phi_{22}(t), & \phi_{12}(0)=0,\\ \dfrac{\mathrm{d}\phi_{21}(t)}{\mathrm{d}t}=r_2\phi_{21}(t), & \phi_{21}(0)=0,\\ \dfrac{\mathrm{d}\phi_{22}(t)}{\mathrm{d}t}=r_2\phi_{22}(t), & \phi_{22}(0)=1,\end{cases} \tag{3.4}$$

for $0<t<T$.

Let $\widetilde{z}(t)=(\widetilde{\phi}(t),\widetilde{\psi}(t))$ be any positive solution of system (1.2) with the initial condition $\widetilde{z}(0)=((1-p)h,y_0)$ $(y_0\geq 0)$. Note that $u_0=0$ and $f_2(T)=0$, from (2.2) and (2.4), we have

$$\begin{aligned}v_1&=(1+q)\left[v(T)-\frac{f_2(T)u(T)}{f_1(T)}\right]=(1+q)v(T)\\ &=(1+q)[\phi_{21}(T)u_0+\phi_{22}(T)v_0]=(1+q)\phi_{22}(T)v_0.\end{aligned}$$

On the other hand, from the fourth equation of (3.4) and $T=r_1^{-1}\ln\{[r_1-(1-p)b_1h]/[(1-p)(r_1-b_1h)]\}$, we obtain that

$$\phi_{22}(T)=\left[\frac{r_1-(1-p)b_1h}{(1-p)(r_1-b_1h)}\right]^{\frac{r_2}{r_1}}.$$

Therefore,

$$v_1=(1+q)\phi_{22}(T)v_0=(1+q)v_0\left[\frac{r_1-(1-p)b_1h}{(1-p)(r_1-b_1h)}\right]^{\frac{r_2}{r_1}}.$$

Note that $y_0=0$ is a fixed point of $P_1(q,v_0)$ and

$$D_{v_0}P_1(q,0)=(1+q)\left[\frac{r_1-(1-p)b_1h}{(1-p)(r_1-b_1h)}\right]^{\frac{r_2}{r_1}}.$$

If (3.2) holds, then $0<D_{v_0}P_1(q,0)<1$. Thus, (3.1) be a stable semi-trivial periodic solution of system (1.2) with $\alpha=0$. This completes the proof of this theorem. □

Remark 3.1. We note that system (1.2) with $\alpha=0$ has a fold bifurcation at $q=q_0=\{[r_1-(1-p)b_1h]/[(1-p)(r_1-b_1h)]\}^{-r_2/r_1}-1$, since $D_{v_0}P_1(q_0,0)=1$.

3.2. Existence and stability of positive periodic solutions

From the geometrical construction of the phase space of system (1.2), we note that the trajectory from any initial point (x_0,y_0) with $x_0<h$ intersects the section $\sum^h$ infinite times if $h\leq x^*$ (defined by (1.2)). However, the trajectory from any initial point (x_0,y_0) with $x_0>h$ intersects the section $\sum^h$ finite times if $h>x^*$. So, in this subsection, we give the sufficient conditions for the existence and stability of positive periodic solutions in the cases of $h\leq x^*$ and $h>x^*$, respectively.

Case I: *The case of* $h\leq x^*$

On the existence of positive periodic solution of system (1.2), we have the following theorem.

Theorem 3.2. *For any $q>0$ and $\alpha>0$, system (1.2) has a positive order-1 periodic solution.*

Proof. Let the point $M_1((1-p)h,\beta_0)$ be on the section $\sum^p$, where β_0 is small enough and $\beta_0<\alpha$. Then, the trajectory $O^+(M_1,t_0)$ of system (1.2) starting from the initial point M_1 intersects the section $\sum^h$ at the point $N_1(h,\beta_1)$. At the state N_1, the trajectory $O^+(M_1,t_0)$ is subjected by impulsive effects to jumps to the point $M_2((1-p)h,(1+q)\beta_1+\alpha)$ on the section $\sum^p$, and then reaches the $N_2(h,\beta_2)$ on the section $\sum^h$ again. Since $(1+q)\beta_1+\alpha>\beta_0$, then the point M_2 is above the point M_1. Further, the point N_2 is above the point N_1 and $\beta_2>\beta_1$. So, from (2.6) we have $\beta_2=P_2(q,\alpha,\beta_1)$ and

$$\beta_1-P_2(q,\alpha,\beta_1)=\beta_1-\beta_2<0. \tag{3.5}$$

On the other hand, suppose that the curve $L : r_1 - b_1x - a_1y/(x + k_1) = 0$ intersects the section $\sum^p$ at the point $E_0((1-p)h, [r_1 - (1-p)b_1h][(1-p)h + k_1]/a_1)$. The trajectory $O^+(E_0, t_0)$ from the initial point E_0 intersects the section $\sum^h$ at the point $F_1(h, y_1)$, next jumps to the point $F_1^+((1-p)h, (1+q)y_1 + \alpha)$ on the section $\sum^p$ and then reaches the point $F_2(h, y_2)$ on the section $\sum^h$ again. Suppose that there is a positive constant q^* such that $(1+q)y_1 + \alpha = [r_1 - (1-p)b_1h][(1-p)h + k_1]/a_1$. Then, the point F_1^+ coincides with the point E_0 just for $q = q^*$, the point F_1^+ is above the point E_0 for $q > q^*$ and it is under the point E_0 for $q < q^*$. However, From the geometrical construction of the phase space of system (1.2), we obtain that the point F_2 is under the point F_1 for any $q \in (-1, q^*) \cup (q^*, \infty)$.

From the above discussion, we obtain that

(i) if $y_1 = y_2$, then system (1.2) has a positive order-1 periodic solution;
(ii) if $y_1 > y_2$, then

$$m_1 - P_2(q, \alpha, m_1) = m_1 - m_2 > 0. \tag{3.6}$$

By (3.5) and (3.6), it follows that the Poincaré map (2.5) has a fixed point, that is the system (1.2) has a positive order-1 periodic solution. This completes the proof. □

Next, we state and prove our result on the uniqueness and stability of positive order-1 periodic solutions of system (1.2). It is immediate that if each positive order-1 periodic solution of system (1.2) is stable, then system (1.2) admits a unique positive periodic solution.

Theorem 3.3. *Let $(\phi(t), \psi(t))$ be a positive order-1 T-periodic solution of system (1.2) which starts from the point (h, γ). If the condition*

$$|\mu| = \left|\kappa \exp \int_0^T \left[r_1 - 2b_1\phi(t) - \frac{a_1k_1\psi(t)}{(\phi(t) + k_1)^2} + r_2 - \frac{2a_2\psi(t)}{\phi(t) + k_2}\right] \mathrm{d}t\right| < 1 \tag{3.7}$$

holds, where

$$\kappa = \frac{(1-p)(1+q)[r_1 - (1-p)b_1h - \frac{(1+q)a_1\gamma + a_1\alpha}{(1-p)h + k_1}]}{(r_1 - b_1h - \frac{a_1\gamma}{h+k_2})}.$$

Then $(\phi(t), \psi(t))$ is unique positive order-1 periodic solution of system (1.2) and which is orbitally asymptotically stable and has asymptotic phase property.

Proof. Based on the conclusion of Theorem 3.2, we need only to verify the stability of positive order-1 periodic solutions $(\phi(t), \psi(t))$ of system (1.2). Suppose the solution $(\phi(t), \psi(t))$ intersects sections $\sum^p$ and $\sum^h$ at points $E^+((1-p)h, (1+q)\gamma + \alpha)$ and $E(h, \gamma)$, respectively. Comparing with system (2.6), we have

$$P(x, y) = \left[r_1 - b_1x - \frac{a_1y}{x + k_1}\right]x, \qquad Q(x, y) = \left[r_2 - \frac{a_2y}{x + k_2}\right]y$$

and $\xi(x, y) = -px$, $\eta(x, y) = qy + \alpha$, $\varphi(x, y) = x - h$, $(\phi(T), \psi(T)) = (h, \gamma)$ and $(\phi(T^+), \psi(T^+)) = ((1-p)h, (1+q)\gamma + \alpha)$. Thus

$$\frac{\partial P}{\partial x} = r_1 - 2b_1x - \frac{a_1k_1y}{(x + k_1)^2}, \qquad \frac{\partial Q}{\partial y} = r_2 - \frac{2a_2y}{x + k_2} \tag{3.8}$$

and

$$\frac{\partial \xi}{\partial x} = -p, \qquad \frac{\partial \eta}{\partial y} = q, \qquad \frac{\partial \varphi}{\partial x} = 1, \qquad \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial y} = \frac{\partial \varphi}{\partial y} = 0. \tag{3.9}$$

Further, from (3.8) and (3.9), it follows that

$$\begin{aligned}
\kappa &= \frac{(\frac{\partial \eta}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial \eta}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial x})P_+ + (\frac{\partial \xi}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y})Q_+}{\frac{\partial \varphi}{\partial x}P + \frac{\partial \varphi}{\partial y}Q} \\
&= \frac{P^+(\phi(T^+), \psi(T^+))(1+q)}{P(\phi(T), \psi(T))} \\
&= \frac{(1-p)(1+q)[r_1 - (1-p)b_1h - \frac{(1+q)a_1\gamma + a_1\alpha}{(1-p)h + k_1}]}{(r_1 - b_1h - \frac{a_1\gamma}{h+k_2})}
\end{aligned}$$

and

$$\mu = \kappa \exp \int_0^T \left[r_1 - 2b_1\phi(t) - \frac{a_1k_1\psi(t)}{(\phi(t)+k_1)^2} + r_2 - \frac{2a_2\psi(t)}{\phi(t)+k_2} \right] \mathrm{d}t.$$

By the condition (3.7), we see that system (1.2) satisfies all conditions of Lemma 2.1. Therefore, by Lemma 2.1, the order-1 periodic solution $(\phi(t), \psi(t))$ of system (1.2) is orbitally asymptotically stable and has asymptotic phase property. This completes the proof. □

Remark 3.2. From the proof of Theorem 3.3, we note that if there is a positive constant q_0 such that $|\mu| = 1$, then a flip bifurcation may occur at $q = q_0$.

Case II: *The case of* $h > x^*$

On the existence and stability of positive periodic solution of system (1.2), we have the following theorem.

Theorem 3.4. *Suppose that $h > x^*$, then there is a positive constant $\alpha^* = \alpha(h) > 0$ such that for any $q > -1$ and $\alpha > \alpha^*$, system (1.2) only has a orbitally asymptotically stable positive order-1 or order-2 periodic solution, which is asymptotic orbital stability.*

Proof. In view of the geometrical construction of the phase space of the system (2.1), there is a trajectory Γ which crosses $\sum^p$ at the points $D_1((1-p)h, \widetilde{y}_1)$ and $D_2((1-p)h, \widetilde{y}_2)$ $(\widetilde{y}_2 < \widetilde{y}_1)$, and tangents to the line $L_2 : x = h$ at the point $D_3(h, (r_1 - b_1h)(h+k_1)/a_1)$. So, the trajectory of system (1.2) which starts from the point $((1-p)h, y)$ with $y \in (\widetilde{y}_1, \widetilde{y}_2)$ will not intersect $\sum^h$.

Let $\alpha^* = \widetilde{y}_1 + (r_1 - b_1h)(h+k_1)/a_1$. If the trajectory of system (1.2) which starts from the point $E((1-p), \tilde{y})$ $(\tilde{y} \in (0, \widetilde{y}_1) \cup \widetilde{y}_2)$ will intersect with $\sum^h$ infinite times due to the impulsive effects with $\alpha > \alpha^*$ and $h > x^*$. Suppose the orbit $O^+(E, t_0)$ which intersects $\sum^h$ at the point (h, y_0), then $y_0 \in (0, (r_1 - b_1h)(h+k_1)/a_1)$. Further, by the Poincaré map (2.5) of the section $\sum^h$, it follows that $y_1 = P_2(q, \alpha, y_0)$ and $y_2 = P_2(q, \alpha, y_0)$. Repeating the above process, we have $y_{n+1} = P_2(q, \alpha, y_n)(n = 1, 2, \ldots)$. On the other hand, for any two points $A_i(h, y_i)$ and $A_j(h, y_j)$ on $\sum^h$, where y_i, $y_j \in (0, (r_1 - b_1h)(h+k_1)/a_1)$ and $y_i < y_j$. In view of the impulsive effects, the points $A_i^+((1-p)h, (1+q)y_i + \alpha)$ and $A_j^+((1-p)h, (1+q)y_j + \alpha)$ are above the point D_1. Therefore, from the geometrical construction of the phase space of the system (1.2) we have

$$0 < y_{j+1} < y_{i+1} < \frac{(r_1 - b_1h)(h+k_1)}{a_1} \tag{3.10}$$

for $\alpha > \alpha^*$.

Now, we suppose the trajectory of system (1.2) which starts from the point $E_1((1-p)h, y)$ $(y \in (0, \infty))$ intersects with $\sum^h$ at the point $E_2(h, y_0)$ for the fist time, then $y_0 \in (0, (r_1 - b_1h)(h+k_1)/a_1]$. If $y_0 = (r_1 - b_1h)(h+k_1)/a_1$ and $y = (1+q)y_0 + \tau$, then system (1.2) has a positive order-1 periodic solution, and for any trajectory which starts form the inner $O^+(E_1, t_0)$ will not intersect with $\sum^h$. Therefore, from the Poincaré map (2.5) of the section $\sum^h$, it follows that $y_1 = P_2(q, \alpha, y_0)$ and $y_2 = P_2(q, \alpha, y_1)$. Repeating the above process, we have $y_{n+1} = P_2(q, \alpha, y_n)$ $(n = 3, 4, \ldots)$. In particular, if $y_0 = y_1$, then system (1.2) has a positive order-1 periodic solution, and if $y_0 \neq y_1$ and $y_0 = y_2$, then system (1.2) has a positive order-2 periodic solution.

Next, we discuss the general circumstance, that is $y_0 \neq y_1 \neq y_2 \neq \cdots \neq y_n$ $(n > 2)$.

(a) If $y_0 < y_1$, from (3.10) we obtain that $y_1 > y_2$. In this case, the relation of y_0, y_1 and y_2 is one of the following:

(i) $y_2 < y_0 < y_1$

If $y_2 < y_0 < y_1$, it follows that $y_3 > y_1 > y_2$ by (3.10). Repeating the above process, we have

$$0 < \cdots < y_{2n} < \cdots < y_2 < y_0 < y_1 < \cdots < y_{2n+1} < \cdots < \frac{(r_1 - b_1h)(h+k_1)}{a_1}.$$

(ii) $y_0 < y_2 < y_1$

If $y_0 < y_2 < y_1$, similar to (i) we have

$$y_0 < y_2 < \cdots < y_{2n} < \cdots < y_{2n+1} < \cdots < y_3 < y_1 < \frac{(r_1 - b_1h)(h+k_1)}{a_1}.$$

(b) If $y_0 > y_1$, from (3.10) we obtain that $y_1 < y_2$. In this case, the relation of y_0, y_1 and y_2 is one of the following:

(i) $y_1 < y_0 < y_2$

If $y_1 < y_0 < y_2$, it follows that $y_2 > y_1 > y_3$ by (3.10). Repeating the above process, we have

$$0 < \cdots < y_{2n+1} < \cdots < y_1 < y_0 < y_2 < \cdots < y_{2n} < \cdots < \frac{(r_1 - b_1h)(h+k_1)}{a_1}.$$

(ii) $y_1 < y_2 < y_0$
If $y_1 < y_2 < y_0$, similar to (i) we have

$$0 < y_1 < \cdots < y_{2n+1} < \cdots < y_{2n} < \cdots < y_2 < y_0 < \frac{(r_1 - b_1 h)(h + k_1)}{a_1}.$$

Further, in case (i) of (a), it follows that $\lim_{n\to\infty} y_{2n} = \theta_2$ and $\lim_{n\to\infty} y_{2n+1} = \theta_1$, where $0 < \theta_2 < \theta_1 < (r_1 - b_1 h)(h + k_1)/a_1$. Therefore, we have $\theta_1 = P_2(q, \alpha, \theta_2)$ and $\theta_2 = P_2(q, \alpha, \theta_1)$. So, system (1.2) has a orbitally asymptotically stable positive order-2 periodic solution. Similarly, in case (ii) of (a) and (ii) of (b), system (1.2) has an orbitally asymptotically stable positive order-1 periodic solution. In case (i) of (b), system (1.2) has a orbitally asymptotically stable positive order-2 periodic solution. The proof is completed. □

Remark 3.3. From the proof of Theorem 3.4, we note that the trajectory of system (1.2) passing through the point $((1-p)h, y)$ will not intersect with $\sum^h$ as time increasing and will tend to the focus (x^*, y^*) for any $y \in (\widetilde{y}_2, \widetilde{y}_1)$. Therefore, if all trajectories of system (1.2) will pass through the points $((1-p)h, y)$ with $y \in (\widetilde{y}_2, \widetilde{y}_1)$ after several impulsive effects, they all tend to the focus finally and then there is no positive periodic solution in this case. Therefore, $\alpha > \widetilde{y}_1 + (r_1 - b_1 h)(h + k_1)/a_1$ is a sufficient condition for the trajectory of system (1.2) intersects with $\sum^h$ infinite times in view of the impulsive effects.

4. Example and numerical simulation

In this paper, we investigate a class of a modified Leslie–Gower and Holling-type II predator–prey model with state dependent impulsive effects. By using Poincaré map we give the criteria for the existence and stability of semi-trivial solution and positive periodic solution of system (1.2).

In order to testify the validity of our results, we consider the following two species predator–prey systems with state dependent impulsive effects:

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= \left[0.6 - 0.2x(t) - \frac{0.2y(t)}{x(t)+1.2}\right]x(t) \\ \frac{dy(t)}{dt} &= \left[0.3 - \frac{0.15y(t)}{x(t)+1}\right]y(t) \end{aligned}\right\} & x \neq h, \\ \left.\begin{aligned} \Delta x(t) &= x(t^+) - x(t) = -0.5x(t) \\ \Delta y(t) &= y(t^+) - y(t) = qy(t) + \alpha \end{aligned}\right\} & x = h, \end{cases} \tag{4.1}$$

where $q \geq 0$, $\alpha \geq 0$ and $h > 0$. By (2.1) we obtain that system (4.1) has a focus $(1.1689, 4.3377)$. Now, we consider the impulsive effects influences on the dynamics of system (4.1).

Example 4.1. Existence and stability of semi-trivial periodic solution with $\alpha = 0$.

Let $h = 0$ and $\alpha = 0$, it is easy to compute that system (4.1) has the following semi-trivial periodic solution for $(k-1)T < t \leq kT$ $(k = 1, 2, \ldots)$,

$$\begin{cases} \phi(t) = \dfrac{21\exp[0.6(t-(k-1)T)]}{7\exp[0.6(t-(k-1)T)] + 53}, \\ \psi(t) = 0, \end{cases} \tag{4.2}$$

where $T = 5/3\ln(53/23)$. From Remark 3.1, it follows $q_0 \approx -0.3412$. Therefore, by Theorem 3.1, we have that (4.2) is stable for $-1 < q < q_0$ and is unstable for $q > q_0$. The typical stable and unstable periodic solution of semi-trivial periodic solution (4.2) are shown in Fig. 1(a) and Fig. 1(b), respectively.

Example 4.2. Existence and stability of positive periodic solutions for $h \leq x^*$.

For any $q > 0$, $\alpha > 0$ and $h \leq x^*$, by Theorem 3.2, we obtain that system (4.1) has a positive order-1 periodic solution. Further, if the conditions of Theorem 3.3 hold, then the positive order-1 periodic solution is orbitally asymptotically stable and has asymptotic phase property. For example, we choose $h = 0.7$, $q = 1$ and $\alpha = 6$ in system (4.1), the conditions of Theorem 3.2 hold. So, system (4.1) has a positive order-1 periodic solution. From numerical simulations, we also note that the solution $(\phi(t), \psi(t))$ which starts from $(0.7000, 3.0039)$ is an order-1 periodic solution, which is locally orbitally asymptotically stable and has asymptotic phase property. These are shown in Fig. 2(a) and Fig. 2(b), respectively.

Example 4.3. Existence and stability of positive periodic solutions for $h > x^*$.

In system (4.1), let $h = 1.3$ and $q = 1$. In view of numerical results of system (4.1) without state dependent impulsive effects, the trajectory Γ which starts from the point $(0.650, 39.295)$ and tangents to the line $L_2 : x = 1.3$ at the point $(1.30, 4.25)$. Applying the Theorem 3.4, we have $\alpha^* = 39.295$. So, system (4.1) has no positive periodic solution for any $\alpha < 39.295$ and only has orbitally asymptotically stable positive order-1 or order-2 periodic solutions for any $\alpha > 39.295$ — which are shown in Fig. 3.

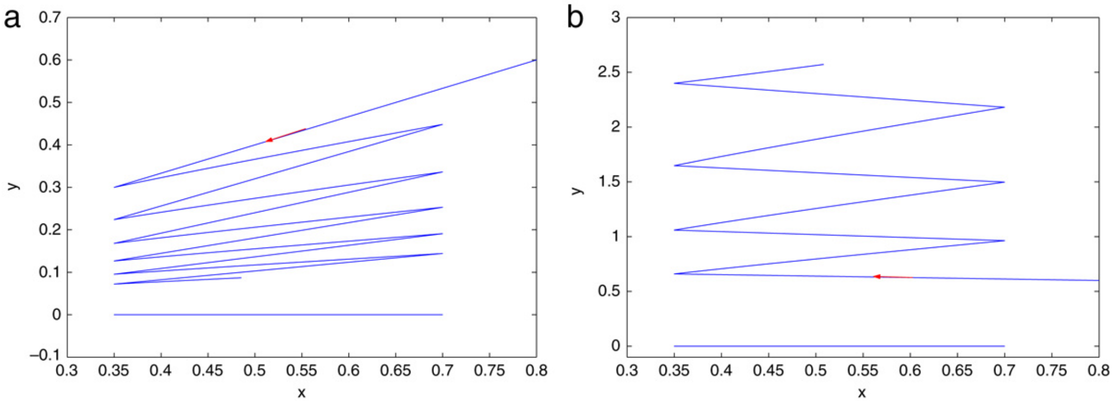

Fig. 1. The trajectory from the initial point (0.8, 0.6) of system (4.1) with $h = 0.7$: (a) $q = -0.5$, (b) $q = 0.1$.

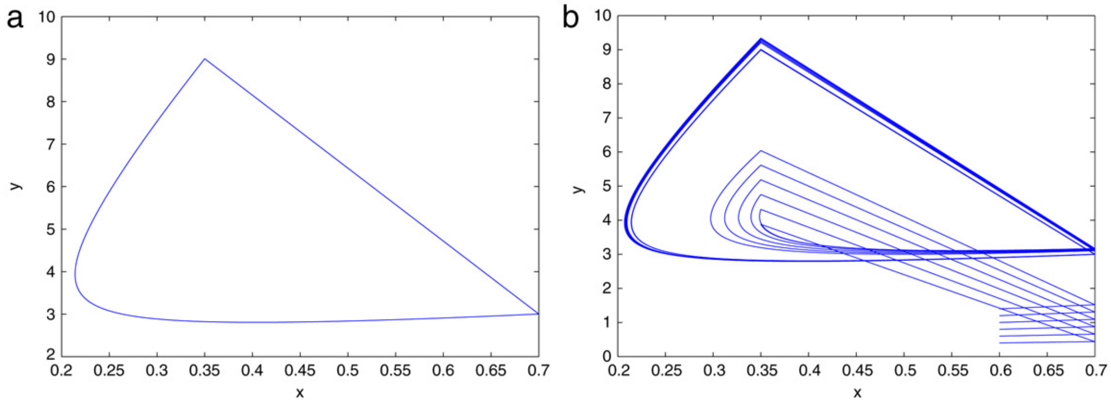

Fig. 2. The trajectory of system (4.1) with $h = 0.7$, $q = 1$ and $\alpha = 6$.

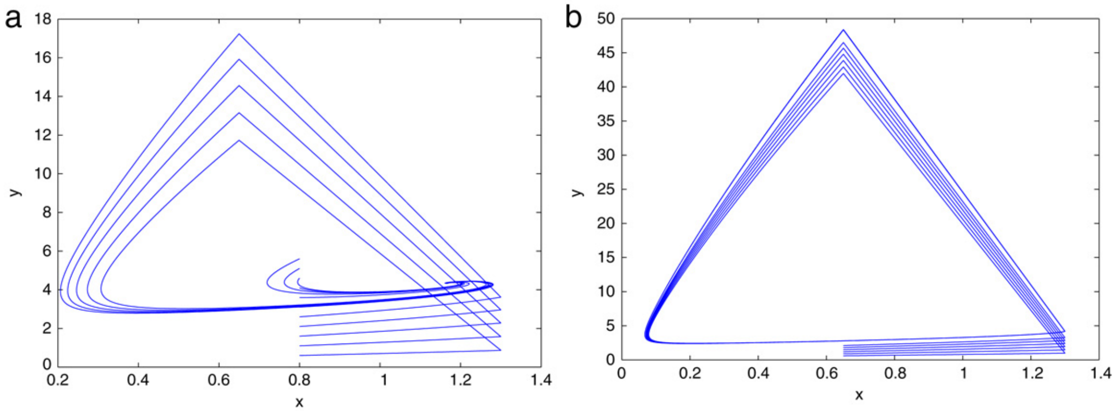

Fig. 3. The trajectory of system (4.1) with $h = 1.3$ and $q = 1$: (a) $\alpha = 10$, (b) $\alpha = 40$.

Acknowledgments

We would like to thank the referees for their careful reading of the original manuscript and many valuable comments and suggestions that greatly improved the presentation of this paper.

References

[1] S. Tang, L. Chen, Density-dependent birth rate birth pulses and their population dynamic consequences, J. Math. Biol. 44 (2002) 185–199.

[2] X. Song, Y. Li, Dynamic behaviors of the periodic predator–prey model with modified Leslie–Gower Holling-type II schemes and impulsive effect, Nonlinear Anal. RWA 9 (2008) 64–79.
[3] P.S. Simeonov, D.D. Bainov, Orbital stability of periodic solutions of autonomous systems with impulse effect, Internat. J. Systems Sci. 19 (1988) 2561–2585.
[4] G. Zeng, L. Chen, L. Sun, Existence of periodic solution of order one of planar impulsive autonomous system, J. Comput. Appl. Math. 186 (2006) 466–481.
[5] S. Gao, L. Chen, Z. Teng, Impulsive vaccination of an SEIRS model with time delay and varying total population size, Bull. Math. Biol. (2006) doi:10.1007/s11538-006-9149-x.
[6] S. Gao, Z. Teng, J.J. Nieto, A. Torres, Analysis of an SIR epidemic model with pulse vaccination and distributed time delay, J. Biomedicine Biotechnology (2007) Article No.: 64870.
[7] W. Wang, J. Shen, J.J. Nieto, Permanence and periodic solution of predator–prey system with Holling type functional response and impulses, Discrete Dyn. Nat. Soc. (2007) doi:10.1155/2007/81756.
[8] T. Zhang, Z. Teng, Extinction and permanence for a pulse vaccination delayed SEIRS epidemic model, Chaos Solitons Fractals (2007) doi:10.1016/j.chaos.2007.07.012.
[9] X. Liu, K. Rohlf, Impulsive control of Lotka–Volterra system, IMA J. Math. Control Inform. 15 (1998) 269–284.
[10] X. Liu, Stability results for impulsive differential systems with application to population growth models, Dyn. Stab. Syst. 9 (2) (1994) 163–174.
[11] B. Liu, Z. Teng, L. Chen, Analysis of a predator–prey model with Holling II functional response concerning impulsive control strategy, J. Comput. Appl. Math. 193 (2006) 347–362.
[12] S. Ahmad, I.M. Stamova, Asymptotic stability of competitive systems with delays and impulsive perturbations, J. Math. Anal. Appl. (2007) doi:10.1016/j.jmaa.2006.12.068.
[13] A. D'onof, Stability properties of pulse vaccination strategy in SEIR epidemic model, Math. Biosci. 179 (2002) 57–72.
[14] G. Ballinger, X. Liu, Permanence of population growth models with impulsive effects, Math. Comput. Modelling 26 (1997) 59–72.
[15] G. Jiang, Q. Lu, Impulsive state feedback control of a predator–prey model, J. Comput. Appl. Math. 200 (2007) 193–207.
[16] G. Jiang, Q. Lu, L. Qian, Complex dynamics of a Holling type II prey–predator system with state feedback control, Chaos Solitons Fractals 31 (2007) 448–461.
[17] F. Wang, G. Pang, L. Chen, Qualitative analysis and applications of a kind of state-dependent impulsive differential equations, J. Comput. Appl. Math. (2007) doi:10.1016/j.cam.2007.05005.
[18] S. Tang, Y. Xiao, L. Chen, R.A. Cheke, Integrated pest management models and their dynamical behaviour, Bull. Math. Biol. 67 (2005) 115–135.
[19] L. Nie, J. Peng, Z. Teng, L. Hu., Existence and stability of periodic solution of a Lotka–Volterra predator–prey model with state dependent impulsive effects, J. Comput. Appl. Math. 224 (2009) 544–555.
[20] L. Nie, Z. Teng, L. Hu, J. Peng, Existence and stability of periodic solution of a predator–prey model with state-dependent impulsive effects, Math. Comput. Simulation 79 (2009) 2122–2134.
[21] M.A. Aziz-Alaoui, M. Daher Okiye, Boundedness and global stability for a predator–prey model with modified Leslie–Gower and Holling-type II schemes, Appl. Math. Lett. 16 (2003) 1069–1075.
[22] A.F. Nindjin, M.A. Aziz-Alaoui, Analysis of a predator–prey model with modified Leslie–Gower and Holling-type II schemes with time delay, Nonlinear Anal. RWA 7 (2006) 1104–1118.
[23] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, Singapore, World Scientific, 1989.
[24] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, vol. 66, Longman, 1993.